

# ON HYPERBOLIC GROUPS WITH SPHERES AS BOUNDARY

ARTHUR BARTELS, WOLFGANG LÜCK, AND SHMUEL WEINBERGER

*Dedicated to Steve Ferry on the occasion of his 60th birthday*

ABSTRACT. Let  $G$  be a torsion-free hyperbolic group and let  $n \geq 6$  be an integer. We prove that  $G$  is the fundamental group of a closed aspherical manifold if the boundary of  $G$  is homeomorphic to an  $(n - 1)$ -dimensional sphere.

## INTRODUCTION

If  $G$  is the fundamental group of an  $n$ -dimensional closed Riemannian manifold with negative sectional curvature, then  $G$  is a hyperbolic group in the sense of Gromov (see for instance [6], [7], [21], [22]). Moreover such a group is torsion-free and its boundary  $\partial G$  is homeomorphic to a sphere. This leads to the natural question whether a torsion-free hyperbolic group with a sphere as boundary occurs as fundamental group of a closed aspherical manifold (see Gromov [23, page 192]). We settle this question if the dimension of the sphere is at least 5.

**Theorem A.** *Let  $G$  be a torsion-free hyperbolic group and let  $n$  be an integer  $\geq 6$ . The following statements are equivalent:*

- (i) *the boundary  $\partial G$  is homeomorphic to  $S^{n-1}$ ;*
- (ii) *there is a closed aspherical topological manifold  $M$  such that  $G \cong \pi_1(M)$ , its universal covering  $\widetilde{M}$  is homeomorphic to  $\mathbb{R}^n$  and the compactification of  $\widetilde{M}$  by  $\partial G$  is homeomorphic to  $D^n$ ;*

The aspherical manifold  $M$  appearing in our result is unique up to homeomorphism. This is a consequence of the validity of the Borel Conjecture for hyperbolic groups [2], see also Section 3.

The proof depends on the surgery theory for homology ANR-manifolds due to Bryant-Ferry-Mio-Weinberger [9] and the validity of the  $K$ - and  $L$ -theoretic Farrell-Jones Conjecture for hyperbolic groups due to Bartels-Reich-Lück [4] and Bartels-Lück [2]. It seems likely that this result holds also if  $n = 5$ . Our methods can be extended to this case if the surgery theory from [9] can be extended to the case of 5-dimensional homology ANR-manifolds – such an extension has been announced by Ferry-Johnston. We also hope to give a treatment elsewhere by more algebraic methods.

We do not get information in dimensions  $n \leq 4$  for the usual problems about surgery. For instance, our methods give no information in the case, where the boundary is homeomorphic to  $S^3$ , since virtually cyclic groups are the only hyperbolic groups which are known to be good in the sense of Friedman [19]. In the case  $n = 3$  there is the conjecture of Cannon [11] that a group  $G$  acts properly, isometrically and cocompactly on the 3-dimensional hyperbolic plane  $\mathbb{H}^3$  if and only if it is a hyperbolic group whose boundary is homeomorphic to  $S^2$ . Provided that the

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infinite hyperbolic group  $G$  occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina-Mess [5, Theorem 4.1] have shown that its universal cover is homeomorphic to  $\mathbb{R}^3$  and its compactification by  $\partial G$  is homeomorphic to  $D^3$ , and the Geometrization Conjecture of Thurston implies that  $M$  is hyperbolic and  $G$  satisfies Cannon's conjecture. The problem is solved in the case  $n = 2$ , essentially as a consequence of Eckmann's theorem that 2 dimensional Poincaré duality groups are surface groups (see [16]). Namely, for a hyperbolic group  $G$  its boundary  $\partial G$  is homeomorphic to  $S^1$  if and only if  $G$  is a Fuchsian group (see [12], [18], [20]).

In general the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary  $\partial G$  of an infinite hyperbolic group  $G$  contains an open subset homeomorphic to Euclidean  $n$ -space, then it is homeomorphic to  $S^n$ . This is proved in [25, Theorem 4.4], where more information about the boundaries of hyperbolic groups can be found.

We also prove the following result.

**Theorem B.** *Let  $G$  and  $H$  be a torsion-free hyperbolic groups such that  $\partial G \cong \partial H$ . Then  $G$  can be realized as the fundamental group of a closed aspherical manifold of dimension at least 6 if and only if  $H$  can be realized as the fundamental group of such a manifold.*

*Moreover, even in case that neither can be realized by a closed aspherical manifold, they can both be realized by closed aspherical homology ANR-manifolds, which both have the same Quinn obstruction [30] (see Theorem 1.3 for a review of this notion).*

In particular, if  $G$  is hyperbolic and realized as the fundamental group of a closed aspherical manifold of dimension at least 6, then any torsion-free group  $H$  that is quasi-isometric to  $G$  can also be realized as the fundamental group of such a manifold. This follows from Theorem B, because the homeomorphism type of the boundary of a hyperbolic group is invariant under quasi-isometry (and so is the property of being hyperbolic). The attentive reader will realize that most of the content of Theorem A can also be deduced from Theorem B, as every sphere appears as the boundary of the fundamental group of some closed hyperbolic manifold.

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The techniques and ideas of this paper are very closely related to the work of Steve Ferry; indeed his unpublished work could have been used to simplify some parts of this work. It is a pleasure to dedicate this paper to him on the occasion of his 60th birthday.

## 1. HOMOLOGY MANIFOLDS

A topological space  $X$  is called an *absolute neighborhood retract* or briefly an ANR if it is normal and for every normal space  $Z$ , every closed subset  $Y \subseteq Z$  and every (continuous) map  $f: Y \rightarrow X$  there exists an open neighborhood  $U$  of  $Y$  in  $Z$  together with an extension  $F: U \rightarrow X$  of  $f$  to  $U$ .

**Definition 1.1** (Homology ANR-manifold). *An  $n$ -dimensional homology ANR-manifold  $X$  is an absolute neighborhood retract satisfying:*

- $X$  has a countable base for its topology;
- the topological dimension of  $X$  is finite;
- $X$  is locally compact;
- for every  $x \in X$  the  $i$ -th singular homology group  $H_i(X, X - \{x\})$  is trivial for  $i \neq n$  and infinite cyclic for  $i = n$ .

Notice that a normal space with a countable basis for its topology is metrizable by the Urysohn Metrization Theorem (see [29, Theorem 4.1 in Chapter 4-4 on page 217]) and is separable, i.e., contains a countable dense subset [29, Theorem 4.1]. Notice furthermore that every metric space is normal (see [29, Theorem 2.3 in Chapter 4-4 on page 198]), and has a countable basis for its topology if and only if it is separable (see [29, Theorem 1.3 in Chapter 4-1 on page 191 and Exercise 7 in Chapter 4-1 on page 194]). Hence a homology ANR-manifold in the sense of Definition 1.1 is the same as a generalized manifold in the sense of Daverman [14, page 191]. A closed  $n$ -dimensional topological manifold is an example of a closed  $n$ -dimensional homology ANR-manifold (see [14, Corollary 1A in V.26 page 191]). A homology ANR-manifold  $M$  is said to have the *disjoint disk property (DDP)*, if for any  $\varepsilon > 0$  and maps  $f, g: D^2 \rightarrow M$ , there are maps  $f', g': D^2 \rightarrow M$  so that  $f'$  is  $\varepsilon$ -close to  $f$ ,  $g'$  is  $\varepsilon$ -close to  $g$  and  $f'(D^2) \cap g'(D^2) = \emptyset$ , see for example [9, page 435]. We recall that a *Poincaré duality group*  $G$  is a finitely presented group satisfying the following two conditions: firstly, the  $\mathbb{Z}G$ -module  $\mathbb{Z}$  (with the trivial  $G$ -action) admits a resolution of finite length by finitely generated projective  $\mathbb{Z}G$ -modules; secondly, there is  $n$  such that  $H^i(G; \mathbb{Z}G) = 0$  for  $n \neq i$  and  $H^n(G; \mathbb{Z}G) \cong \mathbb{Z}$ . In this case  $n$  is the formal dimension of the Poincaré duality group  $G$ .

**Theorem 1.2.** *Let  $G$  be a torsion-free group.*

(i) *Assume that*

- *the (non-connective)  $K$ -theory assembly map*

$$H_i(BG; \mathbf{K}_{\mathbb{Z}}) \rightarrow K_i(\mathbb{Z}G)$$

*is an isomorphism for  $i \leq 0$  and surjective for  $i = 1$ ;*

- *the (non-connective)  $L$ -theory assembly map*

$$H_i(BG; {}^w \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \rightarrow L_i^{\langle -\infty \rangle}(\mathbb{Z}G, w)$$

*is bijective for every  $i \in \mathbb{Z}$  and every orientation homomorphism  $w: G \rightarrow \{\pm 1\}$ .*

*Then for  $n \geq 6$  the following are equivalent:*

- (a)  *$G$  is a Poincaré duality group of formal dimension  $n$ ;*
  - (b) *there exists a closed ANR-homology manifold  $M$  homotopy equivalent to  $BG$ . In particular,  $M$  is aspherical and  $\pi_1(M) \cong G$ ;*
- (ii) *If the statements in assertion (i) hold, then the homology ANR-manifold  $M$  appearing there can be arranged to have the DDP;*
- (iii) *If the statements in assertion (i) hold, then the homology ANR-manifold  $M$  appearing there is unique up to  $s$ -cobordism of ANR-homology manifolds.*

*Proof.* (i) The assumption on the  $K$ -theory assembly map implies that  $\text{Wh}(G) = 0$ ,  $\tilde{K}_0(\mathbb{Z}G) = 0$  and  $K_i(\mathbb{Z}G) = 0$  for  $i < 0$ , compare [27, Conjecture 1.3 on page 653 and Remark 2.5 on page 679]. This implies that we can change the decoration in the above  $L$ -theory assembly map from  $\langle -\infty \rangle$  to  $s$  (see [27, Proposition 1.5 on page 664]). Thus the assembly map  $A$  in the algebraic surgery exact sequence [31, Definition 14.6] (for  $R = \mathbb{Z}$  and  $K = BG$ ) is an isomorphism. This implies in particular that the quadratic structure groups  $\mathbb{S}_i(\mathbb{Z}, BG)$  are trivial for all  $i \in \mathbb{Z}$ .

Assume now that  $G$  is a Poincaré duality group of dimension  $n \geq 3$ . We conclude from Johnson-Wall [24, Theorem 1] that  $BG$  is a finitely dominated  $n$ -dimensional Poincaré complex in the sense of Wall [35]. Because  $\tilde{K}_0(\mathbb{Z}G) = 0$  the finiteness obstruction vanishes and hence  $BG$  can be realized as a finite  $n$ -dimensional simplicial complex (see [34, Theorem F]). We will now use Ranicki's (4-periodic) total surgery obstruction  $\bar{s}(BG) \in \bar{\mathbb{S}}_n(BG)$  of the Poincaré complex  $BG$ , see [31, Definition 25.6]. The main result of [9] asserts that this obstruction vanishes if and only

if there is a closed  $n$ -dimensional homology ANR-manifold  $M$  homotopy equivalent to  $BG$ . The groups  $\overline{\mathbb{S}}_k(BG)$  arise in a 0-connected version of the algebraic surgery sequence [31, Definition 15.10]. It is a consequence of [31, Proposition 15.11(iii)] (and the fact that  $L_{-1}(\mathbb{Z}) = 0$ ) that  $\overline{\mathbb{S}}_n(BG) = \mathbb{S}_n(\mathbb{Z}, BG)$ . Since  $\mathbb{S}_n(\mathbb{Z}, BG) = 0$ , we conclude  $\overline{\mathbb{S}}(BG) = 0$ . This shows that (i)a implies (i)b. (In this argument we ignored that the orientation homomorphism  $w: G \rightarrow \{\pm 1\}$  may be non-trivial. The argument however extends to this case, compare [31, Appendix A].) Homology manifolds satisfy Poincaré duality and therefore (i)b implies (i)a.

(ii) It is explained in [9, Section 8] that this homology manifold  $M$  appearing above can be arranged to have the DDP. (Alternatively, we could appeal to [10] and resolve  $M$  by an  $n$ -dimensional homology ANR-manifold with the DDP.)

(iii) The uniqueness statement follows from Theorem 3.1 (ii).  $\square$

In order to replace homology ANR-manifolds by topological manifolds we will later use the following result that combines work of Edwards and Quinn, see [14, Theorems 3 and 4 on page 288], [30]).

**Theorem 1.3.** *There is an invariant  $\iota(M) \in 1 + 8\mathbb{Z}$  (known as the Quinn obstruction) for homology ANR-manifolds with the following properties:*

- (i) *if  $U \subset M$  is an open subset, then  $\iota(U) = \iota(M)$ ;*
- (ii) *let  $M$  be a homology ANR-manifold of dimension  $\geq 5$ . Then the following are equivalent*
  - *$M$  has the DDP and  $\iota(M) = 1$ ;*
  - *$M$  is a topological manifold.*

**Definition 1.4.** *An  $n$ -dimensional homology ANR-manifold  $M$  with boundary  $\partial M$  is an absolute neighborhood retract which is a disjoint union  $M = \text{int } M \cup \partial M$ , where*

- *$\text{int } M$  is an  $n$ -dimensional homology ANR-manifold;*
- *$\partial M$  is an  $(n - 1)$ -dimensional homology ANR-manifold;*
- *for every  $z \in \partial M$  the singular homology group  $H_i(M, M \setminus \{z\})$  vanishes for all  $i$ .*

**Lemma 1.5.** *If  $M$  is an  $n$ -dimensional homology ANR-manifold with boundary, then  $\widehat{M} := M \cup_{\partial M} \partial M \times [0, 1)$  is an  $n$ -dimensional homology ANR-manifold.*

*Proof.* Suppose that  $Y$  is the union of two closed subsets  $Y_1$  and  $Y_2$  and set  $Y_0 := Y_1 \cap Y_2$ . If  $Y_0$ ,  $Y_1$  and  $Y_2$  are ANRs, then  $Y$  is an ANR, see [14, Theorem 7 on page 117]. If  $Y_1$  and  $Y_2$  have countable bases  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of the topology, then sets  $U_1 \setminus Y_2$  with  $U_1 \in \mathcal{U}_1$ ,  $U_2 \setminus Y_1$  with  $U_2 \in \mathcal{U}_2$  and  $(U_1 \cup U_2)^\circ$  with  $U_i \in \mathcal{U}_i$  form a countable basis of the topology of  $Y$ . (Here  $(\ )^\circ$  is the operation of taking the interior in  $Y$ .) If  $Y_1$  and  $Y_2$  are both finite dimensional, then  $Y$  is finite dimensional [29, Theorem 9.2 on page 303]. If  $Y_1$  and  $Y_2$  are both locally compact, then  $Y$  is locally compact.

Thus the only non-trivial requirement is that for  $x = (z, 0) \in \widehat{M}$  with  $z \in \partial M$ , we have  $H_i(\widehat{M}, \widehat{M} \setminus \{x\}) = 0$  if  $i \neq n$  and  $\cong \mathbb{Z}$  if  $i = n$ . Let  $I_z := \{z\} \times [0, 1/2)$ . Because of homotopy invariance we can replace  $\{x\}$  by  $I_z$ . Let  $U_1 := M \cup_{\partial M} \partial M \times [0, 1/2) \subset \widehat{M}$  and  $U_2 := \partial M \times (0, 1) \subset \widehat{M}$ . Then  $H_i(U_1, U_1 \setminus I_z) \cong H_i(M, M \setminus \{z\}) = 0$  and  $H_i(U_2, U_2 \setminus I_z) = 0$ . Because  $U_1$  and  $U_2$  are both open, we can use a Mayer-Vietoris sequence to deduce

$$H_i(\widehat{M}, \widehat{M} \setminus I_z) \cong H_{i-1}(U_1 \cap U_2, U_1 \cap U_2 \setminus I_z) \cong H_{i-1}(\partial M, \partial M \setminus \{z\}).$$

The result follows as  $\partial M$  is an  $(n - 1)$ -dimensional homology ANR-manifold.  $\square$

**Corollary 1.6.** *Let  $M$  be an homology ANR-manifold with boundary  $\partial M$ . If  $\partial M$  is a manifold, then  $\iota(\text{int } M) = 1$ .*

*Proof.* We use  $\widehat{M}$  from Lemma 1.5. If  $\partial M$  is a manifold then so is  $\partial M \times (0, 1)$ . The result follows now from Theorem 1.3.  $\square$

## 2. HYPERBOLIC GROUPS AND ASPHERICAL MANIFOLDS

For a hyperbolic group we write  $\overline{G} := G \cup \partial G$  for the compactification of  $G$  by its boundary, compare [7, III.H.3.12], [5]. Left multiplication of  $G$  on  $G$  extends to a natural action of  $G$  on  $\overline{G}$ . We will use the following properties of the topology on  $\overline{G}$ .

**Proposition 2.1.** *Let  $G$  be a hyperbolic group. Then*

- (i)  $\overline{G}$  is compact;
- (ii)  $\overline{G}$  is finite dimensional;
- (iii)  $\partial G$  has empty interior in  $\overline{G}$ ;
- (iv) the action of  $G$  on  $\overline{G}$  is small at infinity: if  $z \in \partial G$ ,  $K \subset G$  is finite and  $U \subset \overline{G}$  is a neighborhood of  $z$ , then there exists a neighborhood  $V \subseteq \overline{G}$  of  $z$  with  $V \subseteq U$  such that for any  $g \in G$  with  $gK \cap V \neq \emptyset$  we have  $gK \subseteq U$ ;
- (v) if  $z \in \partial G$  and  $U$  is an open neighborhood of  $z$  in  $\overline{G}$ , then for every finite subset  $K \subseteq G$  there is an open neighborhood  $V$  of  $z$  in  $\overline{G}$  such that  $V \subseteq U$  and  $(V \cap G) \cdot K \subseteq U \cap G$ .

*Proof.* (i) see for instance [7, III.H.3.7(4)].

(ii) see for instance [3, 9.3.(ii)].

(iii) is obvious from the definition of the topology in [5].

(iv) see for instance [32, page 531].

(v) follows from (iv): We may assume  $1_G \in K$ . Pick  $V$  as in (iv). If  $g \in V \cap G$  and  $k \in K$ , then  $g \in gK \cap V$ . Thus  $gK \subseteq U$ . Therefore  $gK \in U \cap G$ .  $\square$

Let  $X$  be a locally compact space with a cocompact and proper action of a hyperbolic group  $G$ . Then we equip  $\overline{X} := X \cup \partial G$  with the topology  $\mathcal{O}_{\overline{X}}$  for which a typical open neighborhood of  $x \in X$  is an open subset of  $X$  and a typical (not necessarily open) neighborhood of  $z \in \partial G$  is of the form

$$(U \cap \partial G) \cup (U \cap G) \cdot K$$

where  $U$  is an open neighborhood of  $z$  in  $\overline{G}$  and  $K$  is a compact subset of  $X$  such that  $G \cdot K = X$ . We observe that we could fix the choice of  $K$  in the definition of  $\mathcal{O}_{\overline{X}}$ : let  $U$ ,  $z$  and  $K$  be as above and let  $K'$  be a further compact subset of  $X$  such that  $G \cdot K' = X$ . Because the  $G$ -action is proper, there is a finite subset  $L$  of  $G$  such that  $K' \subseteq L \cdot K$ . By Proposition 2.1 (v) there is an open neighborhood  $V \subseteq U$  of  $z \in \overline{G}$  such that  $(V \cap G) \cdot L \subseteq U \cap G$ . Thus

$$(V \cap \partial G) \cup (V \cap G) \cdot K' \subseteq (U \cap \partial G) \cup (V \cap G) \cdot L \cdot K \subseteq (U \cap \partial G) \cup (U \cap G) \cdot K.$$

If  $f: X \rightarrow Y$  is a  $G$ -equivariant continuous map where  $Y$  is also a locally compact space with a cocompact proper  $G$ -action, then we define  $\overline{f}: \overline{X} \rightarrow \overline{Y}$  by  $\overline{f}|_X := f$  and  $\overline{f}|_{\partial G} := \text{id}_{\partial G}$ .

**Lemma 2.2.** *Let  $G$  be a hyperbolic group and  $X$  be a locally compact space with a cocompact and proper  $G$ -action.*

- (i)  $\overline{X}$  is compact;
- (ii)  $\partial G$  is closed in  $\overline{X}$  and its interior in  $\overline{X}$  is empty;
- (iii) if  $\dim X$  is finite, then  $\dim \overline{X}$  is also finite;
- (iv) if  $f: X \rightarrow Y$  is a  $G$ -equivariant continuous map where  $Y$  is also a locally compact space with a cocompact proper  $G$ -action, then  $\overline{f}$  is continuous.

*Proof.* These claims are easily deduced from the observation following the definition of the topology  $\mathcal{O}_{\overline{X}}$  and Proposition 2.1.  $\square$

We recall that for a hyperbolic group  $G$  equipped with a (left invariant) word-metric  $d_G$  and a number  $d > 0$  the Rips complex  $P_d(G)$  is the simplicial complex whose vertices are the elements of  $G$ , and a collection  $g_1, \dots, g_k \in G$  spans a simplex if  $d_G(g_i, g_j) \leq d$  for all  $i, j$ . The action of  $G$  on itself by left translation induces an action of  $G$  on  $P_d(G)$ . Recall that a closed subset  $Z$  in a compact ANR  $Y$  is a  $Z$ -set if for every open set  $U$  in  $Y$  the inclusion  $U \setminus Z \rightarrow U$  is a homotopy equivalence. An important result of Bestvina-Mess [5] asserts that (for sufficiently large  $d$ )  $\overline{P_d(G)}$  is an ANR such that  $\partial G \subset \overline{P_d(G)}$  is  $Z$ -set. The proof uses the following criterion [5, Proposition 2.1]:

**Proposition 2.3.** *Let  $Z$  be a closed subspace of the compact space  $Y$  such that*

- (i) *the interior of  $Z$  in  $Y$  is empty;*
- (ii)  *$\dim Y < \infty$ ;*
- (iii) *for every  $k = 0, \dots, \dim Y$ , every  $z \in Z$  and every neighborhood  $U$  of  $z$ , there is a neighborhood  $V$  of  $z$  such that every map  $\alpha: S^k \rightarrow V \setminus Z$  extends to  $\tilde{\alpha}: D^{k+1} \rightarrow U \setminus Z$ ;*
- (iv)  *$Y \setminus Z$  is an ANR.*

*Then  $Y$  is an ANR and  $Z \subset Y$  is a  $Z$ -set.*

Condition (iii) is sometimes abbreviated by saying that  $Z$  is  $k$ -LCC in  $Y$ , where  $k = \dim Y$ .

**Theorem 2.4.** *Let  $X$  be a locally compact ANR with a cocompact and proper action of a hyperbolic group  $G$ . Assume that there is a  $G$ -equivariant homotopy equivalence  $X \rightarrow P_d(G)$ . If  $d$  is sufficiently large, then  $\overline{X}$  is an ANR,  $\partial G$  is  $Z$ -set in  $\overline{X}$  and  $Z$  is  $k$ -LCC in  $X$  for all  $k$ .*

*Proof.* Bestvina-Mess [5, page 473] show that (for sufficiently large  $d$ )  $\overline{P_d(G)}$  satisfies the assumptions of Proposition 2.3. Moreover, they show that  $Z$  is  $k$ -LCC in  $\overline{X}$  for all  $k$ . Using this, it is not hard to show, that  $\overline{X}$  satisfies these assumptions as well: Assumptions (i) and (ii) hold because of Lemma 2.2. Assumption (iv) holds because  $X$  is an ANR. Because  $f \mapsto \bar{f}$  is clearly functorial, the homotopy equivalence  $X \rightarrow P_d(G)$  induces a homotopy equivalence  $\overline{X} \rightarrow \overline{P_d(G)}$  that fixes  $\partial G$ . Using this homotopy equivalence it is easy to check that  $\partial G$  is  $k$ -LCC in  $\overline{X}$ , because it is  $k$ -LCC in  $\overline{P_d(G)}$ . Thus Assumption (iii) holds.  $\square$

**Proposition 2.5.** *Let  $M$  be a finite dimensional locally compact ANR which is the disjoint union of an  $n$ -dimensional ANR-homology manifold  $\text{int } M$  and an  $(n-1)$ -dimensional ANR-homology manifold  $\partial M$  such that  $\partial M$  is a  $Z$ -set in  $M$ . Then  $M$  is an ANR-homology manifold with boundary  $\partial M$ .*

*Proof.* The  $Z$ -set condition implies that there exists a homotopy  $H_t: M \rightarrow M$ ,  $t \in [0, 1]$  such that  $H_0 = \text{id}_M$  and  $H_t(M) \subseteq \text{int } M$  for all  $t > 0$ , see [5, page 470].

Let  $z \in \partial M$ . Then the restriction of  $H_1$  to  $M \setminus \{z\}$  is a homotopy inverse for the inclusion  $M \setminus \{z\} \rightarrow M$ . Thus  $H_i(M, M \setminus \{z\}) = 0$  for all  $i$ .  $\square$

There is the following (harder) manifold version of Proposition 2.5 due to Ferry and Seebeck [17, Theorem 5 on page 579].

**Theorem 2.6.** *Let  $M$  be a locally compact with a countable basis of the topology. Assume that  $M$  is the disjoint union of an  $n$ -dimensional manifold  $\text{int } M$  and an  $(n-1)$ -dimensional manifold  $\partial M$  such that  $\text{int } M$  is dense in  $M$  and  $\partial M$  is  $(n-1)$ -LCC in  $M$ . Then  $M$  is an  $n$ -manifold with boundary  $\partial M$ .*

**Theorem 2.7.** *Let  $G$  be a torsion-free word-hyperbolic group. Let  $n \geq 6$ .*

- (i) *The following statements are equivalent:*
  - (a) *the boundary  $\partial G$  has the integral Čech cohomology of  $S^{n-1}$ ;*
  - (b)  *$G$  is a Poincaré duality group of formal dimension  $n$ ;*
  - (c) *there exists a closed ANR-homology manifold  $M$  homotopy equivalent to  $BG$ . In particular,  $M$  is aspherical and  $\pi_1(M) \cong G$ ;*
- (ii) *If the statements in assertion (i) hold, then the homology ANR-manifold  $M$  appearing there can be arranged to have the DDP;*
- (iii) *If the statements in assertion (i) hold, then the homology ANR-manifold  $M$  appearing there is unique up to  $s$ -cobordism of ANR-homology manifolds.*

*Proof.* By [21, page 73] torsion-free hyperbolic groups admit a finite  $CW$ -model for  $BG$ . Thus the  $\mathbb{Z}G$ -module  $\mathbb{Z}$  admits a resolution of finite length of finitely generated free  $\mathbb{Z}G$  modules. By [5, Corollary 1.3] the  $(i-1)$ -th Čech cohomology of the boundary  $\partial G$  agrees with  $H^i(G; \mathbb{Z}G)$ . This shows that the statements (i)a and (i)b in assertion (i) are equivalent.

The Farrell-Jones Conjecture in  $K$ - and  $L$ -theory holds by [2, 4]. This implies that the assumptions of Theorem 1.2 are satisfied, compare [27, Proposition 2.2 on page 685]. This finishes the proof of Theorem 2.7.  $\square$

*Proof of Theorem A.* (i) Let  $G$  be a torsion-free hyperbolic group. Assume that  $\partial G \cong S^{n-1}$  and  $n \geq 6$ . Theorem 2.7 implies that there is a closed  $n$ -dimensional homology ANR-manifold  $N$  homotopy equivalent to  $BG$ . Moreover, we can assume that  $N$  has the DDP. The universal cover  $M$  of  $N$  is an  $n$ -dimensional ANR-homology manifold with a proper and cocompact action of  $G$ . The homotopy equivalence  $N \rightarrow BG$  lifts to a  $G$ -homotopy equivalence  $M \rightarrow EG$ . For sufficiently large  $d$ ,  $P_d(G)$  is a model for  $EG$  (see [21, page 73]). Thus there is a  $G$ -homotopy equivalence  $M \rightarrow P_d(G)$ . Theorem 2.4 implies that  $\overline{M}$  is an ANR and  $\partial G$  is a  $Z$ -set in  $\overline{M}$ . We conclude from Lemma 2.2 that  $\overline{M}$  is compact and has finite dimension. Thus we can apply Proposition 2.5 and deduce that  $\overline{M}$  is a homology ANR-manifold with boundary. Its boundary is a sphere and in particular a manifold. Corollary 1.6 implies that  $\iota(M) = 1$ . By Theorem 1.3 (i) this implies  $\iota(N) = 1$ . Using Theorem 1.3 (ii) we deduce that  $N$  is a topological manifold. By Theorem 2.4 the boundary  $\partial G \cong S^{n-1}$  is  $k$ -LCC in  $M$  for all  $k$ . Therefore we can apply Theorem 2.6 and deduce that  $\overline{M}$  is a manifold with boundary  $S^{n-1}$ . The  $Z$ -condition implies that  $\overline{M}$  is contractible, because  $M$  is contractible as the universal cover of the aspherical manifold  $N$ . The  $h$ -cobordism theorem for topological manifolds implies that  $\overline{M} \cong D^n$ . In particular,  $M \cong \mathbb{R}^n$ . This shows that (i) implies (ii). The converse is obvious.  $\square$

### 3. RIGIDITY

The uniqueness question for the manifold appearing in our result from the introduction is a special case of the Borel Conjecture that asserts that aspherical manifolds are topological rigid: any isomorphism of fundamental groups of two closed aspherical manifolds should be realized (up to inner automorphism) by a homeomorphism. The connection of this rigidity question to assembly maps is well-known and one of the main motivations for the Farrell-Jones Conjecture. For homology ANR-manifolds the corresponding rigidity statement is (because of the lack of an  $s$ -cobordism theorem) somewhat weaker.

**Theorem 3.1.** *Let  $G$  be a torsion-free group. Assume that*

- *the (non-connective)  $K$ -theory assembly map*

$$H_i(BG; \mathbf{K}_{\mathbb{Z}}) \rightarrow K_i(\mathbb{Z}G)$$

- *is an isomorphism for  $i \leq 0$  and surjective for  $i = 1$ ;*
- *the (non-connective)  $L$ -theory assembly map*

$$H_i(BG; {}^w \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \rightarrow L_i^{\langle -\infty \rangle}(\mathbb{Z}G, w)$$

*is bijective for every  $i \in \mathbb{Z}$  and every orientation homomorphism  $w: G \rightarrow \{\pm 1\}$ .*

*Then the following holds:*

- (i) *Let  $M$  and  $N$  be two aspherical closed  $n$ -dimensional manifolds together with isomorphisms  $\phi_M: \pi_1(M) \xrightarrow{\cong} G$  and  $\phi_N: \pi_1(N) \xrightarrow{\cong} G$ . Suppose  $n \geq 5$ .*

*Then there exists a homeomorphism  $f: M \rightarrow N$  such that  $\pi_1(f)$  agrees with  $\phi_N \circ \phi_M^{-1}$  (up to inner automorphism);*

- (ii) *Let  $M$  and  $N$  be two aspherical closed  $n$ -dimensional homology ANR-manifolds together with isomorphisms  $\phi_M: \pi_1(M) \xrightarrow{\cong} G$  and  $\phi_N: \pi_1(N) \xrightarrow{\cong} G$ . Suppose  $n \geq 6$ .*

*Then there exists an  $s$ -cobordism of homology ANR-manifolds  $W = (W, \partial_0 W, \partial_1 W)$ , homeomorphisms  $u_0: M_0 \rightarrow \partial_0 W$  and  $u_1: M_1 \rightarrow \partial_1 W$  and an isomorphism  $\phi_W: \pi_1(W) \rightarrow G$  such that  $\phi_W \circ \pi_1(i_0 \circ u_0)$  and  $\phi_W \circ \pi_1(i_1 \circ u_1)$  agree (up to inner automorphism), where  $i_k: \partial_k W \rightarrow W$  is the inclusion for  $k = 0, 1$ .*

*Proof.* (i) As discussed in the proof of Theorem 1.2 the assumptions imply that  $\text{Wh}(G) = 0$ . Therefore it suffices to show that the structure set  $\mathbb{S}^{TOP}(M)$  (see [31, Definition 18.1]) in the Sullivan-Wall geometric surgery exact sequence consists of precisely one element. This structure set is identified with the quadratic structure group  $\mathbb{S}_{n+1}(M) = \mathbb{S}_{n+1}(BG)$  in [31, Theorem 18.5]. A discussion similar to the one in the proof of Theorem 1.2 shows that our assumptions imply that the quadratic structure group is trivial.

(ii) This follows from a similar argument that uses the surgery exact sequences for homology ANR-manifolds due to Bryant-Ferry-Mio-Weinberger [9, Main Theorem on page 439].  $\square$

#### 4. THE QUINN OBSTRUCTION DEPENDS ONLY ON THE BOUNDARY

Let  $G$  be a torsion-free hyperbolic group. Assume that  $\partial G$  has the integral Čech cohomology of a sphere  $S^{n-1}$  with  $n \geq 6$ . By Theorem 2.7 there is a closed aspherical ANR-homology manifold  $N$  whose fundamental group is  $G$ .

**Proposition 4.1.** *In the above situation the Quinn obstruction (see Theorem 1.3)  $\iota(N)$  depends only on  $\partial G$ .*

*Proof.* Let  $H$  be a further torsion-free hyperbolic group such that  $\partial H \cong \partial G$ . Let  $N'$  be a closed aspherical ANR-homology manifold whose fundamental group is  $H$ . Then both the universal covers  $M$  of  $N$  and  $M'$  of  $N'$  can be compactified to  $\overline{M}$  and  $\overline{M}'$  such that  $\partial G \cong \partial H$  is a  $Z$ -set in both, see Theorem 2.4. Now set  $X := \overline{M} \cup_{\partial G} \overline{M}'$ . We claim that  $X$  is a connected ANR-homology manifold. Thus

$$\iota(N) = \iota(M) = \iota(X) = \iota(M') = \iota(N')$$

by Theorem 1.3 (i). To prove the claim we refer to [1], see in particular pp.1270-1271. Both,  $M$  and  $M'$  are homology manifolds in the sense of this reference. By fact 6 of this reference,  $X$  is also a homology manifold. It remains to show that  $X$  is an ANR. This follows from an argument given during the proof of Theorem 9 of this reference.  $\square$

*Proof of Theorem B.* Let  $G$  and  $H$  be torsion-free hyperbolic groups, such that  $\partial G \cong \partial H$ . Assume that  $G$  is the fundamental group of a closed aspherical manifold of dimension at least 6. Theorem 2.7 (i) implies that  $\partial G \cong \partial H$  has the integral Čech cohomology of a sphere  $S^{n-1}$  with  $n \geq 6$  and that  $H$  is the fundamental group of a closed aspherical ANR-homology manifold  $M$  of dimension  $n$ . Because of Theorem 2.7 (ii) this ANR-homology manifold can be arranged to have the DDP. Now by Proposition 4.1 (and Theorem 1.3 (ii)) we have  $\iota(M) = 1$ . Using Theorem 1.3 (ii) again, it follows that  $M$  is a manifold.

A similar argument works if  $G$  is the fundamental group of closed aspherical homology ANR-manifold that is not necessary a closed manifold.  $\square$

## 5. EXOTIC EXAMPLES

In light of the results of this paper one might be tempted to wonder if for a torsion-free hyperbolic group  $G$ , the condition  $\partial G \cong S^n$  is equivalent to the existence of a closed aspherical manifold whose fundamental group is  $G$ . This is however not correct: Davis-Januszkiewicz and Charney-Davis constructed closed aspherical manifolds whose fundamental group is hyperbolic with boundary not homeomorphic to a sphere. We review these examples below.

**Example 5.1.** (i) For every  $n \geq 5$  there exists an example of an aspherical closed topological manifold  $M$  of dimension  $n$  which is a piecewise flat, non-positively curved polyhedron such that the universal covering  $\widetilde{M}$  is not homeomorphic to Euclidean space (see [15, Theorem 5b.1 on page 383]). There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [13] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic.

(ii) For every  $n \geq 5$  there exists a strictly negative curved polyhedron of dimension  $n$  whose fundamental group  $G$  is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to  $\mathbb{R}^n$ , but the boundary  $\partial G$  is not homeomorphic to  $S^{n-1}$ , see [15, Theorem 5c.1 on page 384 and Remark on page 386].

On the other hand, one might wonder if assertion (ii) in Theorem A can be strengthened to the existence of more structure on the aspherical manifold. Strict hyperbolization [13] can be used to show that in general there may be no smooth closed aspherical manifold in this situation.

**Example 5.2.** Let  $M$  be a closed oriented triangulated PL-manifold. It follows from [13, Theorem 7.6] that there is a hyperbolization  $\mathcal{H}(M)$  of  $M$  has the following properties:

- (i)  $\mathcal{H}(M)$  is a closed oriented PL-manifold. (This uses properties (2) and (4) from [13, p.333].)
- (ii) There is a degree 1-map  $\mathcal{H}(M) \rightarrow M$  under which the rational Pontrjagin classes of  $M$  pull back to those of  $\mathcal{H}(M)$ . In particular, the Pontrjagin numbers of  $M$  and  $\mathcal{H}(M)$  coincide. (See properties (5) and (6) from [13, p.333].)
- (iii)  $M$  is a negatively curved piece-wise hyperbolic polyhedra. In particular  $G := \pi_1(\mathcal{H}(M))$  is hyperbolic. Moreover, by [15, p. 348] the boundary of  $\partial G$  is a sphere.

Suppose that some Pontrjagin number of  $M$  is not an integer. Then the same is true for  $\mathcal{H}(M)$ . In particular  $\mathcal{H}(M)$  does not carry the structure of a smooth manifold. If in addition  $\dim \mathcal{H}(M) = \dim M \geq 5$ , then by Theorem 3.1 (i) any other closed

aspherical manifold  $N$  with  $\pi_1(N) = G$  is homeomorphic to  $M$  and does not carry a smooth structure either. Such manifolds  $M$  exist in all dimensions  $4k$ ,  $k \geq 2$ , see Lemma 5.3. This shows that there are for all  $k \geq 2$  torsion-free hyperbolic groups  $G$  with  $\partial G \cong S^{4k-1}$  that are not fundamental groups of smooth closed aspherical manifolds. In particular such a  $G$  is not the fundamental group of a Riemannian manifold of non-positive curvature.

In the previous example we needed  $PL$ -manifolds that do not carry a smooth structure. Such manifolds are classically constructed using Hirzebruch's Signature Theorem.

**Lemma 5.3.** *Let  $k \geq 2$ . There is an oriented closed  $4k$  dimensional  $PL$ -manifold  $M^{4k}$  whose top Pontrjagin number  $\langle p_k(M^{4k}) \mid [M^{4k}] \rangle$  is not an integer.*

*Proof.* For all  $k \geq 2$  there are smooth framed compact manifolds  $N^{4k}$  whose signature is 8 and whose boundary is a  $4k - 1$ -homotopy sphere, see [8] and [26, Theorem 3.4]. By [33] this homotopy sphere is  $PL$ -isomorphic to a sphere. We can now cone off the boundary and obtain a  $PL$ -manifold  $M^{4k}$  (often called the Milnor manifold) whose only nontrivial Pontrjagin class is  $p_k$  and whose signature  $\sigma(M^{4k})$  is 8. Hirzebruch's Signature Theorem implies that

$$8 = \sigma(M^{4k}) = \frac{2^{2k}(2^{2k-1} - 1)B_k}{2k!} \langle p_k(M^{4k}) \mid [M^{4k}] \rangle$$

where  $B_k$  is the  $k$ -th Bernoulli number, see [26, p. 75]. For  $k = 2, 3$  we have then

$$8 = \frac{7}{45} \langle p_2(M^8) \mid [M^8] \rangle = \frac{62}{945} \langle p_3(M^{12}) \mid [M^{12}] \rangle$$

compare [28, p.225]. This yields examples for  $k = 2, 3$ . Taking products of these examples we obtain examples for all  $k \geq 2$ .  $\square$

## 6. OPEN QUESTIONS

We conclude this paper with two open questions.

- (i) Can the boundary of a hyperbolic group be a ANR-homology sphere that is not a sphere?
- (ii) Can one give an example of a hyperbolic group (with torsion) whose boundary is a sphere, such that the group does not act properly discontinuously on some contractible manifold?

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*E-mail address:* bartelsa@math.uni-muenster.de  
*URL:* <http://www.math.uni-muenster.de/u/bartelsa>

WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, MATHEMATISCHES INSTITUT, EINSTEINSTR. 62,  
 D-48149 MÜNSTER, GERMANY

*E-mail address:* lueck@math.uni-muenster.de  
*URL:* <http://www.math.uni-muenster.de/u/lueck>

WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, MATHEMATISCHES INSTITUT, EINSTEINSTR. 62,  
D-48149 MÜNSTER, GERMANY

*E-mail address:* [shmuel@math.uchicago.edu](mailto:shmuel@math.uchicago.edu)

*URL:* <http://www.math.uchicago.edu/%7Eshmuel/>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVENUE CHICAGO,  
IL 60637-151, U.S.A.