Geometrische Gruppentheorie I

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Contents

0.1 Introduction

These are the lecture notes for a first course on geometric group theory, taught at Universität Münster in Wintersemester 2022/23. The main focus is the Bass-Serre theory of actions of groups on simplicial trees. We take a leisurely route towards it, first considering free groups, free and amalgamated products, and HNN extensions, before proving the main theorems of Bass-Serre theory in full generality. As applications, we consider Kurosh's theorem and the theory of FA groups, ending with the proof that $\mathrm{SL}_3(\mathbb{Z})$ is not a non-trivial amalgam.

0.2 Sources

We mostly follow [Ser03], with some material adapted from [Bog08]. We follow [Chi79] for the proof that the universal cover of a graph of groups is a tree. The overall structure and choice of topics, especially in the earlier sections, owes much to lecture notes of Katrin Tent.

1 Groups and actions

1.1 Elementary notions

A *group* is a set equipped with an associative binary operation (which we write as $a \cdot b$ or ab or, when the group is commutative, $a + b$) with a 2-sided identity element (1 or e or, when commutative, 0), such that every element has a 2-sided inverse (x^{-1} or, when commutative, $-x$). We often write 1 for the trivial subgroup $\{1\}$.

A **left action** of a group G on a set X is a binary operation $G \times X \to X$ (we write $g * x$ or just gx) satisfying $(gh)x = g(hx)$ and $1x = x$.

In other words, a left action is a homomorphism $G \to \mathrm{Sym}(X)$; here $\mathrm{Sym}(X)$ is the group of all permutations of X, with the group operation $\sigma \cdot \tau := \sigma \circ \tau$.

A **right action** is defined analogously; the axiom becomes $x(gh) = (xg)h$. Given a right action, we can define a corresponding left action by $gx := xg^{-1}$ (indeed, then $(gh)x = x(gh)^{-1} = (xh^{-1})g^{-1} = g(hx)$).

We often write $G \circlearrowleft X$ to denote an action of G on X.

Definition 1.1. Let $G \circlearrowleft X$ be a group action.

- The **orbit** of $x \in X$ is $Gx = \{gx : g \in G\} \subseteq X$.
- The **stabiliser** of $x \in X$ is $G_x = \{g \in G : gx = x\} \leq G$.
- The **kernel** of the action is the kernel of the homomorphism $G \to \mathrm{Sym}(X)$, namely $\{g \in G : \forall x \in X.\ gx = x\} = \bigcap_{x \in X} G_x \trianglelefteq G$.

The action is called

- **transitive** if $Gx = X$ for some (equivalently all) $x \in X$;
- **faithful** if the kernel is trivial, i.e. $\forall g, h \in G.\ (\forall x \in X.\ gx = hx \Rightarrow g = h)$;
- **free** if $G_x = 1$ for all $x \in X$, i.e. no $g \in G \setminus \{1\}$ fixes any $x \in X$;
- **regular** if it is transitive and free.

Example 1.2.

- Let X be a set. The group $\mathrm{Sym}(X)$ acts naturally on X on the left.

  Up to isomorphism, $\mathrm{Sym}(X)$ depends only on $|X|$; we write S_n for $\mathrm{Sym}(X)$ when $|X| = n$.
- Let G be a group. The **left regular action** of G on G is defined by $g * x := g \cdot x$.

  This is a regular action, and up to isomorphism every regular left action is of this form.
- Let G be a group. The (right) **conjugation action** of G on G is defined by $x * g := x^g = g^{-1}xg$.

  This is a right action: $x^{gh} = (x^g)^h$.

  The orbit g^G of $g \in G$ is the **conjugacy class** of g.

  The stabiliser of $g \in G$ is its **centraliser** $C(g) = \{h : hg = gh\}$.

  The kernel of the action is the **centre** $Z(G) = \{g : \forall h \in G.\ hg = gh\}$.
- Let $H \leq G$ be a subgroup of a group G. Let G/H be the set of left cosets $\{gH : g \in G\}$.

  Then G acts on G/H by $g * xH := gxH$.

  The kernel of this action is $\bigcap_{g \in G} H^g$, since $G_{xH} = H^{x^{-1}}$, since

$$gxH = xH \Leftrightarrow g^x \in H \Leftrightarrow g \in H^{x^{-1}}.$$

Lemma 1.3 (Orbit-Stabiliser Theorem)**.** *The map*

$$G/G_x \to Gx;\ gG_x \mapsto gx$$

is a bijection. In particular, $|Gx| = |G : G_x|$.

Proof. Well-defined and injective: $gx = hx \Leftrightarrow (h^{-1}g)x = x \Leftrightarrow h^{-1}g \in G_x \Leftrightarrow gG_x = hG_x$.
Surjective: clear. □

Lemma 1.4. $(G_x)^g = G_{g^{-1}x}$ *for* $g \in G$ *and* $x \in X$.

Proof. $h \in G_x \Leftrightarrow hx = x \Leftrightarrow g^{-1}hgg^{-1}x = g^{-1}x \Leftrightarrow h^g \in G_{g^{-1}x}$. □

1.1.1 Quotients

Definition 1.5.

- Recall: a subgroup $N \leq G$ is **normal** in G, written $N \trianglelefteq G$, if $\forall g \in G.\ N^g = N$. Then G/N has the induced structure of a group, and we call it the **quotient** of G by N.
- Given $H \leq G$ and $N \trianglelefteq G$, we write H/N for $\{h/N : h \in H\} = (NH)/N \leq G/N$.
- A group G is **simple** if it has no proper non-trivial normal subgroup $1 \lneq N \ntrianglelefteq G$.

Lemma 1.6 ("Isomorphism Theorems")**.** *Let G be a group.*

(1) $G/\ker\theta \cong \operatorname{im}\theta$ *for* $\theta : G \to H$ *a homomorphism.*

(2) $H/N = (NH)/N \cong H/(N \cap H)$ *for* $N \trianglelefteq G$ *and* $H \leq G$.

(3) $(G/M)/(N/M) \cong G/N$ *for* $M \leq N \leq G$ *with* $M, N \trianglelefteq G$.

1.2 Solvability

Definition 1.7. Let G be a group.

- The **commutator** of $g \in G$ and $h \in G$ is $[g, h] := g^{-1}h^{-1}gh = g^{-1}g^h = (h^{-1})^g h$.
- The **commutator subgroup** (or **derived subgroup**) of G is the subgroup generated by the commutators, $G' := \langle [g, h] : g, h \in G\rangle \trianglelefteq G$.
- G is **perfect** if $G = G'$.

Remark 1.8. $[g, h] = 1 \Leftrightarrow gh = hg$, so a group G is abelian if and only if G' is trivial.

Lemma 1.9. G/G' *is abelian, and it is the "Abelianisation" of* G*, which means that any homomorphism* $\theta : G \to A$ *with* A *abelian factorises uniquely through the quotient map* $\pi : G \to G/G'$*, i.e.* $\theta = \phi \circ \pi$ *for some unique homomorphism* $\phi : G/G' \to A$.

Proof. G/G' is abelian, since $[x/G', y/G'] = [x, y]/G' = 1$.

Given $\theta : G \to A$ with A abelian, we have $G' \leq \ker\theta$, since $\theta([x, y]) = [\theta(x), \theta(y)] = 1$, and so $\phi : G/G' \to A$; $x/G' \mapsto \theta(x)$ is well-defined, and is as required. □

Definition 1.10. Let G be a group.

- We inductively define for $n \in \mathbb{N}$: $G^{(0)} := G$, $G^{(n+1)} := (G^{(n)})'$.
- G is **solvable** if $G^{(n)} = 1$ for some $n \in \mathbb{N}$.

Lemma 1.11.

(i) Every quotient of a solvable group is solvable.

(ii) Every quotient of a perfect group is perfect.

Proof. (i) Let $N \trianglelefteq G$. Then

$$(G/N)' = \langle [g/N, h/N] : g, h \in G\rangle = \langle [g, h]/N : g, h \in G\rangle = G'/N.$$

Then by induction: $(G/N)^{(i)} = G^{(i)}/N$ for $i \geq 0$. So if $G^{(n)}$ is trivial, then also $(G/N)^{(n)}$ trivial.

(ii) Similar: if G is perfect, then $(G/N)' = G'/N = G/N$, so G/N is perfect. □

1.3 Direct and semidirect products

Let G and H be groups. Their direct **product** is the group $G \times H$ with group operation $(g, h) \cdot (g', h') := (gg', hh')$.

More generally, the direct product of a family of groups $(G_i)_{i\in I}$ is the group $\prod_{i\in I} G_i$ with group operation $(g_i)_{i\in I} \cdot (h_i)_{i\in I} := (g_i h_i)_{i\in I}$.

Lemma 1.12. *Let G be a group and let $N, M \trianglelefteq G$ be normal subgroups, and suppose $NM = G$ and $N \cap M = 1$. Then $\theta : N \times M \to G$; $(n, m) \mapsto nm$ is an isomorphism.*

Proof.

- Injectivity:

$$\begin{aligned} nm = n'm' &\Rightarrow (n')^{-1}n = m'm^{-1} \\ &\Rightarrow (n')^{-1}n = 1 = m'm^{-1} \\ &\Rightarrow (n, m) = (n', m'). \end{aligned}$$

- Surjectivity: $NM = G$.
- Homomorphicity: N and M commute, since for $n \in N$ and $m \in M$, $n^{-1}n^m = [n, m] = (m^{-1})^n m$, so by normality $[n, m] \in N \cap M = 1$.

  Hence $\theta((n, m)(n', m')) = nn'mm' = nmn'm' = \theta(n, m)\theta(n', m')$.

□

Now let N and H be groups, and suppose H acts on the left on N by automorphisms, i.e. $n \mapsto h * n$ is an automorphism of N for any $h \in H$. The **semidirect product** of N and H with respect to this action is the group $N \rtimes H$ with underlying set $N \times H$ and group operation

$$(m, g)(n, h) = (m(g * n), gh).$$

The identity element is $(1, 1)$, and $(m, g)^{-1} = (g^{-1} * m^{-1}, g^{-1})$, and the operation is associative:

$$\begin{aligned}((l, f)(m, g))(n, h) &= (l(f * m), fg)(n, h) \\ &= (l(f * m)(fg * n), fgh) \\ &= (l, f)(m(g * n), gh) \\ &= (l, f)((m, g)(n, h)).\end{aligned}$$

Remark 1.13. $N \cong (N, 1) \trianglelefteq N \rtimes H$ and $(N \rtimes H)/(N, 1) \cong H$.

Lemma 1.14. *Let G be a group, suppose $N \trianglelefteq G$ and $H \leq G$, with $NH = G$ and $N \cap H = 1$.*

Then $\theta : N \rtimes H \to G$; $(n, h) \mapsto nh$ is an isomorphism, where $N \rtimes H$ is the semidirect product with respect to the left conjugation action $h * n := n^{h^{-1}} = hnh^{-1}$.

Proof. Bijectivity: as in the previous lemma.

Homomorphicity: $\theta((m, g)(n, h)) = m(g * n)gh = mgnh = \theta(m, g)\theta(n, h)$. □

Lemma 1.15. *Let $G \circlearrowleft X$ be a left action and $N \trianglelefteq G$ a normal subgroup, and suppose the induced action $N \circlearrowleft X$ is regular. Let $x \in X$. Then $G \cong N \rtimes G_x$ with respect to left conjugation.*

Proof. We apply Lemma 1.14.

- $N \cap G_x = 1$ since N acts freely on X.
- $NG_x = G$: Let $g \in G$. Since N acts transitively on X, we have $ngx = x$ for some $n \in N$. Then $ng \in G_x$, so $g \in NG_x$.

□

Example 1.16. Let K be a field. Let A be the group of affine linear transformations of K,

$$A := \{x \mapsto ax + b : a \in K^* := K \setminus \{0\},\ b \in K\}.$$

Then the subgroup of translations $N := \{x \mapsto x + b : b \in K\} \trianglelefteq A$ acts regularly on K, and is normal:

$$(x \mapsto x + b)^{x \mapsto cx+d} = x \mapsto (c^{-1}((cx + d) + b) - dc^{-1}) = x + c^{-1}b.$$

Now N is isomorphic to the additive group K^+, and the stabiliser $G_0 = \{x \mapsto ax : a \in K^*\} \leq A$ is isomorphic to the multiplicative group K^*.

So by Lemma 1.15, $A \cong K^+ \rtimes K^*$ with the multiplication action.

Remark 1.17. Let $G = A \rtimes B$ be a semidirect product, where A and B are abelian (e.g. $K^+ \rtimes K^*$.) Then $G' \leq A$ and $G'' = 1$, so G is solvable.

1.4 Primitive actions and Iwasawa's Criterion

Definition 1.18. Given an action $G \circlearrowleft X$, an equivalence relation $\sim$ on X is **G-equivariant** if

$$\forall g \in G.\ (x \sim y \Leftrightarrow gx \sim gy)$$

(so then G induces an action on the set $X/\sim$ of equivalence classes).

The **trivial** equivalence relations on a set are equality and the equivalence relation with only one equivalence class.

An action $G \circlearrowleft X$ is **primitive** if it is transitive and every G-equivariant equivalence relation is trivial.[1]

Lemma 1.19. *A transitive action $G \circlearrowleft X$ is primitive if and only some (equivalently any) stabiliser $G_x \leq G$ is a maximal subgroup, i.e. there is no $G_x \lneq K \lneq G$.*

Proof.

- $\Rightarrow$: Suppose $G_x \lneq K \lneq G$. Then $gx \sim hx \Leftrightarrow gK = hK$ defines a non-trivial G-equivariant equivalence relation (it is well-defined since: $gx = g'x \Leftrightarrow gG_x = g'G_x \Rightarrow gK = g'K$)).
- $\Leftarrow$: If $\sim$ is a non-trivial G-equivariant equivalence relation, then $\{x\} \subsetneq (x/\sim) \subsetneq X$ holds for some (and hence by transitivity every) x, and then by transitivity $G_x \lneq G_{x/\sim} \lneq G$.

□

Remark 1.20. Only the trivial group is both solvable and perfect.

Theorem 1.21 (Iwasawa's Criterion)**.** *Let $G \circlearrowleft X$ be faithful and primitive, and suppose G is perfect. Let $x \in X$, and suppose $A \trianglelefteq G_x$ is solvable with $G = \langle A^g : g \in G \rangle$.*

Then G is simple.

Proof. Suppose (for a contradiction) $1 \lneq N \ntrianglelefteq G$.

If $N \leq G_x$, then $N \leq G_y$ for all $y \in X$ since G is transitive and $N \trianglelefteq G$. So N lies in the kernel of the action, contradicting faithfulness.

So $N \not\leq G_x$. But G_x is maximal by Lemma 1.19, so $G_x N = G$. It follows that $NA \trianglelefteq G$; indeed, for $j \in G_x$ and $n \in N$ we have

$$\begin{aligned}(NA)^{jn} &= NA^n && (N \trianglelefteq G,\ A \trianglelefteq G_x)\\ &= Nn^{-1}An = NAn = ANn = AN = NA && (N \trianglelefteq G).\end{aligned}$$

Then $G = \langle A^g : g \in G \rangle \leq \langle (NA)^g : g \in G \rangle = NA$, so $NA = G$.

By Lemma 1.11, $G/N = NA/N \cong A/(A \cap N)$ is perfect (since G is) and solvable (since A is), hence trivial by Remark 1.20, contradicting $N \neq G$. □

Example 1.22. A_n is simple for $n \geq 5$, and $\mathrm{PSL}_2(K)$ is simple for K a field with $|K| \geq 4$.

Proof. Apply Iwasawa's criterion to appropriate actions. See exercises. □

[1] In fact, transitivity follows from the second condition. Indeed, suppose the action is not transitive. The orbit equivalence relation $Gx = Gy$ is G-equivariant, so it must be equality. Then G acts trivially, $g * x = x$ for all g, x, so any equivalence relation is G-equivariant and hence trivial. Then we must have $|X| \leq 1$, because otherwise we could define a non-trivial equivalence relation. But this contradicts non-transitivity.

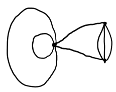
2 Graphs

Definition 2.1.

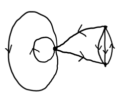
- A **graph** (in the sense of Serre) X consists of
  - two sets, X^0 and X^1,
  - a map $\alpha : X^1 \to X^0$, and
  - a map $\bar{\cdot} : X^1 \to X^1$ such that $\overline{\overline{e}} = e \neq \overline{e}$ for all $e \in X^1$.

  Then
  - X^0 is the set of **vertices** of the graph,
  - X^1 is the set of **edges**,
  - $\overline{e}$ is the **inverse** of $e \in X^1$,
  - $\alpha(e)$ is the **initial vertex** of e,
  - $\omega(e) := \alpha(\overline{e})$ is the **terminal vertex** of e.
- An **oriented graph** is a graph with a distinguished subset $X^+ \subseteq X^1$ of edges, called **positive** edges, such that $\bar{\cdot} : X^+ \to X^- := X^1 \setminus X^+$ is a bijection.

Remark 2.2. We can draw a (finite) graph by drawing the vertices and an arc for each pair $\{e, \overline{e}\}$ of edges.

 .

We can indicate an orientation of a graph by drawing arrows on the edges.

 .

Formally, the **realisation** of a graph X is the topological space real(X) which is the quotient of $X^0 \dot{\cup} (X^1 \times [0,1])$, where X^0 and X^1 have the discrete topology, by the finest equivalence relation such that $(e, 0) \sim \alpha(e)$, $(e, 1) \sim \omega(e)$, and $(e, t) \sim (\overline{e}, 1-t)$. (This is a CW-complex of dimension ≤ 1.)

Remark 2.3. To define an oriented graph, it suffices to specify the sets X^0 and X^+ and maps $\alpha, \omega : X^+ \to X^0$; this extends uniquely up to isomorphism to an oriented graph:

- let $\bar{\cdot} : X^+ \to X^-$ be a bijection with a disjoint set,
- set $X^1 := X^+ \cup X^-$,
- set $\overline{\overline{e}} := e$ and $\alpha(\overline{e}) := \omega(e)$ for $e \in X^+$.

Example 2.4. For $n \in \mathbb{N}$, we define an oriented graph C_n by $(C_n)^0 := \{0, \dots, n-1\}$ and $(C_n)^+ := \{0, \dots, n-1\}$ with $\alpha(i) := i$ and $\omega(i) := i + 1 \mod n$.

Similarly, C_∞ is the oriented graph with $C_\infty^0 := \mathbb{Z}$ and $(C_\infty)^+ := \mathbb{Z}$ with $\alpha(i) := i$ and $\omega(i) := i + 1$.

Definition 2.5.

- A **morphism** of graphs $p : X \to Y$ consists of maps $X^0 \to Y^0$ and $X^1 \to Y^1$ (both denoted by p) such that $\overline{p(e)} = p(\overline{e})$ and $\alpha(p(e)) = p(\alpha(e))$ for all $e \in X^1$.
- As usual, a morphism is an **isomorphism** if it has an inverse morphism; equivalently, if it is bijective.
- A **subgraph** of a graph X is a graph Y with $Y^0 \subseteq X^0$ and $Y^1 \subseteq X^1$ such that the inclusion is a morphism.

  In other words, $Y^0 \subseteq X^0$ and $Y^1 \subseteq X^1$ form a subgraph if and only if $\alpha(Y^1) \subseteq Y^0$ and $\overline{Y^1} = Y^1$.

2.1 Actions on graphs

Definition 2.6. A left **action** $G \circlearrowleft X$ of a group G on a graph X consists of left actions on X^0 and X^1 such that $\overline{ge} = g\overline{e}$ and $\alpha(ge) = g\alpha(e)$ (hence also $\omega(ge) = g\omega(e)$) for all $g \in G$ and $e \in X^1$.

The action is **non-inversive** if $ge \neq \overline{e}$ for all $e \in X^1$ and $g \in G$.

Remark 2.7. An action is non-inversive if and only if it preserves some orientation.

Definition 2.8. The **quotient graph** of a graph X under a non-inversive action $G \circlearrowleft X$ is the graph ${}_{G}\backslash^{X}$ with

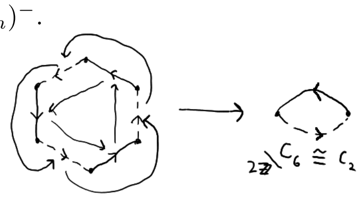
$$\begin{aligned} ({}_{G}\backslash^{X})^0 &:= {}_{G}\backslash^{X^0} = \{Gx : x \in X^0\} \\ ({}_{G}\backslash^{X})^1 &:= {}_{G}\backslash^{X^1} = \{Ge : e \in X^1\} \\ \overline{Ge} &:= G\overline{e} \quad \text{(note } G\overline{e} \neq Ge \text{ since the action is non-inversive)} \\ \alpha(Ge) &:= G\alpha(e). \end{aligned}$$

The natural morphism $p : X \to {}_{G}\backslash^{X}$ (defined by $x \mapsto Gx$ for $x \in X^0 \cup X^1$) is the **quotient morphism**.

If X is oriented and the action preserves the orientation, i.e. $GX^+ = X^+$, then ${}_{G}\backslash^{X}$ has the natural orientation $({}_{G}\backslash^{X})^+ := {}_{G}\backslash^{X^+}$.

Example 2.9. $\mathbb{Z}$ acts non-inversively on C_n by $m * i := i + m \mod n$ for $i \in (C_n)^0$ or $i \in (C_n)^+$, and correspondingly on $(C_n)^-$.

Then ${}_{\mathbb{Z}}\backslash^{C_n} \cong C_1$, and e.g. ${}_{2\mathbb{Z}}\backslash^{C_6} \cong C_2$:



2.2 Barycentric subdivision

Definition 2.10. The **barycentric subdivision** of a graph X is the graph $B(X)$ obtained by dividing each edge in two. Formally:

$$\begin{aligned} B(X)^0 &:= X^0 \dot\cup \{\{e, \overline{e}\} : e \in X^1\} \\ B(X)^1 &:= X^1 \times \{0,1\} \\ \alpha((e,0)) &:= \alpha(e) \\ \alpha((e,1)) &:= \{e, \overline{e}\} \\ \overline{(e,t)} &:= (\overline{e}, 1-t). \end{aligned}$$

Remark 2.11. Topologically, this has no effect: $\mathrm{real}(B(X))$ is homeomorphic to $\mathrm{real}(X)$.

Remark 2.12. Any action $G \circlearrowleft X$ induces a non-inversive action $G \circlearrowleft B(X)$:

$$\begin{aligned} g\{e, \overline{e}\} &:= \{ge, \overline{ge}\} \\ g(e,t) &:= (ge, t). \end{aligned}$$

This is non-inversive, since $g\overline{(e,t)} = g(\overline{e}, 1-t) = (\overline{ge}, 1-t) \neq (e,t)$.

The upshot is that non-inversiveness is not such a restrictive condition: we can always ensure it by barycentrically subdividing.

2.3 Cayley graphs

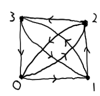
Definition 2.13. Let G be a group, and let $S \subseteq G$.

Then $\Gamma(G,S)$ is the oriented graph with:

$$\begin{aligned} \Gamma(G,S)^0 &:= G \\ \Gamma(G,S)^+ &:= G \times S \\ \alpha(g,s) &:= g \\ \omega(g,s) &:= gs. \end{aligned}$$

If S is a generating set for G (i.e. $\langle S \rangle = G$), $\Gamma(G,S)$ is called the **Cayley graph** of G with respect to S.

Example 2.14.

- $\Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}) \cong C_n$ (for $n \geq 1$),
- $\Gamma(\mathbb{Z}, \{1\}) \cong C_\infty$.
- $\Gamma(\mathbb{Z}/4\mathbb{Z}, \{1,2\})$ is the following oriented graph: 

Remark 2.15. Cayley graphs are connected.

Definition 2.16. The natural left action of G on $\Gamma(G,S)$ is defined by:

$$g * h := gh$$
$$g * (h,s) := (gh,s).$$

Remark 2.17. This action is non-inversive, and ${}_{G}\backslash^{\Gamma(G,S)}$ is the oriented graph with one vertex and a positive edge for each element of S.

The following lemma provides a characterisation which allows us to recognise graphs as being of the form $\Gamma(G,S)$.

Lemma 2.18. *Let X be an oriented non-empty graph and $G \circlearrowleft X$ an orientation-preserving action with $G \circlearrowleft X^0$ regular.*

Suppose:

($*$) *for all $x, y \in X^0$, there is at most one positive edge from x to y.*

Then $X \cong \Gamma(G,S)$ as oriented graphs, where $|S| = |({}_{G}\backslash^{X})^+|$.

Proof. By regularity, we may assume $X^0 = G$ and the action on X^0 is the left regular action (indeed, let $x_0 \in X^0$, then $gx_0 \mapsto g$ defines a suitable bijection $X^0 \to G$).

Let $S := \{\alpha(e)^{-1}\omega(e) : e \in X^+\} \subseteq G$. Note

$$\begin{aligned}\alpha(e)^{-1}\omega(e) = \alpha(e')^{-1}\omega(e') &\Leftrightarrow \alpha(e')\alpha(e)^{-1} = \omega(e')\omega(e)^{-1}\\ &\Leftrightarrow \exists g \in G.\ (\alpha(e'),\omega(e')) = (g\alpha(e), g\omega(e))\\ &\Leftrightarrow \exists g \in G.\ e' = ge \ \ (\text{by } (*))\\ &\Leftrightarrow e' \in Ge,\end{aligned}$$

so $|S| = |({}_{G}\backslash^{X})^+|$.

Define $p : X \to \Gamma(G,S)$ by:

$$\begin{aligned}&p(g) := g \text{ for } g \in G = X^0\\ &p(e) := (\alpha(e), \alpha(e)^{-1}\omega(e)) \text{ for } e \in X^+.\end{aligned}$$

Then p is a morphism, since $\omega(p(e)) = \alpha(e)\alpha(e)^{-1}\omega(e) = \omega(e) = \omega(p(e))$.

To show that p is an isomorphism, it remains to see that $p : X^+ \to \Gamma(G,S)^+$ is a bijection.

- Injectivity:

$$\begin{aligned}p(e) = p(e') &\Rightarrow \alpha(e) = \alpha(e') \text{ and } \alpha(e)^{-1}\omega(e) = \alpha(e')^{-1}\omega(e')\\ &\Rightarrow (\alpha(e),\omega(e)) = (\alpha(e'),\omega(e'))\\ &\Rightarrow e = e' \ \ (\text{by } (*)).\end{aligned}$$

- Surjectivity: $(g, \alpha(e)^{-1}\omega(e)) = p(g\alpha(e)^{-1}e)$.

$\square$

2.4 Trees

Definition 2.19.

- Given a graph and a vertex x, a **path** of *length* $n \in \mathbb{N}$ *from* x is a sequence of edges $(e_i)_{i<n}$ such that $\omega(e_i) = \alpha(e_{i+1})$ for $i < n-1$ and, if $n > 0$, $\alpha(e_0) = x$. The path is *to* $\omega(e_{n-1})$ if $n > 0$, and *to* x if $n = 0$.

  The path is **reduced** if $e_i \neq \overline{e}_{i+1}$ for any $i < n-1$.

  The path is **trivial** if $n = 0$.

- A graph is **connected** if for any vertices x and y, there is a path from x to y.

- A graph is **acyclic** if any reduced path from a vertex to itself has length 0.

- A **tree** is a connected non-empty acyclic graph.

Lemma 2.20. *Let T be a tree. Given $x, y \in T^0$, there is a unique reduced path from x to y.*

Definition 2.21. This path is called the **geodesic** from x to y in T.
Its length is the **distance** $d(x, y)$ between x and y in T.

Proof.

- Existence: By connectedness, there is some path from x to y, and by eliminating any subsequences of the form $(e, \overline{e})$, we obtain a reduced path from x to y.

- Uniqueness: If $(e_0, \dots, e_{n-1})$ and $(f_0, \dots, f_{m-1})$ are two reduced paths from x to y, then $(e_0, \dots, e_{n-1}, \overline{f_{m-1}}, \dots, \overline{f_0})$ is a path from x to x, so either it has length 0, in which case $n = 0 = m$ and we are done, or it is not reduced.

  But then we must have $e_{n-1} = \overline{\overline{f_{m-1}}} = f_{m-1}$, so $(e_0, \dots, e_{n-2})$ and $(f_0, \dots, f_{m-2})$ are reduced paths from x to the same vertex, so inductively they must be equal, and we conclude.

□

2.4.1 Maximal subtrees

Lemma 2.22. *Any subtree of a graph extends to a subtree which is maximal with respect to inclusion.*
In particular, any non-empty graph contains a maximal subtree.

Proof. The union of a chain of subtrees is also a subtree, so the first statement follows from Zorn's lemma.
Any non-empty graph contains a vertex, which forms a subtree and so extends to a maximal subtree. □

Lemma 2.23. *Any maximal subtree of a connected graph contains every vertex.*

Proof. Let T be a maximal subtree of a connected graph X, and suppose $x \in X^0 \setminus T^0$. Now $T^0 \neq \emptyset$, so by connectedness there is a path from x to a vertex of T, some edge e of which is from some $y \in X^0 \setminus T^0$ to a vertex of T.

But then $T^\dagger := T \cup \{e, \overline{e}, y\}$ is a subtree properly containing T, contradicting maximality of T. □

2.4.2 Lifting trees

Definition 2.24.

- The **star** of a vertex x of a graph X is the set of edges with initial vertex x
$$\mathrm{star}^X(x) := \{e \in X^1 : \alpha(e) = x\}.$$
- A morphism $p : X \to Y$ is **locally injective**, resp. **locally surjective**, if for each $x \in X$ the restriction $p|_{\mathrm{star}^X(x)} : \mathrm{star}^X(x) \to \mathrm{star}^Y(p(x))$ is injective, resp. surjective.

Lemma 2.25. *If $p : X \to T$ is a locally injective map from a connected graph to a tree, then p is injective and X is a tree.*

Proof. Exercise. □

Lemma 2.26. *The quotient morphism of a non-inversive action $G \circlearrowleft X$ is locally surjective.*

Proof. Let $x \in X^0$ and let $Ge \in ({}_G\backslash^X)^1$ with $\alpha(Ge) = Gx$. Then $\alpha(e) \in Gx$, so $\alpha(ge) = x$ for some $g \in G$. □

Lemma 2.27. *Let $p : X \to Y$ be a surjective, locally surjective morphism of graphs, and let $T' \subseteq Y$ be a subtree. Then there exists a subtree $T \subseteq X$ such that p restricts to an isomorphism $p|_T : T \to T'$.*

Definition 2.28. T is then called a **lift** of T' (along p).

Proof. Let T be maximal among the subtrees of X such that $p(T) \subseteq T'$ and $p|_T : T \to T'$ is injective; some such subtree exists since p is surjective, and then a maximal such exists by Zorn's lemma. We conclude by showing that $p(T) = T'$.

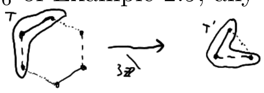
Suppose not. By considering a geodesic from a vertex in $p(T)$ to a vertex outside, we find an edge $e' \in (T')^1 \setminus p(T)^1$ with $\alpha(e') \in p(T)^0$.

Suppose $\omega(e') \in p(T)^0$. Then there is a reduced path from $\alpha(e')$ to $\omega(e')$ in $p(T)$ (the image of a geodesic in T), but then appending $\overline{e'}$ yields a reduced path from $\alpha(e')$ to itself which contradicts acyclicity of T'. So $\omega(e') \notin p(T)^0$.

By local surjectivity, there is $e \in X^1$ with $p(e) = e'$ and $\alpha(e) \in T^0$. Then $\omega(e) \notin T^0$, since $p(\omega(e)) = \omega(e') \notin p(T)^0$, and so $T^\dagger := T \cup \{e, \overline{e}, \omega(e)\}$ is a tree properly extending T, contradicting maximality of T. □

Definition 2.29. A **tree of representatives** for a non-inversive action $G \circlearrowleft X$ on a non-empty connected graph is an arbitrary lift along the quotient morphism of an arbitrary maximal tree in ${}_G\backslash^X$.

(A maximal tree exists by Lemma 2.22, and a lift exists by Lemma 2.26 and Lemma 2.27.)

Example 2.30. For the action $3\mathbb{Z} \circlearrowleft C_6$ of Example 2.9, any subtree of C_6 with 3 vertices is a tree of representatives: 

2.4.3 B_1

Definition 2.31. The **first Betti number** of a non-empty finite connected graph X is

$$B_1(X) := 1 + \frac{1}{2}|X^1| - |X^0|.$$

Lemma 2.32. *Let X be a non-empty finite connected graph.*

Let T be a maximal subtree. Then

$$B_1(X) = \frac{1}{2}|X^1 \setminus T^1|.$$

In particular, $B_1(X) \geq 0$, and $B_1(X) = 0$ iff X is a tree.

Proof. First, we prove $B_1(T) = 0$ for any finite tree T by induction on $|T^1|$. If $|T^1| = 0$, then $|T^0| = 1$ so $B_1(T) = 0$. If $|T^1| > 0$, let $e_0, \dots, e_{n-1}$ be a geodesic of maximal length. Then $T' := T \setminus \{\omega(e_{n-1}), e_{n-1}, \overline{e_{n-1}}\}$ is connected: otherwise some reduced path must include but not end with e_{n-1}, and we could extend the geodesic, contradicting its maximality. So T' is a tree, so $B_1(T') = 0$ by induction. Then $B_1(T) = B_1(T') + \frac{2}{2} - 1 = B_1(T') = 0$.

Now let X and T be as in the statement. We have $X^0 = T^0$ by Lemma 2.23, so

$$B_1(X) = 1+\frac{1}{2}|X^1|-|X^0| = 1+\frac{1}{2}|T^1|+\frac{1}{2}|X^1\setminus T^1|-|T^0| = B_1(T)+\frac{1}{2}|X^1\setminus T^1| = \frac{1}{2}|X^1\setminus T^1|.$$

□

2.4.4 Contracting subtrees

Definition 2.33. Let X be a graph, and let $Y \subseteq X$ be the union $Y = \bigcup_{i \in I} T_i$ of disjoint subtrees $T_i \subseteq X$.

We define the graph X/Y resulting from **contracting** the trees in Y as follows.

Let $\sim$ be the equivalence relation on X^0 whose equivalence classes are: T_i^0 for $i \in I$, and $\{x\}$ for $x \in X^0 \setminus Y^0$. Then:

$$\begin{aligned}
(X/Y)^0 &:= {}^{X^0}/_{\sim} \\
(X/Y)^1 &:= X^1 \setminus Y^1 \\
\alpha^{X/Y}(e) &:= {}^{\alpha(e)}/_{\sim} \\
\overline{e}^{X/Y} &:= \overline{e}
\end{aligned}$$

Lemma 2.34. *Let X be a finite connected graph, and let $Y \subseteq X$ be a union of disjoint subtrees. Then $B_1(X) = B_1(X/Y)$.*

Proof. Inductively, it suffices to consider the case that Y is a single subtree. Then

$$\begin{aligned} B_1(X/Y) &= 1 + \frac{1}{2}|(X/Y)^1| - |(X/Y)^0| \\ &= 1 + \frac{1}{2}(|X^1| - |Y^1|) - (|X^0| - |Y^0| + 1) \\ &= B_1(X) - B_1(Y) = B_1(X). \end{aligned}$$

□

Lemma 2.35. *A non-empty graph X is a tree if and only if $X = \bigcup_{i\in I} T_i$ for some collection $(T_i)_{i\in I}$ of finite subtrees forming a* directed system*, meaning that for any T_i and T_j there exists T_k with $T_i, T_j \subseteq T_k$.*

Proof. If X is a tree, the collection of all finite subtrees is directed, by connectedness of X.

Conversely, $\bigcup_i T_i$ is connected since, by directedness, any two points are contained in a tree, and acyclic because, again by directedness, any finite path is contained in a tree. □

Lemma 2.36. *Let T be a tree, and let $Y \subseteq T$ be a union of disjoint subtrees. Then T/Y is a tree.*

Proof. For finite T, this follows from Lemma 2.34 and Lemma 2.32.

For an arbitrary tree T, we apply Lemma 2.35: write T as the union of a directed system of finite subtrees $T = \bigcup_i T_i$; then $(T_i/(Y \cap T_i))_{i\in I}$ is a directed system of finite subtrees of T/Y with union T/Y, so T/Y is a tree.

More explicitly: here we consider $T_i/(Y \cap T_i)$ as a subgraph of T/Y by restricting the equivalence relation in the definition of the contraction to T_i; this does agree with the equivalence relation in the definition of $T_i/(Y \cap T_i)$, since each tree in Y corresponds to at most one tree in $Y \cap T_i$ – this is because the intersection of two subtrees of a tree is connected, by uniqueness of geodesics. The directedness follows from directedness of the T_i. □

Remark 2.37. If X is a finite connected graph and T is a maximal subtree, then $\mathrm{real}(X/T)$ is a bouquet of $B_1(X)$ circles.

3 Free groups

Definition 3.1. Let X be a subset of a group F. Then F is **free** with **basis** X if for any group G, any map $f : X \to G$ extends uniquely to a homomorphism $f^* : F \to G$.

Lemma 3.2. *If F is free with basis $X \subseteq F$ then $F = \langle X\rangle$.*

Proof. The identity embedding $\iota : X \to \langle X\rangle$ extends to a homomorphism $\iota^* : F \to \langle X\rangle$. Then the composition with the inclusion of $\langle X\rangle$ is a homomorphism $F \to F$ with image $\langle X\rangle$. But this must coincide with the identity map $F \to F$, since both extend the identity embedding $X \to F$. So $\langle X\rangle = F$. □

Lemma 3.3. *If F is free with basis X and F' is free with basis X', then any bijection $X \to X'$ extends to a unique isomorphism $F \to F'$.*

Proof. Let $f : X \to X'$ be a bijection and let $g : X' \to X$ be its inverse. Let $f^* : F \to F'$ and $g^* : F' \to F$ be the unique extensions to homomorphisms.

Then $g^* \circ f^* : F \to F$ extends $\mathrm{id}_X : X \to X$, but so does id_F, so $g^* \circ f^* = \mathrm{id}_F$. Similarly, $f^* \circ g^* = \mathrm{id}_{F'}$. So f^* is an isomorphism. □

So a free group with a given basis is uniquely determined up to unique isomorphism over that basis.

We will now show how to construct, for any set X, a free group $F(X) \supseteq X$ with basis X. By the uniqueness of Lemma 3.3, we will be justified in calling $F(X)$ "the" free group with basis X.

Definition 3.4. Let X be a set.

- Let $X^{\pm}$ be the union of X and a disjoint set X^{-1} which is in bijection with X via a map $\cdot^{-1} : X \to X^{-1}$.

  We extend $\cdot^{-1}$ to an involution $\cdot^{-1} : X^{\pm} \to X^{\pm}$ by defining $(x^{-1})^{-1} := x$ for $x \in X$.

  In other words, $X^{\pm}$ is obtained by adjoining to X a disjoint set of "formal inverses" of the elements of X.

- A **group word** in X is a finite sequence of elements of $X^{\pm}$.

  We denote a word by concatenating these elements, so an arbitrary word of length $n \in \mathbb{N}$ is written $w = {x_0}^{\epsilon_0} \ldots {x_{n-1}}^{\epsilon_{n-1}}$ with $x_i \in X$ and $\epsilon_i \in \{1, -1\}$ (where $x^1 := x$). We also write this word as $\prod_{i<n} {x_i}^{\epsilon_i}$.

  (We often abbreviate "group word" to "word", despite the potential ambiguity.)

- A group word is **reduced** if it contains no subword of the form aa^{-1} for $a \in X^{\pm}$. In other words, $\prod_{i<n} {x_i}^{\epsilon_i}$ is reduced if $x_i = x_{i+1}$ implies $\epsilon_i = \epsilon_{i+1}$ (for all i).

- An **elementary reduction** of a word w is a word which results from deleting from w a subword of the form aa^{-1} for some $a \in X^{\pm}$.

- A **reduction** of a group word w is a reduced word w' obtained by successive elementary reductions, i.e. such that there is a chain $w = w_0, w_1, \ldots, w_n = w'$ of words ($n \in \mathbb{N}$) where each w_{i+1} is an elementary reduction of w_i.

Example 3.5. The group word $ab^{-1}aa^{-1}ba$ in $\{a, b\}$ has reduction aa.

Lemma 3.6. *Any group word has a unique reduction.*

Proof. Existence: since words are finite and elementary reductions decrease the length, any word reduces to a reduced word.

For uniqueness, we first prove:

Claim 3.7. Let w_1 and w_2 be elementary reductions of a group word w. Then w_1 and w_2 have a common reduction.

Proof. Say w_i is formed by deleting a subword $a_i a_i^{-1}$ from w ($i = 1, 2$).

If the subwords are disjoint in w, then we obtain a common subword w' by deleting $a_2 a_2^{-1}$ from w_1 and $a_1 a_1^{-1}$ from w_2. Then any reduction of w' is a common reduction of the w_i.

Otherwise, either they are the same subword, or $a_2 = a_1^{-1}$ and the union of the subwords is a subword $a_1 a_1^{-1} a_1$ or $a_2 a_2^{-1} a_2$. In these cases, $w_1 = w_2$, and any reduction is a common reduction. $\square_{3.7}$

Now suppose w is a word of minimal length with two distinct reductions w_1' and w_2'. Then w has elementary reductions w_1 and w_2 such that w_i' is a reduction of w_i. But by the claim, w_1 and w_2 have a common reduction w', and then by the minimality of w we have

$$w_1' = w' = w_2'.$$

$\square$

Definition 3.8. Let X be a set. Then $F(X)$ is the group on the set of reduced words in X with group operation:

$$ww' := \text{ the reduction of the concatenation of } w \text{ and } w'.$$

Remark 3.9. This does define a group. Associativity follows from uniqueness of reductions: $(w_1w_2)w_3 = w_1(w_2w_3)$ because the order in which we reduce the concatenation of the w_i doesn't affect the reduction. The identity element is the empty word, and

$$({x_0}^{\epsilon_0} \ldots {x_{n-1}}^{\epsilon_{n-1}})^{-1} = {x_{n-1}}^{-\epsilon_{n-1}} \ldots {x_0}^{-\epsilon_0}.$$

Note that our notation is coherent: a word $\prod_{i<n} {x_i}^{\epsilon_i}$ is equal to the corresponding product in $F(X)$.

Theorem 3.10. *$F(X)$ is free with basis X.*

Proof. Given a map $f : X \to G$, we can extend f to a homomorphism $f^* : F(X) \to G$ by

$$f^*(\prod_{i<n} {x_i}^{\epsilon_i}) := \prod_{i<n} f(x_i).$$

Since any homomorphism extending f must satisfy this equality, f^* is unique. $\square$

Lemma 3.11. *Let $X \subseteq G$ be a subset of a group. Let $\iota^* : F(X) \to G$ be the homomorphism extending the inclusion $\iota : X \to G$.*

(i) ι^* is the "evaluation homomorphism" which maps a reduced word $\prod_{i<n} {x_i}^{\epsilon_i} \in F(X)$ to the element of G obtained by computing this product in G. The image of ι^* is $\langle X \rangle \leq G$.

(ii) G is free with basis X if and only if $\iota^* : F(X) \to G$ is an isomorphism.

Proof. (i) Immediate.

(ii) $\Leftarrow$: Immediate from freeness of $F(X)$.

$\Rightarrow$: By Lemma 3.3, ι extends to an isomorphism $F(X) \to G$, which must be ι^* by uniqueness of the latter.

$\square$

Proposition 3.12. *Any group is a quotient of a free group.*

Proof. Let G be a group, and let $X \subseteq G$ be any generating set (e.g. $X := G$). Then the identity embedding $\iota : X \to G$ extends to an epimorphism $\iota^* : F(X) \to G$. □

Theorem 3.13. *Any two bases for a given free group F have the same cardinality.*

Definition 3.14. This cardinality is the **rank** of the free group, $\mathrm{rk}(F)$.

Proof. Let $\mathrm{Hom}(F, \mathbb{Z}/2\mathbb{Z})$ be the set of homomorphisms $F \to \mathbb{Z}/2\mathbb{Z}$.

Let $X \subseteq F$ be a basis. Then any map $X \to \mathbb{Z}/2\mathbb{Z}$ extends uniquely to a homomorphism $F \to \mathbb{Z}/2\mathbb{Z}$. By the uniqueness, any homomorphism $F \to \mathbb{Z}/2\mathbb{Z}$ is determined by its restriction to X. So $|\mathrm{Hom}(F, \mathbb{Z}/2\mathbb{Z})| = 2^{|X|}$.

So if Y is another basis, then $2^{|X|} = 2^{|Y|}$. So if either X or Y is finite, then $|X| = |Y|$.

If X and Y are infinite, then $|X| = |F| = |Y|$ by the following claim.

Claim 3.15. Let X be an infinite set. Then $|F(X)| = |X|$.

Proof.

$$\begin{aligned}
|X| &\leq |F(X)| \\
&= |\text{reduced words in } X| \\
&\leq |\text{words in } X| \\
&= \left| \bigcup_n \{\text{words of length } n \text{ in } X\} \right| \\
&= \sup_n \left| (X^{\pm})^n \right| \\
&= \sup_n |X| \\
&= |X|.
\end{aligned}$$

$\square_{3.15}$

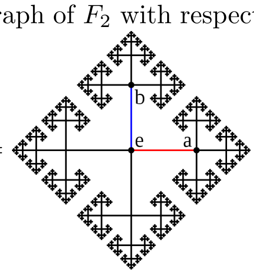
□

Proposition 3.16. *For each cardinal κ there exists a free group of rank κ, and it is unique up to isomorphism.*

Definition 3.17. For $n \in \mathbb{N}$, we write F_n for "the" free group of rank n.

Proof. Existence is by Theorem 3.10. Uniqueness is by Lemma 3.3. □

3.1 Cayley graphs and free groups

Example 3.18. The Cayley graph of F_2 with respect to a basis can be drawn as follows: $\Gamma(F(\{a, b\}), \{a, b\}) =$ 

Theorem 3.19. *Let $S \subseteq G$ be a subset of a group. Then $\Gamma(G,S)$ is a tree if and only if G is free with basis S.*

Proof. Let $\iota^* : F(S) \to G$ be the homomorphism extending the inclusion $\iota : S \to G$.

Claim 3.20. There is a bijection π from $F(S)$ to the set of reduced paths in $\Gamma(G,S)$ from 1, such that for any $w \in F(S)$, $\pi(w)$ is a path from 1 to $\iota^*(w)$.

Proof. Let $\pi(\prod_{i<n} s_i^{\epsilon_i})$ be the path $e_0 \dots e_{n-1}$ from 1, where

$$e_k := \begin{cases} (g_k, s_k) & \text{if } \epsilon_k = 1 \\ \overline{(g_{k+1}, s_k)} & \text{if } \epsilon_k = -1. \end{cases}$$

and $g_k := \iota^*(\prod_{i<k} s_i^{\epsilon_i})$. Inductively, $e_0 \dots e_{k-1}$ is a reduced path from 1 to g_k for $k \leq n$.

π is a bijection since any reduced path from 1 uniquely determines a corresponding word via $(g,s) \mapsto s$ and $\overline{(g,s)} \mapsto s^{-1}$. $\square_{3.20}$

Since G acts transitively on $(\Gamma(G,S))^0$, $\Gamma(G,S)$ is acyclic iff there is no non-trivial reduced path from 1 to 1, iff (by the claim) $\ker \iota^* = 1$.

Similarly, $\Gamma(G,S)$ is connected iff 1 is connected to every vertex, iff $\operatorname{im} \iota^* = G$.

So $\Gamma(G,S)$ is a tree iff $\iota^* : F(S) \to G$ is an isomorphism, iff (by Lemma 3.11(ii)) G is free with basis S. $\square$

3.2 Free actions on trees

Definition 3.21. An action $G \circlearrowleft X$ on a graph is **free** if the corresponding action $G \circlearrowleft X^0$ on the vertices is free.

Theorem 3.22. *Let $G \circlearrowleft X$ be a free non-inversive action of a group on a tree. Then G is free. If ${}_G\backslash^X$ is finite, then $\mathrm{rk}(G) = B_1({}_G\backslash^X)$.*

Proof. Equip X with an orientation preserved by G (such exists by Remark 2.7). Let T be a tree of representatives for the action $G \circlearrowleft X$. Consider an image gT of T under the action of $g \in G$. Then gT is also a tree.

Now T contains exactly one vertex of each orbit of $G \circlearrowleft X_0$ (using that X and hence ${}_G\backslash^X$ is connected, so a maximal subtree contains every vertex). So $GT := \bigcup_{g \in G} gT$ contains every vertex of X, and these trees are disjoint: if gT shares a vertex x with $g'T$, then $g^{-1}x = g'^{-1}x$ is the unique vertex of T in the orbit Gx, so $g'g^{-1}x = x$, so $g' = g$ by freeness.

Let $Y := X/(GT)$, the graph obtained by contracting each tree gT to a point ${}^{gT}/_{\sim}$. So since $(GT)^0 = X^0$, we have

$$Y^0 = \{ {}^{gT}/_{\sim} : g \in G \}$$
$$Y^1 = X^1 \setminus (GT)^1.$$

So the orientation X^+ of X induces an orientation $X^+ \setminus (GT^1)$ of Y, and the action of G on X induces an orientation-preserving action on Y which on vertices is the regular action $h * ({}^{gT}/_{\sim}) = {}^{hgT}/_{\sim}$.

By Lemma 2.36, Y is a tree. In particular, by acyclicity, for any $x, y \in Y^0$ there is at most one positive edge from x to y.

So by Lemma 2.18, $Y \cong \Gamma(G, S)$ where $|S| = |({}_G\backslash^Y)^+| = \frac{1}{2}|({}_G\backslash^Y)^1|$.

So $\Gamma(G, S)$ is a tree, and we conclude by Theorem 3.19 that G is free with basis S.

Now suppose ${}_G\backslash^X$ is finite. Let $T' := {}_G\backslash^T$, the maximal tree in ${}_G\backslash^X$ of which T is a lift. Then

$$({}_G\backslash^Y)^1 = {}_G\backslash^{Y^1} = {}_G\backslash^{X^1 \backslash (GT)^1} = ({}_G\backslash^X)^1 \setminus (T')^1,$$

so

$$\text{rk}G = |S| = \frac{1}{2}|({}_G\backslash^Y)^1| = \frac{1}{2}|({}_G\backslash^X)^1 \setminus (T')^1| = B_1({}_G\backslash^X)$$

by Lemma 2.32. □

Corollary 3.23 (The Nielson-Schreier Theorem)**.** *Any subgroup of a free group is free.*

Proof. Let F be free with basis S, and let $G \leq F$ be a subgroup. The natural action of F on $\Gamma(F, S)$ is free and non-inversive, so also the induced action of G is. But $\Gamma(F, S)$ is a tree by Theorem 3.19, so G is free by Theorem 3.22. □

Corollary 3.24 (Schreier's formula)**.** *If F is free of finite rank and $G \leq F$ has finite index, then* $\text{rk}(G) - 1 = [F : G](\text{rk}(F) - 1)$.

Proof. Let $S \subseteq F$ be a basis, so $|S| = \text{rk}(F)$, and let $Y := {}_G\backslash^{\Gamma(F,S)}$. Then $Y^0 = {}_G\backslash^F$, the set of right cosets of G, and $Y^+ = {}_G\backslash^F \times S$.

Then Y is finite, and so by Theorem 3.22,

$$\text{rk}(G) = B_1(Y) = 1 + |Y^+| - |Y^0| = 1 + [F : G](\text{rk}(F) - 1).$$

□

4 Group presentations

Definition 4.1.

- If G is a group and $R \subseteq G$, the **normal closure** of R is the subgroup

  $$\langle\langle R\rangle\rangle = \langle\langle R\rangle\rangle^G := \left\langle R^G \right\rangle = \langle\{r^g : r \in R, g \in G\}\rangle \trianglelefteq G,$$

  the smallest normal subgroup of G containing R.

- Given X and a set $R \subseteq F(X)$ of reduced words in X, the group **generated by X with relators** R is

  $$\langle X \mid R\rangle := F(X)/\langle\langle R\rangle\rangle.$$

  We often write $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ as shorthand for $\langle\{x_1, \ldots, x_n\} \mid \{r_1, \ldots, r_m\}\rangle$.

- We call $\langle X \mid R\rangle$ a **presentation**, and we call an isomorphism $G \cong \langle X \mid R\rangle$ a **presentation of** G. The presentation is **finitely generated** if X is finite, and **finite** if both X and R are finite.

Remark 4.2. Every group has a presentation, by Proposition 3.12.

Example 4.3.

- $\langle X \mid \rangle = \langle X \mid \emptyset \rangle = F(X)$
- If $X \cap Y = \emptyset$, then $\langle X \cup Y \mid Y \rangle \cong F(X)$.
- For $n \in \mathbb{N}$, $\langle x \mid x^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$.
- $\langle x, y \mid xy, yx^2 \rangle \cong 1$.

  One way to see this: $x = ((xy)^x)^{-1}(yx^2)$ and $y = x^{-1}(xy)$.

  Another: if $\phi : F(X) \to \langle x, y \mid xy, yx^2 \rangle$ is the quotient homomorphism, then $\phi(xy) = 1 = \phi(yx^2)$, so $\phi(y) = \phi(x)^{-1}$ and $1 = \phi(yx^2) = \phi(x)$, so $\phi(x) = 1 = \phi(y)$, so $\phi(F(X)) = 1$.

Notation 4.4.

- When writing a presentation, we often write a **relation** $r = s$ as alternative notation for the corresponding relator rs^{-1}. In particular, a relator r corresponds to the relation $r = 1$.
- We often denote an element $g \in \langle X \mid R \rangle$ by a word $w \in F(X)$ whose image under the quotient map $F(X) \to \langle X \mid R \rangle$ is g.

The following generalises the defining property of a free group.

Lemma 4.5. *Let $\langle X \mid R \rangle$ be a presentation and G be a group, and let $f : X \to G$ be a map which respects the relations, meaning that for any relator $\pi_{i<n} x_i{}^{\epsilon_i} \in R$, we have $\pi_{i<n} f(x_i)^{\epsilon_i} = 1$.*

Then f extends uniquely to a homomorphism $\langle X \mid R \rangle \to G$.

Proof. f extends to $f^* : F(X) \to G$. By the assumption, $f^*(\langle\langle R \rangle\rangle) = 1$, so f^* induces a well-defined homomorphism $\langle X \mid R \rangle \to G$. The uniqueness follows from the fact that (the image of) X generates $\langle X \mid R \rangle$. □

Example 4.6. For $n, m \in \mathbb{N}$, $x \mapsto x$ extends to a homomorphism $\langle x \mid x^n \rangle \to \langle x \mid x^m \rangle$ if and only if $m | n$.

Remark 4.7. If $w_1 = w_2$ is a relation in $\langle X \mid R \rangle$ (i.e. $w_i \in F(X)$ and $w_1 w_2^{-1} \in \langle\langle R \rangle\rangle$), then we can substitute w_2 for w_1 within words in $\langle X \mid R \rangle$, i.e. $uw_1v = uw_2v$ in $\langle X \mid R \rangle$ for any $u, v \in F(X)$.

Indeed $uw_1v(uw_2v)^{-1} = uw_1w_2^{-1}u^{-1} = (w_1w_2^{-1})^{u^{-1}} \in \langle\langle R \rangle\rangle$.

In other words: if $\phi : F(X) \to \langle X \mid R \rangle$ is the quotient map, then $\phi(w_1) = \phi(w_2)$, so since ϕ is a homomorphism, $\phi(uw_1v) = \phi(uw_2v)$.

Example 4.8. $\mathrm{FA}(X) := \langle X \mid \{xy = yx : x, y \in X\} \rangle = \langle X \mid \{[x, y] : x, y \in X\} \rangle$ is the abelianisation of $F(X)$. Indeed,

- $\mathrm{FA}(X)$ is abelian: by induction on their lengths, any two words in X commute in $\mathrm{FA}(X)$.

  More explicitly: for any $x \in X$, any word w is in the centraliser of x by induction on the length of w, so x is central. But the centre is a subgroup, and X generates $\mathrm{FA}(X)$, so $\mathrm{FA}(X)$ is abelian.

- $\mathrm{FA}(X)$ satisfies the universal property of the abelianisation (defined in Lemma 1.9) by Lemma 4.5.

Since abelianisations are unique (by the usual arguments), $\mathrm{FA}(X) \cong F(X)/F(X)'$. One can also see directly that $\langle\langle\{[x,y] : x,y \in X\}\rangle\rangle = F(X)'$:

$\leq$: $[x,y]^w = [x^w, y^w] \in F(X)'$.

$\geq$: Any commutator $[w,v]$ has trivial image in the abelian quotient $\mathrm{FA}(X) = F(X)/\langle\langle\{[x,y] : x,y \in X\}\rangle\rangle$.

$\mathrm{FA}(X)$ is the *free abelian group* (or *free $\mathbb{Z}$-module*) on generators X. If X is finite, $\mathrm{FA}(X) \cong \mathbb{Z}^{|X|}$.

In particular, $\mathbb{Z}^2 \cong \langle x, y \mid [x,y]\rangle$.

Example 4.9. S_3 has the presentation $P := \left\langle x, y \;\middle|\; x^2, y^2, (xy)^3 \right\rangle$.

Indeed, $x \mapsto (12), y \mapsto (23)$ respects the relations so extends to a homomorphism $\theta : P \to S_3$, which is surjective since $\langle (12), (23)\rangle = S_3$, and by applying the relations one sees that $P = \{1, x, y, xy, yx, xyx\}$, so θ is an isomorphism on cardinality grounds.

Proposition 4.10. *A finite index subgroup of a finitely presented (resp. finitely generated) group is finitely presented (resp. finitely generated).*

Proof. Suppose X is a finite set, $R \subseteq F(X)$, and $H \leq \langle X \mid R\rangle$ has finite index. Let $G := \pi^{-1}H \leq F(X)$ where $\pi : F(X) \to \langle X \mid R\rangle = F(X)/\langle\langle R\rangle\rangle$ is the quotient map.

Now $[F(X) : G] = [\langle X \mid R\rangle : H]$ is finite (since $ab^{-1} \in \pi^{-1}H \Leftrightarrow \pi(a)\pi(b)^{-1} \in H$). So say $F(X)/G = \{tG : t \in T\}$ where $T \subseteq F(X)$ is finite.

Claim 4.11. Let $R^T := \{r^t : r \in R, t \in T\}$. Then $\langle\langle R\rangle\rangle^{F(X)} = \left\langle\left\langle R^T\right\rangle\right\rangle^G$.

Proof. (Note $R^T \subseteq \langle\langle R\rangle\rangle^{F(X)} = \ker\pi \subseteq \pi^{-1}(H) = G$, so the right hand side does make sense.)

$\supseteq$: Immediate.

$\subseteq$: If $r \in R$ and $f \in F(X)$, say $f = tg$ with $t \in T$ and $g \in G$, then $r^f = r^{tg} = (r^t)^g \in \left\langle\left\langle R^T\right\rangle\right\rangle^G$. Since the $\langle\langle R\rangle\rangle^{F(X)}$ is generated by such r^f, we conclude.

$\square_{4.11}$

By Corollary 3.24, G is free of finite rank. Let $Y \subseteq G$ be a finite basis, and let $\tau : G \to F(Y)$ be the corresponding isomorphism. Then by the claim,

$$H \cong G/\ker\pi = G/\left\langle\left\langle R^T\right\rangle\right\rangle^G \cong \left\langle Y \;\middle|\; \tau(R^T)\right\rangle.$$

So H is finitely generated, and finitely presented if R is finite. $\square$

4.1 Tietze transformations

Lemma 4.12. *Let $\langle X \mid R\rangle$ be a presentation.*

(i) If $r \in \langle\langle R\rangle\rangle$, then $\langle X \mid R\rangle \cong \langle X \mid R \cup \{r\}\rangle$.

(ii) If $w \in F(X)$ and $y \notin X$, then $\langle X \mid R\rangle \cong \langle X \cup \{y\} \mid R \cup \{y = w\}\rangle$.

Proof. (i) $\langle\langle R \cup \{r\}\rangle\rangle = \langle\langle R\rangle\rangle$.

(ii) By Lemma 4.5, $f := \mathrm{id}_X \cup \{y \mapsto w\}$ and $g := \mathrm{id}_X$ extend to homomorphisms $f^* : \langle X \cup \{y\} \mid R \cup \{y = w\}\rangle \to \langle X \mid R\rangle$ and $g^* : \langle X \mid R\rangle \to \langle X \cup \{y\} \mid R \cup \{y = w\}\rangle$.

Now $(f^* \circ g^*)|_X = \mathrm{id}$ and $(g^* \circ f^*)|_{X\cup\{y\}} = \mathrm{id}$ (since $g^*(f^*(y)) = g^*(w) = w = y$), so $f^* \circ g^* = \mathrm{id}$ and $g^* \circ f^* = \mathrm{id}$, so f^* and g^* are inverse isomorphisms.

□

Definition 4.13. The two operations on presentations in Lemma 4.12, along with their inverses, are the **Tietze transformations**. We call (i) and its inverse *adding/deleting a relation*, and (ii) and its inverse *adding/deleting a generator*.

Example 4.14.

$$\begin{aligned} S_3 &\cong \left\langle x, y \mid x^2, y^2, (xy)^3 \right\rangle \\ &\cong \left\langle x, y \mid x^2, y^2, (xy)^3, (yx)^3 \right\rangle \quad [(yx)^3 = ((xy)^3)^x] \\ &\cong \left\langle x, y \mid x^2, y^2, (yx)^3 \right\rangle \\ &\cong \left\langle x, y, z \mid x^2, y^2, (yx)^3, z = yx \right\rangle \\ &\cong \left\langle x, y, z \mid x^2, y^2, (yx)^3, z = yx, z^3, x = yz, (yz)^2 \right\rangle \\ &\cong \left\langle x, y, z \mid y^2, z^3, x = yz, (yz)^2 \right\rangle \\ &\cong \left\langle y, z \mid y^2, z^3, (yz)^2 \right\rangle \\ &\cong \left\langle y, z \mid y^2, z^3, y^{-1}zy = z^{-1} \right\rangle \\ &\cong D_3. \end{aligned}$$

Theorem 4.15. *Finite presentations $\langle X \mid R\rangle$ and $\langle Y \mid S\rangle$ are isomorphic if and only if a finite sequence of Tietze transformations transforms one into the other.*

Proof.

$\Leftarrow$: Immediate.

$\Rightarrow$: It suffices to prove this in the case $X \cap Y = \emptyset$ (the general case then follows by going via a third disjoint presentation).

Fix an isomorphism $\theta : \langle X \mid R\rangle \xrightarrow{\cong} \langle Y \mid S\rangle$. Then for $x \in X$, say $\theta(x) = w_x/\langle\langle S\rangle\rangle$ and $\theta^{-1}(y) = w_y/\langle\langle R\rangle\rangle$, where $w_x \in F(Y)$ and $w_y \in F(X)$.

Then we transform $\langle X \mid R\rangle$ to $\langle X \cup Y \mid R \cup \{y = w_y : y \in Y\}\rangle$ by adding generators, then transform this to $\langle X \cup Y \mid R \cup S \cup \{x = w_x : x \in X\} \cup \{y = w_y : y \in Y\}\rangle$ by adding relations: Indeed, since θ is an isomorphism, if we expand an element of S as a word in X by substituting w_y for y, we obtain an element of $\langle\langle R\rangle\rangle$; similarly for $x^{-1}w_x$. So $S \cup \{x = w_x : x \in X\} \subseteq \langle\langle R \cup \{y = w_y : y \in Y\}\rangle\rangle$.

We conclude by symmetry.

□

4.2 Aside: algorithmic problems

4.2.1 The word problem for finitely generated groups

If G is a finitely generated group, the **word problem** for G is the algorithmic problem of deciding, given a finite set of generators X, which group words in X are trivial in G.

More precisely, G has **decidable word problem** if for some (equivalently any) finitely generated presentation $G \cong \langle X \mid R \rangle$, there exists an algorithm which takes as input a word $w \in F(X)$ and returns True if $w \in \langle\langle R \rangle\rangle$, and returns False otherwise.

One might hope that any finitely *presented* group has decidable word problem. This turns out to be false. Nor must a group with decidable word problem be finitely presented[2], although it must embed in a finitely presented group.

These facts can be proven using the Higman Embedding Theorem, which says that a finitely generated group G embeds in some finitely presented group if and only if G is recursively presented, meaning $G \cong \langle X \mid R \rangle$ where there is an algorithm to determine membership of R.

4.2.2 The isomorphism problem for finitely presented groups

The isomorphism problem asks for an algorithm which would take as input two finite presentations $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$, and would determine whether $\langle X \mid R \rangle \cong \langle Y \mid S \rangle$.

Again, it turns out that no such algorithm exists: a result of Adian and Rabin implies that even the special case of testing triviality, i.e. determining whether $\langle X \mid R \rangle \cong 1$, is unsolvable.

So even though $\langle X \mid R \rangle \cong \langle Y \mid S \rangle$ is guaranteed to be witnessed by a sequence of Tietze transformations, there is no algorithm to produce such a sequence.

4.3 The fundamental group of a graph

Definition 4.16. Let X be a graph. Let $F(X) := \left\langle X^1 \,\middle|\, \{\bar{e} = e^{-1} : e \in X^1\} \right\rangle$.

- Let $x \in X^0$. The **fundamental group of X with base-point x** is the group

  $$\pi_1(X, x) := \{e_0 \dots e_{n-1} : (e_0, \dots, e_{n-1}) \text{ is a path from } x \text{ to } x\} \leq F(X).$$

- If X is connected and non-empty, and $T \subseteq X$ is a maximal subtree, the **fundamental group of X with respect to T** is

  $$\pi_1(X, T) := F(X) / \langle\langle T^1 \rangle\rangle = \left\langle X^1 \,\middle|\, \{\bar{e} = e^{-1} : e \in X^1\} \cup T^1 \right\rangle.$$

Remark 4.17. One can see that this definition of $\pi_1(X, x)$ agrees with the usual topological definition, i.e. $\pi_1(X, x) \cong \pi_1(\mathrm{real}(X), x)$. Homotopic paths yield the same element of $\pi_1(X, x)$.

[2]The lamplighter group provides a counterexample.

Remark 4.18. $\pi_1(X,T)$ is free of rank $\frac{1}{2}|X^1 \backslash T^1|$. Indeed, if X^+ is an orientation, deleting generators[3] shows

$$\pi_1(X,T) \cong \left\langle X^1 \setminus T^1 \;\middle|\; \{\overline{e} = e^{-1} : e \in X^1 \setminus T^1\} \right\rangle \cong F(X^+ \setminus T^1).$$

Theorem 4.19. *Let X be a connected non-empty graph, let $x \in X^0$, and let $T \subseteq X$ be a maximal subtree. Then $\pi_1(X,x) \cong \pi_1(X,T)$.*

In particular, if X is finite, $\pi_1(X,x) \cong F_{B_1(X)}$.

Proof. For $y \in X^0 = T^0$, let $\gamma_y := e_0 \dots e_{n-1} \in F(X)$ where $(e_0, \dots, e_{n-1})$ is the geodesic in T from x to y.

Setting $f(e) := \gamma_{\alpha(e)} e \gamma_{\omega(e)}^{-1} \in \pi_1(X,x)$ for $e \in X^1$, we have $f(e) = 1$ for $e \in T^1$, and $f(\overline{e}) = f(e)^{-1}$ for $e \in X^1$, so f respects the relations of $\pi_1(X,T)$ and so extends uniquely to a homomorphism $f^* : \pi_1(X,T) \to \pi_1(X,x)$.

Let $p : \pi_1(X,x) \to \pi_1(X,T)$ be the restriction of the quotient map $F(X) \to \pi_1(X,T)$.

Then $p(f(e)) = e$ for $e \in X^1$, so $p \circ f^* = \mathrm{id}$, and if $(e_0, \dots, e_{n-1})$ is a path from x to x, then $f^*(p(e_0 \dots e_{n-1})) = \gamma_{\alpha(e_0)} e_0 \gamma_{\omega(e_0)}^{-1} \gamma_{\alpha(e_1)} e_1 \dots e_{n-1} \gamma_{\omega(e_{n-1})} = e_0 \dots e_{n-1}$ (since $\alpha(e_0) = x = \omega(e_{n-1})$ and $\omega(e_i) = \alpha(e_{i+1})$).

So p and f^* are mutually inverse homomorphisms, so they are isomorphisms.

The "in particular" clause follows from Remark 4.18 and Lemma 2.32. □

Proposition 4.20. *If G is a free group with basis S and $H \leq G$ is a subgroup, then $\pi_1({}_H\backslash^{\Gamma(G,S)}, H) \cong H$.*

(This gives an alternative route to the Nielson-Schreier Theorem (Corollary 3.23) and the Schreier formula (Corollary 3.24).)

Proof. Let $X := \Gamma(G,S)$ and $Y := {}_H\backslash^X$. Recall that $X^0 = G$ and $X^+ = G \times S$, and $Y^0 = {}_H\backslash^G$ and $Y^+ = {}_H\backslash^G \times S$.

Let $\phi_X : F(X) \to G$ be (by Lemma 4.5) the unique homomorphism such that $\phi_X((g,s)) = s$ (and $\phi_X(\overline{(g,s)}) = s^{-1}$), and similarly let $\phi_Y : F(Y) \to G$ be the homomorphism such that $\phi_Y((Hg,s)) = s$.

Now any reduced path $(e_0, \dots, e_{n-1})$ in Y from H to H lifts uniquely to a reduced path $(e'_0, \dots, e'_{n-1})$ in X from 1 with $e_i = {}_H\backslash^{e'_i}$. Then

$$\phi_Y(e_0 \dots e_{n-1}) = \phi_X(e'_0, \dots, e'_{n-1}) = \omega(e'_{n-1}) \in \omega(e_{n-1}) = H.$$

Since X is a tree, there is a unique such path in X for any $h \in H$, so we conclude that ϕ_Y restricts to a bijection $\pi_1(Y,H) \to H$. □

5 Colimits of groups

We show that the category of groups has colimits. We assume no prior familiarity with category theory[4].

Definition 5.1.

[3]Possibly infinitely many, so this isn't really a matter of applying a sequence of Tietze transformations, but Lemma 4.12(ii) goes through with an infinite set of new generators.

[4]Those who are familiar with the standard category theory definitions may note that we've taken a shortcut in the definition of diagram, but can confirm that this doesn't affect the resulting notion of colimit.

- A **diagram** of groups consists of a family $(G_i)_{i\in I}$ of groups and, for each pair (i,j), a (possibly empty) set F_{ij} of homomorphisms : $G_i \to G_j$.
- A **co-cone** of such a diagram consists of a group G and morphisms $g_i : G_i \to G$, such that for all (i,j) and $f \in F_{ij}$, $g_j \circ f = g_i$.
- A co-cone $(G,(g_i)_i)$ is a **colimit** of the diagram if it satisfies the following universal property:
  for any co-cone $(H,(h_i)_i)$ there exists a unique homomorphism $\alpha : G \to H$ such that $h_i \circ \alpha = g_i$ for all i.

Lemma 5.2. *Any diagram has a colimit.*

Moreover, it is unique up to unique isomorphism over the diagram: if $(G,(g_i)_i)$ and $(H,(h_i)_i)$ are colimits, then there exists a unique isomorphism $\theta : G \to H$ such that $h_i \circ \theta = g_i$ for all i.

Proof. Say $G_i \cong \langle X_i \mid R_i\rangle$ with the X_i disjoint. Then

$$G := \left\langle \bigcup_i X_i \;\middle|\; \bigcup_i R_i \cup \{x = f(x) : f \in F_{ij}, x \in X_i\} \right\rangle$$

(where $f(x)$ denotes the corresponding word in X_j), along with the homomorphisms induced by the inclusions $X_i \subseteq \bigcup_i X_i$, is a colimit.

Indeed, if $(H,(h_i)_i)$ is another co-cone, then the map $\bigcup_i X_i \to H$ defined by h_i on X_i respects the relations, so it extends uniquely to a homomorphism $G \to H$. This is what the colimit condition requires.

The uniqueness can be verified as in Lemma 3.3: given two colimits, we obtain unique homomorphisms between them, and their compositions must be the identity homomorphisms by uniqueness. □

Notation 5.3. We typically write $G = \varinjlim_i G_i$ to denote the colimit, suppressing the morphisms in the diagram and the limit from the notation.

Example 5.4. To illustrate the notion of colimit, we sketch a couple of special cases.

- Let I be a linear order, let $(G_i)_{i\in I}$, and let $(f_{ij} : G_i \to G_j)_{i<j}$ be a commuting system of embeddings. Let G be the colimit $\varinjlim_i G_i$ of this diagram, and let $g_i : G_i \to G$ be the associated maps. (In this case, G is also called the *direct limit* of the diagram.)

  Then each g_i is an embedding, and $(g_i(G_i))_i$ is a chain of subgroups of G, and $G = \bigcup_i g_i(G_i)$.

  One way to see this is to *define* a group G built as the union of copies of G_i embedded according to the f_{ij}, and check that it satisfies the properties of the colimit.
- Given two homomorphisms $f_1, f_2 : G \to H$, the colimit of this diagram is the quotient $H/\langle\langle\{f_1(g)f_2(g)^{-1} : g \in G\}\rangle\rangle$ (called the *coequaliser* of f_1 and f_2). This can be seen by considering presentations.

5.1 Free products

Definition 5.5. If G_1 and G_2 are groups, the colimit of the diagram consisting of these two groups, with no morphisms, is called the **free product** of G_1 and G_2, denoted $G_1 * G_2$.

Let $G_1 * G_2$ be a free product, and let $f_i : G_i \to G_1 * G_2$ be the homomorphisms of the colimit.

Lemma 5.6. $G_1 * G_2 = \langle f_1(G_1), f_2(G_2)\rangle$.

Proof. By the universal property applied to the homomorphisms $f_i : G_i \to H := \langle f_1(G_1), f_2(G_2)\rangle \leq G_1 * G_2 =: G$, there is $\alpha : G \to H$ with $\alpha \circ f_i = f_i$. Let $\iota : H \to G$ be the inclusion. Then $(\iota \circ \alpha) \circ f_i = f_i$, so $\iota \circ \alpha = \mathrm{id}_G$ by uniqueness, so $H = G$. □

From the proof of Lemma 5.2, we see:

Remark 5.7. If we take presentations $G_i \cong \langle X_i \mid R_i\rangle$ with $X_1 \cap X_2 = \emptyset$, then $G_1 * G_2 \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2\rangle$, and f_i is induced by the inclusion $X_i \subseteq X_1 \cup X_2$.

Example 5.8. For $n, m \in \mathbb{N}$, we have $F_n * F_m \cong F_{n+m}$ by considering presentations.

In particular, $F_2 \cong F_1 * F_1 \cong \mathbb{Z} * \mathbb{Z}$.

Definition 5.9. Let G_1 and G_2 be groups with $G_1 \cap G_2 = 1$. A **normal form** in (G_1, G_2) is a sequence $(g_i)_{i<n}$ where

- $n \in \mathbb{N}$,
- $g_i \in (G_1 \cup G_2) \setminus \{1\}$,
- $g_i \in G_1$ iff $g_{i+1} \in G_2$ $(\forall i)$.

Theorem 5.10 (Normal Form Theorem for free products)**.** *Let $G_1 * G_2$ be a free product with associated homomorphisms $f_i : G_i \to G_1 * G_2$. Then:*

(i) The f_i are embeddings, and the images $\overline{G_i} := f_i(G_i)$ intersect trivially, $\overline{G_1} \cap \overline{G_2} = 1$.

(ii) For all $g \in G_1 * G_2$ there is a unique normal form $(g_i)_{i<n}$ in $(\overline{G_1}, \overline{G_2})$ such that $g = \prod_{i<n} g_i$.

Remark 5.11. By (i), after replacing G_i with its isomorphic copy $\overline{G_i}$, each G_i is a subgroup of $G_1 * G_2$. Then (ii) says that each element of $G_1 * G_2$ has a unique expression as $\prod_{i<n} g_i$ where $(g_i)_{i<n}$ is a normal form in (G_1, G_2).

Proof. Replacing G_1 with an isomorphic copy, we may assume $G_1 \cap G_2 = \{1\}$.

Let X be the set of normal forms in (G_1, G_2).

Claim 5.12. The map $f : X \to G_1 * G_2;\ (g_i)_{i<n} \mapsto \prod_{i<n}(f_1 \cup f_2)(g_i)$ is a bijection.

Proof. Consider the natural left action of G_1 on X, defined for $g \in G_1 \setminus \{1\}$ by:

$$g *_1 (g_0, \ldots, g_{n-1}) := \begin{cases} (g, g_0, \ldots, g_{n-1}) & \text{if } n = 0 \text{ or } g_0 \in G_2 \\ (gg_0, \ldots, g_{n-1}) & \text{if } g_0 \in G_1,\ gg_0 \neq 1 \\ (g_1, \ldots, g_{n-1}) & \text{if } g_0 \in G_1,\ gg_0 = 1, \end{cases}$$

(and $1 *_1 x := x$). By the universal property, this and the analogous action $*_2$ of G_2 induce via f_1, f_2 a left action $*$ of $G_1 * G_2$ on X (i.e. a homomorphism $G_1 * G_2 \to \mathrm{Sym}(X)$); so for $g \in G_i$, $f_i(g) * x = g *_i x$.

Then by induction on n, for any $(g_i)_{i<n} \in X$ we have $f((g_i)_{i<n}) * \emptyset = (g_i)_{i<n}$. So f is injective.

Now from the definition of $*_i$, for $g \in G_i$ and $x \in X$, we have $f(g *_i x) = f_i(g) \cdot f(x)$. So $f(X) \subseteq G$ is closed under left multiplication by each $f_i(G_i)$. and hence by $G_1 * G_2$ (by Lemma 5.6). So $f(X) = G$, and f is surjective. $\square_{5.12}$

By considering normal forms of length 1, it follows from the claim that each f_i is an embedding. If $1 \neq g \in \overline{G_1} \cap \overline{G_2}$, say $f_1(g_1) = g = f_2(g_2)$, then (g_1, g_2^{-1}) is a normal form, but $gg^{-1} = 1$, so this contradicts the claim. So we conclude (i), and then (ii) follows directly from the claim. $\square$

Proposition 5.13. *Let G be a group. Suppose $G_1, G_2 \leq G$ are subgroups such that*

(i) $\langle G_1 \cup G_2 \rangle = G$;

(ii) $G_1 \cap G_2 = 1$;

(iii) if $(g_i)_{i<n}$ is a normal form in G_1, G_2 with $n > 0$, then $\prod_{i<n} g_i \neq 1$.

Then $G \cong G_1 * G_2$.

Proof. Consider the map $G_1 * G_2 \to G$ induced by id_{G_1} and id_{G_2}. It maps normal forms to normal forms, so it is injective by (iii), and surjective by (i). $\square$

Example 5.14. The infinite dihedral group D_∞ is the automorphism group of the tree C_∞. We show $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Recall we defined $(C_\infty)^0 := \mathbb{Z} =: (C_\infty)^+$ with $\alpha(n) := n$, $\omega(n) := n + 1$. Let $a \in D_\infty$ be the reflection through the vertex 0, and let $b \in D_\infty$ be the reflection through the edge 0; so for $n \in (C_\infty)^+$, $a * n = \overline{-n-1}$ and $b * n = \overline{-n}$.

Now an element of D_∞ is determined by its action on $0 \in (C_\infty)^+$, and we calculate

$$\begin{aligned} (ab)^n * 0 &= -n \\ b(ab)^n &= \overline{n} \\ (ba)^n * 0 &= n \\ a(ba)^n * 0 &= \overline{-n-1}, \end{aligned}$$

so we conclude by Proposition 5.13.

Remark 5.15. We can also define the free product of a family $(G_i)_{i\in I}$ of groups as the colimit $\ast_{i\in I} G_i$ of the diagram consisting of the groups G_i and no homomorphisms.

In terms of presentations, $\ast_{i\in I} \langle X_i \mid R_i\rangle \cong \left\langle \bigcup_{i\in I} X_i \mid \bigcup_{i\in I} R_i \right\rangle$ (assuming the X_i are disjoint). With this one can easily verify:

- $\ast_{i\in I} \mathbb{Z} \cong F(I)$.
- The free product of finitely many groups $(G_i)_{i<n}$ is isomorphic over the G_i to the iterated binary free product,

$$\mathop{\ast}_{i<n} G_i \cong G_0 \ast (G_1 \ast (\ldots \ast G_{n-1})\ldots).$$

5.2 Amalgamated free products

Definition 5.16. If A, G_1, G_2 are groups and $\phi_i : A \to G_i$ is an embedding $(i = 1, 2)$, the colimit of the diagram consisting of these groups and embeddings is called the **amalgamated free product** (or just **amalgam**) of G_1 and G_2 over A (with respect to the ϕ_i), denoted $G_1 \ast_A G_2$.

Consider an amalgamated free product $G = G_1 \ast_A G_2$, and let $f_i : G_i \to G$ be the homomorphisms of the colimit.

Exactly as in Lemma 5.6, we have:

Lemma 5.17. $G = \langle f_1(G_1), f_2(G_2)\rangle$.

Remark 5.18. If we take presentations $G_i \cong \langle X_i \mid R_i\rangle$ with $X_1 \cap X_2 = \emptyset$, then by the proof of Lemma 5.2,

$$G_1 \ast_A G_2 \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(a) = \phi_2(a) : a \in A\}\rangle$$

(where $\phi_i(a)$ denotes the corresponding word in X_i). Hence

$$G_1 \ast_A G_2 \cong (G_1 \ast G_2)/\left\langle\left\langle \{\phi_1(a)\phi_2(a)^{-1} : a \in A\}\right\rangle\right\rangle.$$

To simplify notation in the following theorem, assume that the embeddings $\phi_i : A \to G_i$ are inclusions, and $G_1 \cap G_2 = A$.

Theorem 5.19 (Normal Form Theorem for amalgamated free products)**.**

(i) The f_i are embeddings, and $f_1(G_1) \cap f_2(G_2) = f_1(A)$ $(= f_2(A))$.

(ii) Identify G_i with $f_i(G_i)$ via f_i.

Let $S_i \subseteq G_i$ be a set of representatives for the right cosets of A in G_i.

Define a ***normal form*** *to be a sequence $(a; s_0, \ldots, s_{n-1})$ where*

- *$n \in \mathbb{N}$;*
- *$a \in A$;*
- *$s_i \in (S_1 \setminus A) \cup (S_2 \setminus A)$;*
- *$s_i \in S_1$ iff $s_{i+1} \in S_2$ $(\forall i)$.*

Then for all $g \in G$ there is a unique normal form $(a; s_0, \ldots, s_{n-1})$ such that $g = as_0 \ldots s_{n-1}$.

Proof. Let X be the set of normal forms.

Note that $f' := f_1 \cup f_2 : G_1 \cup G_2 \to G$ is well-defined, since $G_1 \cap G_2 = A$.

Claim 5.20. The map $f : X \to G;\ (a; s_0, \ldots, s_{n-1}) \mapsto f'(a)f'(s_0)\ldots f'(s_{n-1})$ is a bijection.

Proof. If $g \in G_i \setminus A$, let $g_A \in A$ and $g_S \in S_i \setminus A$ be the unique elements such that $g = g_A g_S$.

Consider the natural left action of G_1 on X:

$$g{*_1}(a; s_0, \ldots, s_{n-1}) := \begin{cases} (ga; s_0, \ldots, s_{n-1}) & \text{if } g \in A \\ ((ga)_A; (ga)_S, s_0, \ldots, s_{n-1}) & \text{if } g \notin A,\ n = 0 \text{ or } s_0 \in S_2 \\ ((gas_0)_A; (gas_0)_S, s_1, \ldots, s_{n-1}) & \text{if } g \notin A,\ s_0 \in S_1,\ gas_0 \notin A \\ (gas_0; s_1, \ldots, s_{n-1}) & \text{if } g \notin A,\ s_0 \in S_1,\ gas_0 \in A. \end{cases}$$

Let $*_2$ be the analogous action of G_2 on X. Note then:

$$f(g *_i x) = f_i(g) \cdot f(x) \ \ (\text{for } i \in \{1, 2\},\ g \in G_i,\ x \in X). \qquad (*)$$

Now $*_1$ and $*_2$ agree on A, so by the universal property they induce an action $*$ of G on X such that for $g \in G_i$, $f_i(g) * x = g *_i x$.

Then for any $(a; s_0, \ldots, s_{n-1}) \in X$ we have

$$\begin{aligned} f((a; s_0, \ldots, s_{n-1})) * (1;) &= f((a; s_0, \ldots, s_{n-2})) * (1; s_{n-1}) \\ &= f((a; s_0, \ldots, s_{n-3})) * (1; s_{n-2}, s_{n-1}) \\ &= \ldots \\ &= f((a;)) * (1; s_0, \ldots, s_{n-1}) \\ &= (a; s_0, \ldots, s_{n-1}). \end{aligned}$$

So f is injective.

By (*), $f(X) \subseteq G$ is closed under left multiplication by each $f_i(G_i)$. and hence by G (by Lemma 5.17). So $f(X) = G$, and f is surjective.

$\square_{5.20}$

Now for $g \in G_i$,

$$f_i(g) = \begin{cases} f((g_A; g_S)) & \text{if } g \notin A \\ f((g;) & \text{if } g \in A \end{cases},$$

so by the claim $f_i(g) = 1$ iff $g = 1$. So each f_i is an embedding.

If $g \in f_1(G_1 \setminus A) \cap f_2(G_2 \setminus A)$, say $f_1(g_1) = g = f_2(g_2)$, then $f((g_1)_A, (g_1)_S) = g = f((g_2)_A, (g_2)_S)$, contradicting the claim.

So we conclude (i), and then (ii) follows directly from the claim. $\square$

When considering an amalgamated free product $G = G_1 *_A G_2$, we often identify A, G_1, G_2 with their isomorphic images in G, so then we have $A \subseteq G_1, G_2 \subseteq G$ and $G_1 \cap G_2 = A$.

Proposition 5.21. *Let G be a group. Suppose $A \leq G_1, G_2 \leq G$ are subgroups such that $G_1 \cap G_2 = A$.*

Call a sequence $(g_i)_{i<n}$ a ***non-trivial alternating sequence*** *if*

- *$n > 0$;*
- *$g_i \in (G_1 \cup G_2) \setminus A$ for all $i < n$;*
- *$g_i \in G_1 \Leftrightarrow g_{i+1} \in G_2$ for all $i < n-1$.*

Then the homomorphism $\theta : G_1 *_A G_2 \to G$ induced by the inclusions $G_i \to G$ is an isomorphism iff

(i) $\langle G_1 \cup G_2 \rangle = G$, and

(ii) for any non-trivial alternating sequence $(g_i)_{i<n}$, we have $\prod_{i<n} g_i \neq 1$.

Proof.

- (i) holds iff θ is surjective:

  $G_1 *_A G_2$ is generated by $G_1 \cup G_2$ (with the usual identifications), so $\theta(G_1 *_A G_2)$ is generated by $\theta(G_1 \cup G_2) = G_1 \cup G_2$.

- (ii) holds iff θ is injective: Let $G' := G_1 *_A G_2$. Making the usual identifications, we have $G_i \leq G'$ and $\theta|_{G_1 \cup G_2} = \mathrm{id}$. We use superscripts to disambiguate products in G' from products in G.

  We show that $\{\prod^{G'}_{i<n} g_i : (g_i)_{i<n}$ is a non-trivial alternating sequence$\} = G' \setminus A$. Then since $\theta|_A$ is injective, we have: θ is injective iff $1 \notin \theta(G' \setminus A)$ iff $1 \neq \theta(\prod^{G'}_{i<n} g_i) = \prod^{G}_{i<n} \theta(g_i) = \prod^{G}_{i<n} g_i$ for any non-trivial alternating sequence $(g_i)_{i<n}$, as required.

  For the remainder of this proof, all products are in G'. Pick representatives $S_i \subseteq G_i$ for ${}_A\backslash^{G_i}$.

  If $g \in G' \setminus A$, then by existence of normal forms we have $g = as_0 \dots s_{n-1}$ for a normal form $(a; s_0, \dots, s_{n-1})$ with $n > 0$, and then $(as_0, s_1, \dots, s_{n-1})$ is a non-trivial alternating sequence as required.

  Conversely, if $(g_i)_{i<n}$ is a non-trivial alternating sequence, then say $g_i = a_i s_i$ with $s_i \in (S_1 \cup S_2) \setminus A$, then

$$\begin{aligned}\prod_{i<n} g_i &= a_0 s_0 \dots s_{n-3} a_{n-2} s_{n-2} a_{n-1} s_{n-1} \\ &= a_0 s_0 \dots s_{n-3} a_{n-2} (s_{n-2} a_{n-1})_A (s_{n-2} a_{n-1})_S s_{n-1} \\ &= \dots \\ &= a'_0 s'_0 \dots s'_{n-2} s'_{n-1}\end{aligned}$$

  where $(a'_0; s'_0, \dots, s'_{n-1})$ is a normal form (here we use that since $s_{n-2} \in G_i \setminus A$, also $s_{n-2} a_{n-1} \in G_i \setminus A$, and so on). Then $\prod_{i<n} g_i \notin A$ by the normal form theorem, since $n > 0$.

□

6 Trees and amalgams

Definition 6.1. A **segment** is a tree with two vertices.

Theorem 6.2. *Let $G \circlearrowleft X$ be a non-inversive action of a group on a graph, and suppose ${}_G\backslash^X$ is a segment. Let $T = \overset{P}{\circ}\!\overset{y}{\longrightarrow}\!\overset{Q}{\circ}$ be a lift of ${}_G\backslash^X$ (which exists by Lemma 2.27).*

Let $G_P, G_Q, G_y \leq G$ be the stabilisers, so $G_P \cap G_Q \leq G_y$, and so the inclusions $G_P, G_Q \hookrightarrow G$ induce a homomorphism $\theta : G_P *_{G_y} G_Q \to G$.

Then X is a tree if and only if θ is an isomorphism.

Definition 6.3. For $n > 0$, a **circuit** of length n in a graph X is a subgraph isomorphic to C_n.

Lemma 6.4. *A graph X is acyclic if and only if it contains no circuit.*

Proof.

- $\Rightarrow$: If X contains a circuit of length $n > 0$, then the image $(e_0, \ldots, e_{n-1})$ of the path $(0, \ldots, n-1)$ in C_n is a non-trivial reduced path from a vertex to itself.
- $\Leftarrow$: Suppose X is not acyclic but contains no circuit. Suppose $n > 0$ is minimal such that there is a reduced path $(e_0, \ldots, e_{n-1})$ with $\alpha(e_0) = \omega(e_{n-1})$. Since this does not yield a circuit, we must have $\alpha(e_i) = \alpha(e_j)$ for some $i < j$. But then $(e_i, \ldots, e_{j-1})$ is a reduced path from $\alpha(e_i)$ to $\omega(e_{j-1}) = \alpha(e_j) = \alpha(e_i)$, contradicting the minimality.

□

Proof of Theorem 6.2. By Proposition 5.21, it suffices to show:

(i) X is connected iff $\langle G_P \cup G_Q \rangle = G$;

(ii) X is acyclic iff for no non-trivial alternating sequence $(g_i)_{i<n}$ do we have $\prod_{i<n} g_i = 1$.

We prove these in turn.

(i) Let X' be the connected component of X containing T, and let

$$G' := \{g \in G : gX' = X'\} \leq G,$$

so $X = GT$ is connected iff $G' = G$. We conclude by showing $G' = \langle G_P \cup G_Q \rangle =: H \leq G$.

If $h \in G_P \cup G_Q$, then hT shares a vertex with T, so $hT \subseteq X'$. So since hX' is the connected component of X containing hT, we have $hX' = X'$, so $h \in G'$. Hence $H \leq G'$.

Now HT and $(G \setminus H)T$ are disjoint subgraphs of X; indeed, if $h \in H$ and $g \in G \setminus H$, then $h^{-1}g \notin H \supseteq G_P \cup G_Q$, so $hP \neq gP$ and $hQ \neq gQ$ (and of course $hP \neq gQ$ and $hQ \neq gP$, since P and Q are in different orbits).

So since $HT \cup (G \setminus H)T = GT = X$, we must have $X' \subseteq HT$ and so $G' \leq H$.

(ii) We apply Lemma 6.4. So suppose X contains a circuit, and let $(e_0, \ldots, e_{n-1})$ be the image of the path $(0, \ldots, n-1)$ in C_n. Say $e_i = h_i y_i$ where $y_i \in \{y, \overline{y}\}$. Let $P_i := \alpha(y_i) \in \{P, Q\}$.

Let $i < n$. Treat indices modulo n, so $y_n = y_0$ etc. Considering the image path in ${}_G\backslash^X$, we see $y_i = \overline{y_{i+1}}$ and $\{P_i, P_{i+1}\} = \{P, Q\}$. Also

$$h_{i+1}P_{i+1} = h_{i+1}\alpha(y_{i+1}) = \alpha(h_{i+1}y_{i+1}) = \omega(h_i y_i) = h_i\omega(y_i) = h_i P_{i+1},$$

so $g_i := h_i^{-1}h_{i+1} \in G_{P_{i+1}}$. Now by the definition of a circuit, we have $e_i \neq \overline{e_{i+1}}$. So

$$h_i y_i = e_i \neq \overline{e_{i+1}} = h_{i+1}\overline{y_{i+1}} = h_{i+1}y_i,$$

so $g_i \in G_{P_{i+1}} \setminus G_y$.

Now $h_0 = h_n = h_0 g_0 \dots g_{n-1}$, so $\prod_{i<n} g_i = 1$. But $(g_i)_{i<n}$ is a non-trivial alternating sequence, so this establishes the forward direction of the statement.

The converse is obtained by reversing the above construction. Let $(g_i)_{i<n}$ is a non-trivial alternating sequence with $\prod_{i<n} g_i = 1$, and say $g_0 \in G_Q \setminus G_y$. If n is even, then $(y, g_0\overline{y}, g_0g_1y, \dots, \prod_{i<n-1} g_i\overline{y})$ is a non-trivial reduced path from P to P. If n is odd, $n \neq 1$ since θ is an embedding on G_Q and G_P, and $(g_0\overline{y}, g_0g_1y, \dots, \prod_{i<n-1} g_i y)$ is a non-trivial reduced path from Q to Q.

□

Theorem 6.5. *Let $G = G_1 *_A G_2$ be an amalgamated free product. Assume (WLOG) $A \leq G_1, G_2 \leq G$.*

Then there exists a tree X and an action $G \circlearrowleft X$ such that ${}_G\backslash^X$ is a segment, and a lift $\overset{P}{\circ}\xrightarrow{y}\overset{Q}{\circ}$ of ${}_G\backslash^X$ such that $G_P = G_1$, $G_Q = G_2$, and $G_y = A$.

Proof. Let X be the graph:

$$\begin{aligned} X^0 &:= G/G_1 \mathbin{\dot\cup} G/G_2 \\ X^+ &:= G/A \\ \alpha(gA) &:= gG_1 \\ \omega(gA) &:= gG_2, \end{aligned}$$

with the obvious left action of G.

Then ${}_G\backslash^X$ is a segment, and setting $P := G_1$, $Q := G_2$, and $y := A$, $\overset{P}{\circ}\xrightarrow{y}\overset{Q}{\circ}$ is a lift, and the stabilisers are as required.

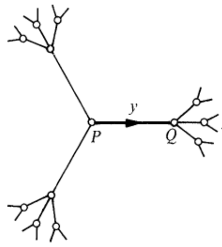
By Theorem 6.2, X is a tree. □

In particular, we deduce from Theorems 6.2 and Theorem 6.5:

Corollary 6.6. *A group G is an amalgamated free product if and only if G has a non-inversive action on a tree X such that ${}_G\backslash^X$ is a segment.*

Example 6.7. The action of $G := D_\infty = \mathrm{Aut}(C_\infty)$ on C_∞ induces a non-inversive action on its barycentric subdivision $B(C_\infty)$ (which is itself isomorphic to C_∞).

Taking a subsegment $\overset{P}{\circ}\xrightarrow{y}\overset{Q}{\circ}$ of $B(C_\infty)$, one sees $G_P \cong \mathbb{Z}/2\mathbb{Z} \cong G_Q$ and $G_y = 1$, so in this case Theorem 6.2 recovers the isomorphism $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ of Example 5.14.

Example 6.8. The tree of Theorem 6.5 for $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$ is: 

.

Corollary 6.9. *Let $\theta : G \twoheadrightarrow G_1 *_A G_2$ be an epimorphism.*
Then $G \cong \theta^{-1}(G_1) *_{\theta^{-1}(A)} \theta^{-1}(G_2)$.

Proof. Let X be the tree of Theorem 6.5 for $G_1 *_A G_2$. Then we conclude by applying Theorem 6.2 to the action of G on X induced by θ, namely $g * x := \theta(g) * x$. □

Corollary 6.10. *Suppose $A \trianglelefteq G_1, G_2$.*
Then $(G_1 *_A G_2)/A \cong (G_1/A) * (G_2/A)$.

Proof. $A \trianglelefteq G_1 *_A G_2 =: G$, since G_1, G_2 generate G.

Let X be the tree of Theorem 6.5 for G. Then A is in the kernel of the action, since A acts trivially on $X^+ = G/A$ (indeed, $agA = ga'A = gA$). So the action induces an action of G/A on X, and we conclude by Theorem 6.2. □

6.1 $\mathrm{SL}_2(\mathbb{Z})$

Definition 6.11. The **upper half plane** is $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. The action of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{1, -1\}$ on $\mathbb{H}$ by **Möbius transformations** is the action induced by the following action of $\mathrm{SL}_2(\mathbb{R})$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * z \mapsto \frac{az+b}{cz+d}.$$

Fact 6.12. *This does define a well-defined faithful action of* $\mathrm{PSL}_2(\mathbb{R})$.

(If we equip $\mathbb{H}$ with the Poincaré metric $ds = \frac{\sqrt{dx^2+dy^2}}{y}$ $(z = x + iy)$, with which $\mathbb{H}$ is a model of the hyperbolic plane, then the action is by isometries.)

Definition 6.13. A **hyperbolic line** in $\mathbb{H}$ is the intersection with $\mathbb{H}$ of a circle with real centre or a vertical line, i.e. $\{z : |z - a| = r,\ \mathrm{Im}(z) > 0\}$ or $\{a + iy : y > 0\}$ with $a \in \mathbb{R}$ and $r > 0$.

Fact 6.14. $\mathrm{PSL}_2(\mathbb{R})$ *maps hyperbolic lines to hyperbolic lines.*

(This follows from the fact that the hyperbolic lines are precisely the maximal geodesics in the hyperbolic plane $\mathbb{H}$.)

Definition 6.15. The **modular group** is $\Gamma := \mathrm{PSL}(\mathbb{Z}) \leq \mathrm{PSL}(\mathbb{R})$.

Fact 6.16.

$$D := \{z : \mathrm{Re}(z) \in [0, \frac{1}{2}],\ |z| \geq 1\} \cup \{z : \mathrm{Re}(z) \in (-\frac{1}{2}, 0),\ |z| > 1\} \subseteq \mathbb{H}$$

is a fundamental domain for the action of Γ on $\mathbb{H}$ by Möbius transformations, i.e. D intersects each orbit of the action in exactly one point.

Let $\alpha, \beta \in \Gamma$ be the images of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, so $\alpha * z = -\frac{1}{z}$ and $\beta * z = \frac{1}{1-z}$.

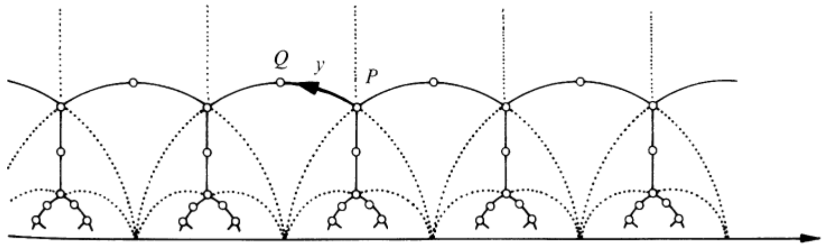
Fact 6.17. *α has order 2, β has order 3, and $\langle \alpha, \beta \rangle = \Gamma$.*

Fact 6.18. *Suppose $\gamma \in \Gamma \setminus 1$, $z \in D$, and $\gamma z = z$. Then either*

- *$z = e^{\frac{1}{2}\pi i}$ and $\gamma \in \langle \alpha \rangle$, or*
- *$z = e^{\frac{1}{3}\pi i}$ and $\gamma \in \langle \beta \rangle$.*

Theorem 6.19. *Let $A := \{e^{\alpha \pi i} : \alpha \in [\frac{1}{3}, \frac{1}{2}]\} \subseteq D$. Then $\Gamma A \subseteq \mathbb{H}$ is homeomorphic to the realisation of a tree X, and the action of Γ on $\mathbb{H}$ induces a non-inversive action on X. The segment T corresponding to A is a lift of the quotient; the stabilisers of the edges of T are trivial, and the stabilisers of its vertices are $\langle \alpha \rangle$ and $\langle \beta \rangle$.*

Hence $\Gamma \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.



Proof. By Facts 6.16 and 6.18, two images gA and hA of A do not intersect outside the endpoints. So ΓA is the realisation of a graph X, and the quotient and stabilisers are as stated. It remains to see that X is a tree.

Suppose X is not acyclic, let $C \subseteq X$ be a circuit, and let $S \subseteq \Gamma A$ be its corresponding realisation. Then S bounds a compact region R in $\mathbb{H}$. Now ΓA does not cover R (because $\mathbb{C}$ is a Baire space), so since D is a fundamental domain, $\gamma z \in R$ for some z in $D \setminus A$ and $\gamma \in \Gamma$. The half-line $L := z + i\mathbb{R}_{\geq 0}$ is contained in $D \setminus A$, so since D is a fundamental domain, also $\gamma L \cap \Gamma A = \emptyset$. So the hyperbolic half-line γL does not intersect $S \subseteq \Gamma A$, so $\gamma L \subseteq R$. But then $\mathrm{Im}(\gamma L)$ is contained in the closed interval $\mathrm{Im}(R)$ in $(0, \infty)$, which contradicts the description of hyperbolic lines.

Finally, X is connected by (i) in the proof of Theorem 6.2, since $\langle \alpha, \beta \rangle = \Gamma$. □

Applying Corollary 6.9, we deduce:

Corollary 6.20. $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \langle A, B \mid A^4, B^6, A^2 = B^3 \rangle$.

7 Bass-Serre theory

We aim to prove a common generalisation of Theorem 3.22 and Theorem 6.2, describing the structure of a group acting on a tree in terms of a generalised notion of fundamental group of the quotient graph, taking into account the stabilisers of the action.

7.1 Graphs of groups

Definition 7.1. A **graph of groups** $(\mathcal{G}, Y)$ consists of

- a connected non-empty graph Y;
- for each $x \in Y^0$ a group G_x;
- for each $e \in Y^1$ a group G_e and an embedding $G_e \to G_{\omega(e)}$ denoted by $a \mapsto a^e$,
- such that $G_e = G_{\overline{e}}$ for all $e \in Y^1$.

($\mathcal{G}$ denotes the family of groups and embeddings.)

We write ${G_e}^e \leq G_{\omega(e)}$ for the image of G_e under the corresponding embedding. We call $(G_x)_x$ the **vertex groups**, and $(G_e)_e$ the **edge groups**.

Example 7.2.

- Let Y be a segment. Then $(\mathcal{G}, Y)$ consists of a diagram

$$\phi_1 : G_e \to G_x,\ \phi_2 : G_e \to G_y$$

  whose colimit is the corresponding amalgamated free product.
- Let $Y = C_1$. Then $(\mathcal{G}, Y)$ consists of a group G_e with two embeddings into a group G_x.

7.2 The fundamental group of a graph of groups

We first define a handy generalisation of the group presentation notation.

Notation 7.3. Given groups $(G_i)_{i \in I}$, a set X, and relators $R \subseteq \ast_i G_i * F(X)$, define

$$\langle (G_i)_{i \in I}, X \mid R \rangle := (\mathop{\ast}_i G_i * F(X)) / \langle\langle R \rangle\rangle\,.$$

As usual, we allow ourselves to write $s = t$ for the relator st^{-1}.

Example 7.4. In this notation, Remark 5.18 becomes:

$$G_1 *_A G_2 \cong \langle G_1, G_2 \mid \phi_1(a) = \phi_2(a) : a \in A \rangle\,.$$

Lemma 7.5. *Let $(G_i)_{i \in I}, X, R$ be as above, and let H be a group, $\theta_i : G_i \to H$ for $i \in I$ homomorphisms, $f : X \to H$ a map of sets.*

Then these maps respect the relations of the presentation, meaning that R is in the kernel of the induced[5] *homomorphism $\ast_i G_i * F(X) \to H$, if and only if they induce a homomorphism $\langle (G_i)_{i \in I}, X \mid R \rangle \to H$.*

Proof. Immediate. □

Definition 7.6. Let $(\mathcal{G}, Y)$ be a graph of groups.

[5](by the universal properties of the free product and the free group)

- The **universal group** of $(\mathcal{G}, Y)$ is the group

  $$F(\mathcal{G}, Y) := \left\langle (G_x)_{x\in Y^0}, Y^1 \;\middle|\; \{ea^e\overline{e} = a^{\overline{e}} : e \in Y^1,\ a \in G_e\} \right\rangle.$$

  WARNING: a^e here is the image of a under the embedding $G_e \to G_{\omega(e)}$, it does not mean conjugation by e! Technically this is not ambiguous, because a itself is not an element of $*_{x\in Y^0} G_x$, but to avoid confusion we will eschew the exponential notation for conjugation in this section.

- For $x \in Y^0$, we identify $g \in G_x$ with its image in $F(\mathcal{G}, Y)$ under the homomorphism of the presentation. We will see later that this homomorphism is an embedding.

- The **fundamental group of** $(\mathcal{G}, Y)$ **with base-point** $x \in Y^0$ is the subgroup

  $$\pi_1(\mathcal{G}, Y, x) := \{g_0e_0g_1 \ldots e_{n-1}g_n : x \in Y^0,\ (e_0, \ldots, e_{n-1}) \text{ is a path from } x \text{ to } x,\ g_i \in G_{\alpha(e_i)}\ (i < n),\ g_n \in G_x\} \leq F(\mathcal{G}, Y).$$

- Let $T \subseteq Y$ be a maximal subtree. The **fundamental group of** $(\mathcal{G}, Y)$ **with respect to** T is the quotient

  $$\begin{aligned}\pi_1(\mathcal{G}, Y, T) &:= F(\mathcal{G}, Y)/\left\langle\left\langle T^1 \right\rangle\right\rangle \\ &\cong \left\langle (G_x)_{x\in Y^0}, Y^1 \;\middle|\; \{ea^e\overline{e} = a^{\overline{e}} : e \in Y^1,\ a \in G_e\} \cup T^1 \right\rangle.\end{aligned}$$

  (Note that we deduce the relations $\overline{e} = e^{-1}$ for $e \in Y^1$ by taking $a = 1 \in G_e$, and $a^e = a^{\overline{e}}$ for $e \in T^1$.)

Example 7.7.

(i) If vertex (and hence edge) groups are all trivial, then these definitions agree with those in Section 4.3 of the fundamental group of a graph.

(ii) If Y is a segment with $Y^0 = \{x, y\}$ and $Y^1 = \{e, \overline{e}\}$, then $\pi_1(\mathcal{G}, Y, Y) = \left\langle G_x, G_y \;\middle|\; \{a^e = a^{\overline{e}} : a \in G_e\} \right\rangle \cong G_x *_{G_e} G_y$.

(iii) Exercise: More generally, if Y is a tree, then $\pi_1(\mathcal{G}, Y, Y)$ is the colimit of the corresponding diagram of groups and embeddings.

(iv) If $Y \cong C_1$ is a loop, say $Y^0 = \{x\}$, $Y^1 = \{e, \overline{e}\}$, then the maximal subtree T has no edges and $\pi_1(\mathcal{G}, Y, T) \cong F(\mathcal{G}, Y) \cong \left\langle G_x, e \;\middle|\; \{ea^e\overline{e} = a^{\overline{e}} : a \in G_e\} \right\rangle$. Section 7.3 below is devoted to this case.

The following theorem and its proof extend Theorem 4.19.

Theorem 7.8. *For any choice of x and T, $\pi_1(\mathcal{G}, Y, x) \cong \pi_1(\mathcal{G}, Y, T)$.*

In particular, the fundamental group is independent (up to isomorphism) of the choice of x or T; we sometimes call it $\pi_1(\mathcal{G}, Y)$.

Proof. For $y \in Y^0 = T^0$, let $\gamma_y := e_0 \ldots e_{n-1} \in F(\mathcal{G}, Y)$ where $(e_0, \ldots, e_{n-1})$ is the geodesic in T from x to y.

We define a homomorphism $f : \pi_1(\mathcal{G}, Y, T) \to \pi_1(\mathcal{G}, Y, x)$ by setting for $e \in Y^1$ and $g \in G_y$ $(y \in Y^0)$:

$$f(e) := \gamma_{\alpha(e)} e \gamma_{\omega(e)}^{-1} \in \pi_1(\mathcal{G}, Y, x)$$
$$f(g) := \gamma_y g \gamma_y^{-1}.$$

To see that this defines such a homomorphism, we must check that it respects the relations (see Lemma 7.5) of $\pi_1(\mathcal{G}, Y, T)$. But indeed, $f(e) = 1$ for $e \in T^1$, and for $e \in Y^1$ and $a \in G_e$,

$$\begin{aligned} f(e)f(a^e)f(\overline{e}) &= \gamma_{\alpha(e)} e \gamma_{\omega(e)}^{-1} \gamma_{\omega(e)} a^e \gamma_{\omega(e)}^{-1} \gamma_{\alpha(\overline{e})} \overline{e} \gamma_{\omega(\overline{e})}^{-1} \\ &= \gamma_{\alpha(e)} e a^e \overline{e} \gamma_{\omega(\overline{e})}^{-1} \\ &= \gamma_{\alpha(\overline{e})} a^{\overline{e}} \gamma_{\omega(e)}^{-1} \\ &= f(a^{\overline{e}}). \end{aligned}$$

Let $p : \pi_1(\mathcal{G}, Y, x) \to \pi_1(\mathcal{G}, Y, T)$ be the restriction of the quotient map $F(\mathcal{G}, Y) \to \pi_1(\mathcal{G}, Y, T)$.

Then $p(f(e)) = e$ for $e \in Y^1$, and $p(f(g)) = g$ for $g \in G_y$ $(y \in Y^0)$, so $p \circ f = \text{id}$. If $(e_0, \dots, e_{n-1})$ is a path from x to x, and $g_i \in G_{\alpha(e_i)}$ and $g_i \in G_{\omega(e_{n-1})}$, then

$$\begin{aligned} &f(p(g_0 e_0 g_1 \dots e_{n-1} g_n)) \\ &\quad = \gamma_{\alpha(e_0)} g_0 \gamma_{\alpha(e_0)}^{-1} \gamma_{\alpha(e_0)} e_0 \gamma_\omega(e_0)^{-1} \gamma_{\alpha(e_1)} g_1 \dots e_{n-1} \gamma_{\omega(e_{n-1})}^{-1} \gamma_{\omega(e_{n-1})} g_n \gamma_{\omega(e_{n-1})}^{-1} \\ &\quad = g_0 e_0 g_1 \dots e_{n-1} g_n \end{aligned}$$

(since $\alpha(e_0) = x = \omega(e_{n-1})$ and $\omega(e_i) = \alpha(e_{i+1})$).

So p and f are mutually inverse homomorphisms, so they are isomorphisms. □

7.3 HNN extensions

Before proceeding with the general theory, we consider in detail the case of a loop, $(\mathcal{G}, C_1)$.

Definition 7.9. Let $\phi_1, \phi_2 : A \to G$ be group embeddings. The **HNN-extension** of this data is the group

$$\text{HNN}(G, A, \phi_1, \phi_2) := \left\langle G, t \;\middle|\; t\phi_1(a)t^{-1} = \phi_2(a) : a \in A \right\rangle.$$

We call t the **stable letter**.

Remark 7.10. The HNN-extension is the fundamental group of the corresponding graph of groups $(\mathcal{G}, C_1)$.

Namely, given $\phi_1, \phi_2 : A \to G$, define a graph of groups $(\mathcal{G}, C_1)$ with vertex group $G_x := G$, edge group $G_e := A =: G_{\overline{e}}$, and ϕ_1, ϕ_2 as the embeddings.

The maximal subtree $T \subseteq C_1$ contains no edges, so

$$\pi_1(\mathcal{G}, C_1, T) \cong F(\mathcal{G}, C_1) \cong \langle G, e \mid e\phi_1(a)\overline{e} = \phi_2(a) : a \in A \rangle \cong \text{HNN}(G, A, \phi_1, \phi_2).$$

We now justify calling an HNN-extension an extension.

Lemma 7.11. *Let $\phi_1, \phi_2 : A \to G$ be group embeddings. For $g \in G$, let $\lambda_g^{G^2} \in \mathrm{Sym}(G^2)$, $\lambda_g^{G^2}(g_1, g_2) := (gg_1, g_2)$ Then there exists $\tau \in \mathrm{Sym}(G^2)$ such that*

$$\tau \circ \lambda_{\phi_1(a)}^{G^2} = \lambda_{\phi_2(a)}^{G^2} \circ \tau$$

for all $a \in A$.

Proof (due to P. Hall).

Claim 7.12. $[G^2 : \phi_1(A) \times 1] = [G^2 : \phi_2(A) \times 1]$.

Proof. Considering the chains

$$\phi_1(A) \times 1 \leq G \times 1 \leq G \times \phi_2(A) \leq G^2 \geq G \times \phi_1(A) \geq G \times 1 \geq \phi_2(A) \times 1,$$

and noting $|\phi_1(A)| = |A| = |\phi_2(A)|$, we have

$$\begin{aligned} [G^2 : \phi_1(A) \times 1] &= [G : \phi_2(A)]|\phi_2(A)|[G : \phi_1(A)] \\ &= [G : \phi_1(A)]|\phi_1(A)|[G : \phi_2(A)] \\ &= [G^2 : \phi_2(A) \times 1]. \end{aligned}$$

$\square_{7.12}$

Identify G with $G \times 1 \leq G^2$ (and hence $\phi_i(A)$ with $\phi_i(A) \times 1$).

Let $R_i \subseteq G^2$ be representatives for the right cosets of $\phi_i(A)$ in G^2. By the claim, let $f : R_1 \to R_2$ be a bijection.

Define $\tau : G^2 \to G^2$ as follows: for $a \in A$ and $r \in R_1$, set

$$\tau(\phi_1(a)r) := \phi_2(a)f(r).$$

Then τ is a well-defined bijection, since any element of G^2 has a unique expression of the form $\phi_1(a)r$, and also a unique expression of the form $\phi_2(a)f(r)$.

Now for $a, a' \in A$ and $r \in R_1$,

$$\tau(\phi_1(a)\phi_1(a')r) = \tau(\phi_1(aa')r) = \phi_2(aa')f(r) = \phi_2(a)(\phi_2(a')f(r)) = \phi_2(a)(\tau(\phi_1(a')r),$$

so

$$\tau \circ (\phi_1(a)\cdot) = (\phi_2(a)\cdot) \circ \tau$$

(using the notation $(g\cdot) : G^2 \to G^2; h \mapsto gh$), as required. $\square$

Theorem 7.13. *Let $H := \mathrm{HNN}(G, A, \phi_1, \phi_2)$ be an HNN-extension. Then the homomorphism $\eta : G \to H$ of the presentation is an embedding.*

Proof. Let τ be given by Lemma 7.11. Then we obtain (by Lemma 7.5) a well-defined homomorphism $\beta : H \to \mathrm{Sym}(G^2)$ with $\beta(\eta(g)) = \lambda_g^{G^2}$ for $g \in G$ and $\beta(t) = \tau$.

But then $\beta \circ \eta : G \to \mathrm{Sym}(G^2)$ is injective (since $\lambda_g^{G^2} \neq \lambda_h^{G^2}$ for $g \neq h$), hence η is injective. $\square$

Corollary 7.14 (Higman-Neumann-Neumann 1949)**.** *Subgroups A and B of a group G are conjugate in some supergroup of G if and only if $A \cong B$.*

Proof.

$\Rightarrow$: Clear.

$\Leftarrow$: Say $\theta : A \to B$ is an isomorphism. Then $B = tAt^{-1}$ in $\mathrm{HNN}(G, A, \mathrm{id}, \theta) \geq G$.

$\square$

Corollary 7.15. *Any group G embeds in a group G^* in which any two elements of the same order are conjugate. If G is countable, G^* can also be taken to be countable.*

Proof for G countable. First note that if $g_1, g_2 \in G$ have the same order, then there is $n \in \mathbb{N}$ and embeddings $\phi_i : \mathbb{Z}/n\mathbb{Z} \to G$ with $\phi_i(1) = g_i$, so g_1, g_2 are conjugate in $\mathrm{HNN}(G, \mathbb{Z}/n\mathbb{Z}, \phi_1, \phi_2) \geq G$.

Claim 7.16. Any countable group H embeds in a countable group $E(H)$ in which any two elements of H of the same order are conjugate.

Proof. Let $(g_i, h_i)_{i\in\mathbb{N}}$ enumerate the pairs of elements of H with the same order. Let $H_0 := H$, and recursively define $H_{i+1} \geq H_i$ to be such that g_i and h_i are conjugate in H_{i+1}, as above. Let $E(H) := \bigcup_{i\in\mathbb{N}} H_i$ be the direct limit of this chain. $\square_{7.16}$

Now let $G_0 := G$, and $G_{i+1} := E(G)$. Then the direct limit $G^* = \bigcup_{i\in\mathbb{N}} G_i$ is as required: if $g, h \in G^*$ have the same order, then $g, h \in G_i$ for some $i \in \mathbb{N}$, so they are conjugate in G_{i+1}, and hence in $G^* \geq G_{i+1}$. $\square$

An alternative proof of Theorem 7.13 goes via the following normal form theorem. We will not use it, so we omit the proof (but we obtain it below (Remark 7.35) as a special case of a more general result).

Fact 7.17 (Britton's Lemma)**.** *Let S_i be a set of representatives for the right cosets of $\phi_i(A)$ in G, with $1 \in S_i$.*

Then every $h \in H$ has a unique expression of the following form: $h = gt^{\epsilon_0}s_0 \ldots t^{\epsilon_{n-1}}s_{n-1}$ where

- $n \in \mathbb{N}$*;*
- $g \in G$*;*
- $\epsilon_i \in \{1, -1\}$*;*
- *if $\epsilon_i = 1$, then $s_i \in S_1$, else $s_i \in S_2$;*
- *if $s_i = 1$ and $i < n - 1$ then $\epsilon_i = \epsilon_{i+1}$.*

The following proposition explains what HNN-extensions have to do with amalgams. We will not use it.

Proposition 7.18. *Let $H := \mathrm{HNN}(G, A, \phi_1, \phi_2)$.*

Let $(\mathcal{G}, C_\infty)$ be the following graph of groups: $G_i = G$ for $i \in (C_\infty)^0$, and for $e \in (C_\infty)^+$:

$$G_e := A,\ a^{\overline{e}} := \phi_1(a),\ a^e := \phi_2(a).$$

Then the action by translations $\mathbb{Z} \circlearrowleft C_\infty$ induces an action $\mathbb{Z} \circlearrowleft \pi_1(\mathcal{G}, C_\infty)$ with respect to which

$$H \cong \pi_1(\mathcal{G}, C_\infty) \rtimes \mathbb{Z}.$$

Sketch proof. For notational purposes, replace the G_i with disjoint isomorphic copies of G; say $\psi_i : G \xrightarrow{\cong} G_i$. Let $\phi_{i,1} := \psi_i \circ \phi_1 : A \to G_i$ and $\phi_{i,2} := \psi_{i+1} \circ \phi_2 : A \to G_{i+1}$ be the resulting embeddings. Let $\theta_i := \psi_{i+1} \circ \psi_i^{-1} : G_i \xrightarrow{\cong} G_{i+1}$.

Note $\theta_i \circ \phi_{i,1} = \phi_{i+1,1}$.

Then we compute presentations as follows, with the action of $\mathbb{Z}$ induced by the θ_i:

$$\begin{aligned}
\pi_1(\mathcal{G}, C_\infty) \rtimes \mathbb{Z} &\cong \langle (G_i)_{i\in\mathbb{Z}} \mid \{\phi_{i,1}(a) = \phi_{i,2}(a) : i \in \mathbb{Z}, a \in A\}\rangle \rtimes \mathbb{Z} \\
&\cong \left\langle (G_i)_{i\in\mathbb{Z}}, t \;\middle|\; \{\phi_{i,1}(a) = \phi_{i,2}(a) : i \in \mathbb{Z}, a \in A\} \cup \{t^{-1} g_i t = \theta_i(g_i) : i \in \mathbb{Z}, g_i \in G_i\}\right\rangle \\
&\cong \left\langle G_1, t \;\middle|\; \{t\phi_{1,1}(a)t^{-1} = \phi_{0,2}(a) : a \in A\}\right\rangle \\
&\cong \mathrm{HNN}(G, A, \phi_1, \phi_2).
\end{aligned}$$

□

7.4 Inclusion of the vertex groups in the fundamental group

Theorem 7.19. *Let $(\mathcal{G}, Y)$ be a graph of groups, and let $x \in Y^0$. Then the natural maps*

(i) $G_x \to F(\mathcal{G}, Y)$

(ii) $G_x \to \pi_1(\mathcal{G}, Y, x)$

(iii) $G_x \to \pi_1(\mathcal{G}, Y, T)$ *(for any maximal subtree $T \subseteq Y$)*

are embeddings.

Proof. (i) Let $K := \prod_{x\in Y^0} G_x$. Identify each G_x with the corresponding subgroup of K.

For $e \in Y^1$, Lemma 7.11 provides $\tau_y \in \mathrm{Sym}(K^2)$ such that

$$\tau_y \circ \lambda^{K^2}_{a^e} = \lambda^{K^2}_{a^{\overline{e}}} \circ \tau_y$$

for $a \in G_e$. So we obtain a homomorphism $\beta : F(\mathcal{G}, Y) \to \mathrm{Sym}(K^2)$ with $\beta(g) = \lambda^{K^2}_g$ for $g \in G_x$, $x \in Y^0$, and the injectivity follows (as in Theorem 7.13).

(ii) This follows from (i), since $\pi_1(\mathcal{G}, Y, x)$ is a subgroup of $F(\mathcal{G}, Y)$ containing G_x (by taking $n = 0$ in the definition of $\pi_1(\mathcal{G}, Y, x)$).

(iii) This follows from (ii), since the quotient map $\pi_1(\mathcal{G}, Y, x) \to \pi_1(\mathcal{G}, Y, T)$ is an isomorphism by (the proof of) Theorem 7.8.

□

7.5 Preview of the main results of Bass-Serre theory

We will associate to a graph of groups $(\mathcal{G}, Y)$ its "universal cover", which will be a tree with an action of $\pi_1(\mathcal{G}, Y)$ such that the quotient is Y. The vertex and edge groups will be recovered as stabilisers of the action.

We will then prove that any action of a group G on a tree X is of this form. So $G \cong \pi_1(\mathcal{G}, {}_G\backslash^X)$ where $\mathcal{G}$ consists of certain stabilisers. Along the way, we will obtain a normal form theorem for fundamental groups of graphs of groups, making this description even more useful.

7.6 Universal covers of graphs

Definition 7.20. A morphism is **locally bijective** if it is both locally injective and locally surjective, i.e. if it restricts to bijections of stars.

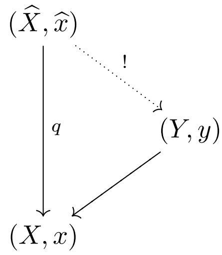
Notation 7.21. If $p : X \to Y$ is a morphism of graphs and $x \in X^0$, $y \in Y^0$, we write $p : (X, x) \to (Y, y)$ to mean $p(x) = y$.

Lemma 7.22. *Let $p : (Y, y) \to (X, x)$ be locally bijective, and let $(e_0, \dots, e_{n-1})$ be a path in X from x. Then there is a unique path (the* **lift***) $(e'_0, \dots, e'_{n-1})$ from y with $p(e'_i) = e_i$.*

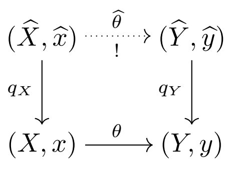
Proof. Immediate, by induction on n. □

Definition 7.23. Let X be a connected graph.

- A **connected cover** of X is a locally bijective morphism $Y \to X$ where Y is a connected graph.
- A **universal cover** of X is a connected cover $q : \widehat{X} \to X$ with the following universal property: for any $\widehat{x} \in \widehat{X}$, if $p : (Y, y) \to (X, x)$ is a connected cover where $x := q(\widehat{x})$, then there exists a unique morphism $r : (\widehat{X}, \widehat{x}) \to (Y, y)$ such that $p \circ r = q$.



Lemma 7.24. *Universal covers are unique up to unique isomorphism of pointed graphs: let $q_X : (\widehat{X}, \widehat{x}) \to (X, x)$ and $q_Y : (\widehat{Y}, \widehat{y}) \to (Y, y)$ be universal covers, and suppose $\theta : (X, x) \to (Y, y)$ is an isomorphism. Then there exists a unique isomorphism $\widehat{\theta} : (\widehat{X}, \widehat{x}) \to (\widehat{Y}, \widehat{y})$ such that $q_Y \circ \widehat{\theta} = \theta \circ q_X$.*



We call $\widehat{\theta}$ the **extension** *of θ sending $\widehat{x}$ to $\widehat{y}$.*

Proof. $\theta^{-1} \circ q_Y : (\widehat{Y}, \widehat{y}) \to (X, x)$ is a connected cover, so by the universal property of q_X there is a unique morphism $\widehat{\theta} : (\widehat{X}, \widehat{x}) \to (\widehat{Y}, \widehat{y})$ making the diagram commute, and it remains only to see that it is an isomorphism.

But $\theta \circ q_X : (\widehat{X}, \widehat{x}) \to (Y, y)$ is also a connected cover, so by the universal property of q_Y there is a (unique) morphism $\widehat{\phi} : (\widehat{Y}, \widehat{y}) \to (\widehat{X}, \widehat{x})$ making the diagram commute.

But then

$$q_Y \circ \widehat{\theta} \circ \widehat{\phi} = \theta \circ q_X \circ \widehat{\phi} = \theta \circ \theta^{-1} \circ q_Y = q_Y \circ \mathrm{id}_{\widehat{Y}} : (\widehat{Y}, \widehat{y}) \to (Y, y),$$

so by the uniqueness in the universal property (with $p := q_Y$) we have $\widehat{\theta} \circ \widehat{\phi} = \mathrm{id}_{\widehat{Y}}$, and similarly $\widehat{\phi} \circ \widehat{\theta} = \mathrm{id}_{\widehat{X}}$. □

Lemma 7.25. *If $q : \widehat{X} \to X$ is a connected cover and $\widehat{X}$ is a tree, then q is a universal cover.*

Proof. Let $\widehat{x} \in \widehat{X}^0$, let $x := q(\widehat{x})$, and let $p : (Y, y) \to (X, x)$ be a connected cover. We construct $r : (\widehat{X}, \widehat{x}) \to (Y, y)$ such that $p \circ r = q$.

Given $z \in \widehat{X}^0$, let $(e_0, \dots, e_{n-1})$ be the geodesic from $\widehat{x}$ to z in $\widehat{X}$, and let (by Lemma 7.22) $(e'_0, \dots, e'_{n-1})$ be the lift of $(q(e_0), \dots, q(e_{n-1}))$ to a path in Y from y. Then to have $p \circ r = q$, we must set $r(z) := \omega(e'_{n-1})$ and, if $n > 0$, $r(e_{n-1}) := e'_{n-1}$ and $r(\overline{e_{n-1}}) := \overline{e'_{n-1}}$.

Defining r this way, ranging over all $z \in \widehat{X}^0$, gives us a unique candidate for a morphism with $p \circ r = q$, and it remains only to check that r is a morphism. But indeed, since in the above $(e_0, \dots, e_{n-2})$ is also a geodesic if $n > 0$, we have

$$\alpha(r(e_{n-1})) = \alpha(e'_{n-1}) = \omega(e'_{n-2}) = r(\omega(e_{n-2}) = r(\alpha(e_{n-1})),$$

and we also have

$$\omega(r(e_{n-1})) = \omega(e'_{n-1}) = r(z) = r(\omega(e_{n-1})).$$

□

Theorem 7.26. *Any connected non-empty graph X has a universal cover $q : \widehat{X} \to X$.*

Proof. Let $x \in X^0$. Let $\widehat{X}^0$ be the set of reduced paths in X from x, and if $p \in \widehat{X}^0$ is a path to y, set $q(p) := y$. Let $\widehat{x} \in \widehat{X}^0$ be the trivial reduced path from x.

Let $\widehat{X}^+$ be the set of non-trivial reduced paths in X from x, and set $q((e_0, \dots, e_{n-1})) := e_{n-1}$. Set $\omega((e_0, \dots, e_{n-1})) := (e_0, \dots, e_{n-1})$ and $\alpha((e_0, \dots, e_{n-1})) := (e_0, \dots, e_{n-2})$.

Then q is a morphism, and it is locally bijective since if $p = (e_0, \dots, e_{n-1}) \in \widehat{X}^0 \setminus \{\widehat{x}\}$ then

$$\begin{aligned} \mathrm{star}^{\widehat{X}}(p) =& \{(e_0, \dots, e_n) : e_n \in \mathrm{star}^X(q(p)) \setminus \{\overline{e_{n-1}}\}\} \\ & \cup \{\overline{(e_0, \dots, e_{n-1})}\} \end{aligned}$$

maps bijectively to $\mathrm{star}^X(q((e_0, \dots, e_{n-1})))$, and similarly $\mathrm{star}^{\widehat{X}}(\widehat{x}) = \{(e) : e \in \mathrm{star}^X(x)\}$ also maps bijectively to $\mathrm{star}^X(x)$.

Considering these formulas for the stars, we also see that the reduced paths in $\widehat{X}$ from $\widehat{x}$ are precisely those of the form

$$((e_0), (e_0, e_1), \dots, (e_0, \dots, e_{n-1}))$$

where $(e_0, \dots, e_{n-1})$ is a reduced path in X. So $\widehat{X}$ is connected and has no non-trivial reduced paths from $\widehat{x}$ to $\widehat{x}$, and it follows that $\widehat{X}$ is a tree. □

7.7 Universal covers of graphs of groups

We say a graph of groups $(\mathcal{G}, Y)$ is **oriented** if Y is oriented, i.e. Y has a specified orientation $Y^+ \subseteq Y^1$.

Definition 7.27. Let $(\mathcal{G}, Y)$ be an oriented graph of groups, and let $T \subseteq Y$ be a maximal subtree (with the induced orientation $T^+ = T^1 \cap Y^+$). Let $\pi := \pi_1(\mathcal{G}, Y, T)$.

For $e \in Y^+$, define

$$\pi_e := G_e^{\overline{e}} \leq G_{\alpha(e)} \leq \pi.$$

The **universal cover** of $(\mathcal{G}, Y)$ with respect to T is the oriented graph $\widetilde{Y}$ defined as follows,

$$\begin{aligned} \widetilde{Y}^0 &:= \dot{\bigcup}_{x \in Y^0} \pi / G_x \\ \widetilde{Y}^+ &:= \dot{\bigcup}_{e \in Y^+} \pi / \pi_e \\ \alpha(g\pi_e) &:= gG_{\alpha(e)} \\ \omega(g\pi_e) &:= geG_{\omega(e)}, \end{aligned}$$

with the obvious action of π, namely $h * (gG_x) := hgG_x$ and $h * (g\pi_e) := hg\pi_e$.

We will actually mostly use the following notation instead. For $x \in Y^0$, set $\widetilde{x} := G_x \in \widetilde{Y}^0$, and for $e \in Y^+$, set $\widetilde{e} := \pi_e \in \widetilde{Y}^+$. So then

$$\begin{aligned} \widetilde{Y}^0 &= \dot{\bigcup}_{x \in Y^0} \pi \widetilde{x} \\ \widetilde{Y}^+ &= \dot{\bigcup}_{e \in Y^+} \pi \widetilde{e} \\ \alpha(g\widetilde{e}) &= g\widetilde{\alpha(e)} \\ \omega(g\widetilde{e}) &= ge\widetilde{\omega(e)}. \end{aligned}$$

We define a morphism $p : \widetilde{Y} \to Y$ by $p(g\widetilde{x}) := x$ and $p(g\widetilde{e}) := e$.

We also define a lift $\widetilde{T} \subseteq \widetilde{Y}$ of T:

$$\begin{aligned} \widetilde{T}^0 &= \{\widetilde{x} : x \in T^0\} \\ \widetilde{T}^+ &= \{\widetilde{e} : e \in T^+\}. \end{aligned}$$

Lemma 7.28.

(i) The graph $\widetilde{Y}$ is well-defined.

(ii) The stabiliser of $\widetilde{x}$ is G_x, and the stabiliser of $\widetilde{e}$ is π_e (for $x \in Y^0$ and $e \in Y^+$).

(iii) $p : \widetilde{Y} \to Y$ is a morphism.

(iv) Y is isomorphic to ${}_{\pi}\backslash^{\widetilde{Y}}$ via $x \mapsto \pi\widetilde{x}$ and $e \mapsto \pi\widetilde{e}$, and p agrees with the quotient map.

(v) $\widetilde{T}$ is a lift of T along p.

Proof. (i) We check that α and ω are well-defined. If $a \in \pi_e \leq G_{\alpha(e)}$, then

$$\alpha(a\pi_e) = aG_{\alpha(e)} = G_{\alpha(e)} = \alpha(\pi_e),$$

and

$$\omega(a\pi_e) = aeG_{\omega(e)} = e\bar{e}aeG_{\omega(e)} = eG_{\omega(e)} = \omega(\pi_e)$$

since $\bar{e}ae \in G_{\omega(e)}$.

(ii) Immediate.

(iii)

$$p(\alpha(g\widetilde{e})) = p(g\widetilde{\alpha(e)}) = \alpha(e) = \alpha(p(g\widetilde{e})).$$

$$p(\omega(g\widetilde{e})) = p(ge\widetilde{\omega(e)}) = \omega(e) = \omega(p(g\widetilde{e})).$$

(iv) Immediate.

(v) $\widetilde{T}$ is a subgraph of $\widetilde{Y}$, since for $e \in T^+$ we have $\alpha(\widetilde{e}) = \widetilde{\alpha(e)}$ and $\omega(\widetilde{e}) = e\widetilde{\omega(e)} = \widetilde{\omega(e)}$. Then it is a lift, since $p|_{\widetilde{T}}$ is a bijection.

□

Example 7.29. If Y is a segment, then $\widetilde{Y}$ is the graph defined in Theorem 6.5.

Remark 7.30. Typically, a universal cover of $(\mathcal{G}, Y)$ is <u>not</u> a universal cover of the graph Y, since p is typically not locally injective.

However, consider the case that $\mathcal{G}$ consists of trivial groups, so $\pi = \pi(\mathcal{G}, Y, T) = \pi(Y, T) \cong F(Y^+ \setminus T^+)$. Then p is locally bijective, since p is a bijection on any

$$\mathrm{star}^{\widetilde{Y}}(g\widetilde{x}) = \{g\widetilde{e} : \alpha(e) = x\} \cup \{\overline{ge^{-1}\widetilde{e}} : \omega(e) = x\}$$

(here we use that since $G_x = 1$, we have $\alpha(h\widetilde{e}) = g\widetilde{\alpha(e)} \Leftrightarrow h = g$ and $\omega(h\widetilde{e}) = g\widetilde{\omega(e)} \Leftrightarrow h = ge^{-1}$).

We will see below that $\widetilde{Y}$ is a tree, so $p : \widetilde{Y} \to Y$ is the universal cover of the graph Y. (We could also prove this directly, giving an alternative proof of Theorem 7.26.)

Lemma 7.31. *Let $(\mathcal{G}, Y)$ be an oriented graph of groups, let $T \subseteq Y$ be a maximal subtree, and let $T^+ := T^1 \cap Y^+$. Then*

$$\pi_1(\mathcal{G}, Y, T) \cong \left\langle (G_x)_{x \in Y^0}, Y^+ \;\middle|\; \{ea^e e^{-1} = a^{\bar{e}} : e \in Y^+,\ a \in G_e\} \cup T^+ \right\rangle.$$

Proof. We obtain this from the original presentation by using the relations $\bar{e} = e^{-1}$ to delete the generators $\{\bar{e} : e \in Y^+\}$. □

Theorem 7.32. *Let $\widetilde{Y}$ be the universal cover of an oriented graph of groups $(\mathcal{G}, Y)$ with respect to a maximal subtree T.*

Then $\widetilde{Y}$ is a tree.

Proof.

- $\widetilde{Y}$ is connected: First, let $W \subseteq \widetilde{Y}$ be the smallest subgraph with $W^+ = \{\widetilde{e} : e \in Y^+\}$. Then W is connected, since $\alpha(\widetilde{e}) = \widetilde{\alpha(e)} \in \widetilde{T}^0$ for $\widetilde{e} \in Y^+$, and $\widetilde{T}$ is connected.

  Let $S := Y^+ \cup \bigcup_{x \in Y^0} G_x \subseteq \pi$. If $s \in S$, then $W^0 \cap sW^0 \neq \emptyset$: indeed, if $g \in G_x$ then $\widetilde{x} \in W^0 \cap gW^0$, and if $e \in Y^+$ then $\omega(\widetilde{e}) = e\widetilde{\omega(e)} \in W^0 \cap eW^0$. So $W \cup sW$ is connected, and hence also $W \cup s^{-1}W = s^{-1}(W \cup sW)$ is connected. It follows that for any $\prod_{i<n} s_i \in \pi$, where $s_i \in S \cup S^{-1}$, $W \cup s_0W \cup \ldots \cup \prod_{i<n} s_iW = W \cup \ldots \cup \prod_{0<i<n}) \cup s_0($ is connected, since each adjacent subunion $\prod_{i<k} s_iW \cup \prod_{i \leq k} s_iW = \prod_{i<k} s_i(W \cup s_kW)$ is connected. Since $\langle S \rangle = G$ and $\pi W = \widetilde{Y}$, it follows that $\widetilde{Y}$ is connected.

- $\widetilde{Y}$ is acyclic (proof due to Chiswell): Let $q : \widehat{Y} \to \widetilde{Y}$ be the universal cover of the graph $\widetilde{Y}$ (which exists by Theorem 7.26). We will conclude by embedding $\widetilde{Y}$ into the tree $\widehat{Y}$.

  Let $\widehat{T}$ be a lift of $\widetilde{T}$ along q (which exists by Lemma 2.27). For $x \in Y^0 = T^0$, let $\widehat{x} \in \widehat{T}$ be the element with $q(\widehat{x}) = \widetilde{x}$. For $e \in Y^+ \setminus T^+$, let $\widehat{e} \in \widehat{Y}^1$ be the unique edge with $q(\widehat{e}) = \widetilde{e}$ and $\alpha(\widehat{e}) = \widehat{\alpha(e)}$.

  For $g \in \pi$, let $\lambda_g \in \mathrm{Aut}(\widetilde{Y})$ be given by the action of π on $\widetilde{Y}$ defined above. Now we extend this action to an action $*$ of π on $\widehat{Y}$.

  For $g \in G_x$, where $x \in Y^0$, we have $g\widetilde{x} = \widetilde{x}$, so by Lemma 7.24 let $(g*) : \widehat{Y} \to \widehat{Y}$ be the unique extension of $\lambda_g : \widetilde{Y} \to \widetilde{Y}$ such that $g * \widehat{x} = \widehat{x}$. This does define an action of G_x, i.e. $(gh*) = (g*) \circ (h*)$ and $(1*) = \mathrm{id}$, by the uniqueness.

  Claim 7.33. Let $e \in Y^+$ and $g \in \pi_e \leq G_{\alpha(e)}$. Then $g * \widehat{e} = \widehat{e}$.

  Proof. $q(g * \widehat{e}) = g\widetilde{e} = \widetilde{e} = q(\widehat{e})$, and $\alpha(g * \widehat{e}) = g * \alpha(\widehat{e}) = g * \widehat{\alpha(e)} = \widehat{\alpha(e)} = \alpha(\widehat{e})$. So since q is locally injective, $g * \widehat{e} = \widehat{e}$. □

  For $e \in Y^+ \setminus T^+$, we have $e\widetilde{\omega(e)} = \omega(\widetilde{e})$, so let $(e*) : \widehat{Y} \to \widehat{Y}$ be the unique extension of $\lambda_e : \widetilde{Y} \to \widetilde{Y}$ such that $e * \widehat{\omega(e)} = \omega(\widehat{e})$.

  We claim that these definitions respect the relations of

  $$\pi = \pi_1(\mathcal{G}, Y, T) \cong \left\langle (G_x)_{x \in Y^0}, Y^+ \;\middle|\; \{ea^e e^{-1} = a^{\overline{e}} : e \in Y^+,\ a \in G_e\} \cup T^+ \right\rangle,$$

  and hence (via Lemma 7.5) define an action $*$ of π on $\widehat{Y}$ (i.e. a homomorphism $\pi \to \mathrm{Aut}(\widehat{Y})$). Indeed,

  $$e * (a^e * (e^{-1} * \omega(\widehat{e}))) = e * (a^e * \widehat{\omega(e)}) = e * \widehat{\omega(e)} = \omega(\widehat{e}),$$

  and also $a^{\overline{e}} * \omega(\widehat{e}) = \omega(a^{\overline{e}} * \widehat{e}) = \omega(\widehat{e})$ by the Claim. So since $\lambda_{ea^e e^{-1}} = \lambda_{a^{\overline{e}}}$, we conclude $(e*) \circ (a^e*) \circ (e^{-1}*) = (a^{\overline{e}}*)$ by uniqueness of extensions.

  Now since $(g*) : \widehat{Y} \to \widehat{Y}$ extends $\lambda_g : \widetilde{Y} \to \widetilde{Y}$ for g a generator, this also holds for any $g \in \pi$. So we have:

  $$q(g * \widehat{x}) = g\widetilde{x},\ q(g * \widehat{e}) = g\widetilde{e}. \tag{*}$$

Now define $r : \widetilde{Y} \to \widehat{Y}$ by $r(g\widetilde{x}) := g * \widehat{x}$ and $r(g\widetilde{e}) := g * \widehat{e}$; this is well-defined, since G_x stabilises $\widehat{x}$ and (by the Claim) π_e stabilises $\widehat{e}$. Then r is a graph morphism, since

$$\alpha(r(g\widetilde{e})) = \alpha(g * \widehat{e}) = g * \alpha(\widehat{e}) = g * \widehat{\alpha(e)} = r(g\widetilde{\alpha(e)}) = r(\alpha(g\widetilde{e}))$$

and

$$\omega(r(g\widetilde{e})) = \omega(g * \widehat{e}) = g * \omega(\widehat{e}) = ge * \widehat{\omega(e)} = r(ge\widetilde{\omega(e)}) = r(\omega(g\widetilde{e})).$$

By (*), $q \circ r = \mathrm{id}_{\widetilde{Y}}$, so r is an embedding as required.

□

We can deduce a normal form theorem for the fundamental group.

Corollary 7.34. *Let $(\mathcal{G}, Y)$ be a graph of groups, and let $x \in Y^0$.*

(i) If $(e_0, \ldots, e_{n-1})$ is a path from x to x, and $g_i \in G_{\alpha(e_i)}$ (for $i < n$) and $g_n \in G_x$, and $g_0 \neq 1$ if $n = 0$, and

$$\forall i < n - 1.\ (e_{i+1} = \overline{e}_i \to g_{i+1} \notin {G_{e_i}}^{e_i}),$$

then in $\pi_1(\mathcal{G}, Y, x)$ we have

$$g_0 e_0 g_1 e_1 \ldots g_{n-1} e_{n-1} g_n \neq 1.$$

(ii) For $e \in Y^1$, let S_e be a set of representatives for the right cosets of ${G_e}^e$ in $G_{\omega(e)}$, with $1 \in S_e$. Then every element $h \in \pi_1(\mathcal{G}, Y, x)$ can be written uniquely in the form $h = g e_0 s_0 \ldots e_{n-1} s_{n-1}$ where $(e_0, \ldots, e_{n-1})$ is a path from x to x, $g \in G_x$, $s_i \in S_{e_i}$, and

$$\forall i < n - 1.\ (e_{i+1} = \overline{e}_i \to s_i \neq 1)$$

*(i.e. there is no subword of the form $e_i 1 \overline{e_i}$). We call this the **normal form** for h.*

Proof. (i) In this proof, we have to consider negative edges. For $e \in Y^+$, define $\widetilde{\overline{e}} := \overline{\widetilde{e}}$, so then this equation holds for any $e \in Y^1$.

For $n = 0$, the result follows from the map $G_x \to \pi_1(\mathcal{G}, Y, x)$ being an embedding (Theorem 7.19(ii)). So suppose $n > 0$.

Let $x_i := \alpha(e_i)$ and $x_n := x = x_0$. Let $P_i := g_0 e_0 \ldots g_{i-1} e_{i-1} \widetilde{x}_i$ for $i \leq n$. For $i < n$, let

$$f_i := \begin{cases} g_0 e_0 \ldots g_{i-1} e_{i-1} g_i \widetilde{e_i} & \text{if } e_i \in Y^+ \\ g_0 e_0 \ldots g_{i-1} e_{i-1} g_i e_i \widetilde{e_i} & \text{if } e_i \in Y^- \end{cases}.$$

Then $\alpha(f_i) = P_i$ and $\omega(f_i) = P_{i+1}$: indeed, if $e_i \in Y^+$ then

$$\alpha(f_i) = g_0 e_0 \ldots g_{i-1} e_{i-1} g_i \widetilde{x}_i = P_i$$

and

$$\omega(f_i) = g_0 e_0 \ldots g_{i-1} e_{i-1} g_i e_i \widetilde{x_{i+1}} = P_{i+1},$$

and if $e_i \in Y^-$ then

$$\begin{aligned}\alpha(f_i) &= \omega(\overline{f_i}) \\ &= \omega(g_0 e_0 \dots g_{i-1} e_{i-1} g_i e_i \widetilde{\overline{e_i}}) \\ &= g_0 e_0 \dots g_{i-1} e_{i-1} g_i e_i \overline{e_i} \widetilde{x}_i \\ &= g_0 e_0 \dots g_{i-1} e_{i-1} g_i \widetilde{x}_i \\ &= P_i\end{aligned}$$

and

$$\begin{aligned}\omega(f_i) &= \alpha(\overline{f_i}) \\ &= \alpha(g_0 e_0 \dots g_{i-1} e_{i-1} g_i e_i \widetilde{\overline{e_i}}) \\ &= g_0 e_0 \dots g_{i-1} e_{i-1} g_i e_i \widetilde{x_{i+1}} \\ &= P_{i+1}.\end{aligned}$$

Suppose $(f_0, \dots, f_{n-1})$ is not reduced, say $f_{i+1} = \overline{f_i}$. Then $e_{i+1} = \overline{e_i}$. Suppose $e_i \in Y^+$, so $e_{i+1} \in Y^-$. Then from $f_{i+1} = \overline{f_i}$ we obtain

$$e_i g_{i+1} e_{i+1} \widetilde{e_{i+1}} = \overline{\widetilde{e_i}} = \widetilde{e_{i+1}},$$

so $e_i g_{i+1} e_{i+1}$ stabilises $\widetilde{e_{i+1}}$ and hence $\widetilde{e_i}$, so $e_i g_{i+1} e_{i+1} \in \pi_{e_i} = {G_{e_i}}^{\overline{e_i}}$, so $g_{i+1} \in \overline{e_i} {G_{e_i}}^{\overline{e_i}} e_i = {G_{e_i}}^{e_i}$, contrary to assumption.

So $e_i \in Y^-$. But then $e_i g_{i+1} \widetilde{e_{i+1}} = e_i \overline{\widetilde{e_i}} = e_i \widetilde{e_{i+1}}$, so $g_{i+1} \in \pi_{e_{i+1}} = {G_{e_i}}^{e_i}$, again contrary to assumption.

So $(f_0, \dots, f_{n-1})$ is a non-trivial reduced path from P_0 to P_n, so since $\widetilde{Y}$ is a tree,

$$g_0 e_0 \dots g_{n-1} e_{n-1} g_n \widetilde{x} = P_n \neq P_0 = \widetilde{x},$$

so $g_0 e_0 \dots g_{n-1} e_{n-1} g_n \neq 1$.

(ii) First we observe that every $h \in \pi_1(\mathcal{G}, Y, x)$ can be written in normal form. Indeed, say $h = g_0 e_0 g_1 \dots e_{n-1} g_n$. Write $g_n = g' s_{n-1}$ where $s_{n-1} \in S_{e_{n-1}}$ and $g' \in {G_{e_{n-1}}}^{e_{n-1}}$. Let $g'' := e_{n-1} g' e_{n-1}^{-1} \in {G_{e_{n-1}}}^{\overline{e_{n-1}}} \leq G_{\alpha(e_{n-1})}$, and let $g'_{n-1} := g_{n-1} g'' \in G_{\alpha(e_{n-1})}$. Then $h = g_0 e_0 g_1 \dots g'_{n-1} e_{n-1} s_{n-1}$. Continuing in this way, we obtain an expression $h = g_0 e_0 s_0 \dots e_{n-1} s_{n-1}$. Iteratively deleting any subwords $e_i 1 \overline{e_i}$, we obtain a normal form.

For the uniqueness, define a "partial normal form" to be an expression $g e_0 s_0 \dots e_{n-1} s_{n-1}$ which can be completed to a normal form $g e_0 s_0 \dots e_{n-1} s_{n-1} \dots e_{n'-1} s_{n'-1}$ for some $n' \geq n$. In other words, it satisfies all the conditions of a normal form, except that the path is not required to be to x.

Suppose $g e_0 s_0 \dots e_{n-1} s_{n-1} = g' e'_0 s'_0 \dots e_{m-1} s_{m-1}$ are both partial normal forms, with equality for the product computed in $F(\mathcal{G}, Y)$. We show that the forms are the same (i.e. $n = m$, $g = g'$, $g_i = g'_i$, $s_i = s'_i$) by induction on $n + m$.

First suppose $n, m > 0$. Then $g e_0 s_0 \dots e_{n-1} s_{n-1} {s'_{m-1}}^{-1} \overline{e'_{n-1}} \dots {s'_0}^{-1} \overline{e'_0} {g'}^{-1} = 1$, so by (i) we must have $e_{n-1} = \overline{\overline{e'_{m-1}}} = e'_{m-1}$ and $s_{n-1} {s'_{m-1}}^{-1} \in {G_{e_{n-1}}}^{e_{n-1}}$. Since $s_{n-1}, s'_{m-1} \in S_{e_{n-1}}$, this implies $s_{n-1} = s'_{m-1}$. We then conclude by the inductive hypothesis.

Now suppose $m = 0$ or $n = 0$; by symmetry, we can assume $m = 0$. Then $ge_0s_0 \dots e_{n-1}s_{n-1}g'^{-1} = 1$, which contradicts (i) unless also $n = 0$ and $g = g'$, as required.

□

Remark 7.35. Considering the cases where Y is a segment or a loop, we recover the normal form theorem for amalgamated free products (Theorem 5.19) and Britton's Lemma (Fact 7.17) on normal forms for HNN-extensions.

7.8 Structure theorem

The universal cover construction associates to a graph of groups a non-inversive action of its fundamental group on a tree. Now we obtain a converse, showing that any non-inversive action on a tree is of this form.

Let $G \circlearrowleft X$ be a non-inversive action of a group on a connected non-empty graph. Let $Y := {}_G\backslash^X$, and let $p : X \to Y$ be the quotient map. Let $Y^+ \subseteq Y^1$ be an orientation. Let $T \subseteq Y$ be a maximal (oriented) subtree, and let $\widehat{T} \subseteq X$ be a lift. (So $\widehat{T}$ is a tree of representatives.) Define lifts $\widehat{x}$ and $\widehat{e}$ of the vertices and positive edges of Y as follows:

- For $x \in T^0 = Y^0$, let $\widehat{x} \in \widehat{T}^0$ be the unique element with $p(\widehat{x}) = x$.
- For $e \in T^+$, let $\widehat{e} \in \widehat{T}^1$ be the unique element with $p(\widehat{e}) = e$.
- For $e \in Y^+ \setminus T^+$, arbitrarily choose $\widehat{e} \in X^1$ such that $p(\widehat{e}) = e$ and $\alpha(\widehat{e}) = \widehat{\alpha(e)}$.

Then for $e \in Y^+$, we have $p(\omega(\widehat{e})) = \omega(p(\widehat{e})) = \omega(e) = p(\widehat{\omega(e)})$, so say $\gamma_e \in G$ is such that

$$\omega(\widehat{e}) = \gamma_e \widehat{\omega(e)},$$

and $\gamma_e = 1$ if $e \in T^+$.

Definition 7.36. The **quotient graph of groups** of $G \circlearrowleft X$ (with respect to the above choices) is the oriented graph of groups ${}_G\backslash\!\!\backslash^X = (\mathcal{G}, Y)$ with $G_x := G_{\widehat{x}}$ and $G_e := G_{\widehat{e}}$ for $x \in Y^0$ and $e \in Y^+$, where the right hand sides denote the stabilisers, and with embeddings $a \mapsto a^e := \gamma_e^{-1} a \gamma_e$ and $a \mapsto a^{\overline{e}} := a$ for $e \in Y^+$ and $a \in G_e$.

(These embeddings are into the right groups, since

$$a^{\overline{e}} = a \in G_{\alpha(\widehat{e})} = G_{\widehat{\alpha(e)}} = G_{\alpha(e)} = G_{\omega(\overline{e})}$$

and

$$a^e = \gamma_e^{-1} a \gamma_e \in \gamma_e^{-1} G_{\omega(\widehat{e})} \gamma_e = G_{\gamma_e^{-1}\omega(\widehat{e})} = G_{\widehat{\omega(e)}} = G_{\omega(e)}.)$$

Remark 7.37. In the case that ${}_G\backslash^X$ is a tree, the vertex and edge groups of ${}_G\backslash\!\!\backslash^X$ are just the vertex and edge stabilisers of a tree of representatives, with the inclusions as the embeddings.

Let $(\mathcal{G}, Y) = {}_G\backslash\!\!\backslash^X$ and $\pi := \pi_1(\mathcal{G}, Y, T)$.

The inclusions $G_x \hookrightarrow G$ and maps $e \mapsto \gamma_e$ induce a homomorphism

$$\phi : \pi \to G;$$

indeed, $\gamma_e = 1$ if $e \in T^+$, and $\gamma_e a^e \gamma_e^{-1} = a^{\overline{e}}$.

Let $\widetilde{Y}$ be the universal cover of $(\mathcal{G}, Y)$ with respect to T. Define

$$\psi : \widetilde{Y} \to X$$
$$\psi(g\widetilde{x}) := \phi(g)\widehat{x}$$
$$\psi(g\widetilde{e}) := \phi(g)\widehat{e}.$$

This is well-defined, since $\phi(\text{stab}(\widetilde{x})) = \phi(G_x) = G_x = G_{\widehat{x}}$ and $\phi(\text{stab}(\widetilde{e})) = \phi(\pi_e) = \phi(G_e^{\overline{e}}) = G_e^{\overline{e}} = G_e = G_{\widehat{e}}$ (using that ϕ is the identity on the vertex groups).

ψ is a graph morphism, since

$$\psi(\alpha(g\widetilde{e})) = \psi(g\widetilde{\alpha(e)}) = \phi(g)\widehat{\alpha(e)} = \phi(g)\alpha(\widehat{e}) = \alpha(\phi(g)\widehat{e}) = \alpha(\psi(g\widetilde{e})),$$
$$\psi(\omega(g\widetilde{e})) = \psi(ge\widetilde{\omega(e)} = \phi(ge)\widehat{\omega(e)} = \phi(g)\gamma_e\widehat{\omega(e)} = \phi(g)\omega(\widehat{e}) = \omega(\phi(g)\omega(\widehat{e})) = \omega(\psi(g\widetilde{e})).$$

Lemma 7.38. *$\phi : \pi \to G$ and $\psi : \widetilde{Y} \to X$ are surjections.*

Proof. Let $H := \phi(\pi) \leq G$. Let $\widehat{Y} \subseteq X$ be the smallest subgraph with $\widehat{Y}^+ = \{\widehat{e} : e \in Y^+\}$. Then $G\widehat{Y} = X$, and $\widehat{Y} \subseteq \psi(\widetilde{Y})$. Also $\widehat{Y}^0 \subseteq H\widehat{T}^0$, since for $e \in Y^+ \setminus T^+$ we have $\omega(\widehat{e}) = \gamma_e\widehat{\omega(e)} \in H\widehat{T}^0$. Since also $\widehat{T}^0 \subseteq \widehat{Y}^0$, we have $H\widehat{Y}^0 = H\widehat{T}^0$.

Claim 7.39. If $g \in G$ and $x, gx \in \widehat{T}^0$, then $g \in H$.

Proof. Since $Gx \cap \widehat{T}^0 = \{x\}$, we have $gx = x$, so $g \in G_x \leq H$. $\square_{7.39}$

Claim 7.40. $H\widehat{Y} = X$.

Proof. Since X is connected, it suffices to show that if $f \in X^1$ and $\alpha(f) \in H\widehat{Y}^0 = H\widehat{T}^0$, then $f \in H\widehat{Y}$. Let f be such. Translating by an element of H, we may assume $\alpha(f) \in \widehat{T}^0$. Say $f = g\widehat{e}$ or $\overline{f} = g\widehat{e}$, and we conclude by showing $g \in H$.

Suppose first $f = g\widehat{e}$, so $\alpha(g\widehat{e}) \in \widehat{T}^0$. Then $g\widehat{\alpha(e)} = \alpha(g\widehat{e}) \in \widehat{T}^0$, so $g \in G_{\widehat{\alpha(e)}} \leq H$ by Claim 7.39.

Otherwise, $\overline{f} = g\widehat{e}$, so $\omega(g\widehat{e}) \in \widehat{T}^0$. Then $g\gamma_e\widehat{\omega(e)} = \omega(g\widehat{e}) \in \widehat{T}^0$ so $g\gamma_e \in G_{\widehat{\omega(e)}} \leq H$ by Claim 7.39, so $g \in H$. $\square$

So $H\widehat{T}^0 = H\widehat{Y}^0 = X^0$.

So if $g \in G$ and $x \in \widehat{T}^0$, then $hgx \in \widehat{T}^0$ for some $h \in H$, so $hg \in H$ by Claim 7.39, so $g \in H$.

Hence $H = G$, and ϕ is surjective. Then also ψ is surjective, since $Y = {}_G\backslash X$. $\square$

Theorem 7.41. *Suppose X is a tree. Then $\phi : \pi \to G$ and $\psi : \widetilde{Y} \to X$ are isomorphisms.*

Proof. Given the previous lemma, it remains to show that ϕ and ψ are injective.

Claim 7.42. ψ is locally injective.

Proof. Suppose $\psi(g\widetilde{e}) = \psi(g'\widetilde{e}')$, i.e. $\phi(g)\widehat{e} = \phi(g')\widehat{e}'$. Then $\widehat{e} = \widehat{e}'$, so $e = e'$, and $\phi(g^{-1}g') \in G_e$.

We must show that if either $\alpha(g\widetilde{e}) = \alpha(g'\widetilde{e})$ or $\omega(g\widetilde{e}) = \omega(g'\widetilde{e})$, then $g\widetilde{e} = g'\widetilde{e}'$, i.e. $g^{-1}g' \in \mathrm{stab}(\widetilde{e}) = \pi_e = G_e^{\overline{e}} = G_e$.

First suppose $\alpha(g\widetilde{e}) = \alpha(g'\widetilde{e})$. Then $g^{-1}g' \in \mathrm{stab}(\alpha(\widetilde{e})) = \mathrm{stab}(\widetilde{\alpha(e)}) = G_{\alpha(e)}$. Then since $\phi|_{G_{\alpha(e)}} = \mathrm{id}$, we have $g^{-1}g' = \phi(g^{-1}g') \in G_e$ as required.

Next suppose $\omega(g\widetilde{e}) = \omega(g'\widetilde{e})$. Then $g^{-1}g' \in \mathrm{stab}(\omega(\widetilde{e})) = \mathrm{stab}(e\widetilde{\omega(e)}) = eG_{\omega(e)}e^{-1}$. But ϕ is injective on $G_{\omega(e)}$ and hence also on $eG_{\omega(e)}e^{-1}$, which contains $eG_e^e e^{-1} = G_e^{\overline{e}} = G_e \ni \phi(g^{-1}g')$. So ϕ is injective on a set containing both $g^{-1}g'$ and $\phi(g^{-1}g')$, but $\phi(\phi(g^{-1}g')) = \phi(g^{-1}g')$ since $\phi|_{G_{\alpha(e)}} = \mathrm{id}$, so we deduce $g^{-1}g' = \phi(g^{-1}g') \in G_e$ as required. $\square_{7.42}$

Since X is a tree, injectivity of ψ follows by Lemma 2.25.

Finally, suppose $g \in \ker(\phi) \setminus 1$. Let $x \in Y^0$. Then $g \notin G_x \leq \pi$, since ϕ is an embedding on G_x by definition, so $g\widetilde{x} \neq \widetilde{x}$. But $\psi(g\widetilde{x}) = \phi(g)\widehat{x} = \widehat{x} = \psi(\widetilde{x})$, contradicting injectivity of ψ. $\square$

Remark 7.43. In particular, the choices in the definition of ${}_G\backslash\backslash^X$ do not affect the isomorphism type of its fundamental group, nor the isomorphism type of its universal cover.

Remark 7.44. With a little more argument, one can strengthen Theorem 7.41 to: X is a tree $\Leftrightarrow$ ϕ is surjective $\Leftrightarrow$ ψ is surjective.

Combining Theorem 7.41 with Theorem 7.32, we obtain what we might call the Fundamental Theorem of Bass-Serre Theory:

Corollary 7.45. *The natural action of the fundamental group of a graph of groups on its universal cover is a non-inversive action of a group of a tree, and conversely every non-inversive action $G \circlearrowleft X$ of a group on a tree is isomorphic to the action of the fundamental group of ${}_G\backslash\backslash^X$ on its universal cover; in particular,*

$$G \cong \pi_1({}_G\backslash\backslash^X).$$

Remark 7.46. Applying this in the case that ${}_G\backslash^X$ is a segment, we recover Theorems 6.2 and 6.5.

Applying it in the case that G acts freely on a tree, we recover Theorem 3.22.

Specialising to the case that ${}_G\backslash^X \cong C_1$, we obtain:

Corollary 7.47. *Let $H := \mathrm{HNN}(G, A, \phi_1, \phi_2) \geq G$. Then the following graph $\widetilde{Y}$ is a tree, and ${}_H\backslash^{\widetilde{Y}} \cong C_1$ for the natural action of H.*

$$\begin{aligned} \widetilde{Y}^0 &= H/G \\ \widetilde{Y}^+ &= H/\phi_1(A) \\ \alpha(h\phi_1(A)) &= hG \\ \omega(h\phi_1(A)) &= htG. \end{aligned}$$

Conversely, if $H \circlearrowleft X$ is a non-inversive action of a group on a tree with ${}_H\backslash^X \cong C_1$, then $H \cong \mathrm{HNN}(G, A, \phi_1, \phi_2)$ where: $G = H_x$ where $x \in X^0$ is arbitrary, $A = H_e \leq G$ where $e \in X^1$ is arbitrary such that $\alpha(e) = x$, and $\phi_1(a) = \gamma_e^{-1} a \gamma_e$ where $\gamma_e \in H$ is arbitrary such that $\omega(e) = \gamma_e x$, and $\phi_2 = \mathrm{id}$.

Note $\phi_2(a) = \gamma_e \phi_1(a) \gamma_e^{-1}$ for $a \in A$, so we can identify γ_e with the stable letter of the HNN extension.

Remark 7.48. The universal cover of a graph of groups is also known as its **Bass-Serre tree**.

Corollary 7.49. *Let $G \circlearrowleft X$ be a non-inversive action of a group on a tree. Then $\pi_1({}_G\backslash X) \cong G/\left\langle G_x : x \in X^0 \right\rangle$.*

Proof. By Corollary 7.45, we may identify G with $\pi_1({}_G\backslash\backslash X)$. Then $\{G_x : x \in X^0\}$ is the set of conjugates in G of the vertex groups of ${}_G\backslash\backslash X$. So setting $Y := {}_G\backslash X$, taking an orientation Y^+ and a maximal oriented subtree T,

$$\begin{aligned} G/\langle G_x : x \in X_0 \rangle &= \pi_1({}_G\backslash\backslash X)/\langle\langle\{G_y : y \in Y^0\}\rangle\rangle \\ &\cong \left\langle (G_y)_{y\in Y^0}, Y^+ \;\middle|\; \{ea^e e^{-1} = a^{\overline{e}} : e \in Y^+,\ a \in G_e\}, T^+, (G_y)_{y\in Y^0} \right\rangle \\ &\cong \left\langle Y^+ \;\middle|\; T^+ \right\rangle \\ &\cong F(Y^+ \setminus T^+) \end{aligned}$$

which is isomorphic to $\pi_1(Y)$ by Remark 4.18. □

7.9 Subgroups of free products

Lemma 7.50. *Let $(\mathcal{G}, X)$ be a graph of groups where each edge group is trivial. Then $\pi_1(\mathcal{G}, X) \cong \pi_1(X) * \ast_{x\in X^0} G_x$.*

Proof. Let $X^+ \subseteq X$ be an orientation, and $T \subseteq X$ a maximal oriented subtree. Then by Lemma 7.31,

$$\pi_1(\mathcal{G}, X, T) = \left\langle (G_x)_{x\in X^0}, X^+ \;\middle|\; T^+ \right\rangle \cong F(X^+ \setminus T^+) * \mathop{\ast}_{x\in X^0} G_x,$$

and $F(X^+ \setminus T^+) \cong \pi_1(X)$ by Remark 4.18. □

Lemma 7.51. *Let $H, K \leq G$ be subgroups of a group G, and let $X \subseteq G$ be a subset. The following are equivalent:*

(i) $X \subseteq G$ is a ***system of representatives for the double cosets*** *${}_H\backslash G/_K$, i.e. each double coset $HgK = \{hgk : h \in H,\ k \in K\} \subseteq G$ contains exactly one element of X.*

(ii) The quotient map induces a bijection $X \to X/K \subseteq G/K$, and X/K contains exactly one element of each orbit of the action of H on G/K by left multiplication.

Proof. The union of the cosets in the orbit under H of a coset gK is precisely the double coset HgK.

So: (i) ⇔ each H-orbit of G/K has union containing exactly one element of X ⇔ $X \to X/K$ is a bijection and each H-orbit of G/K contains exactly one element of X/K ⇔ (ii). □

Theorem 7.52 (Kurosh 1934)**.** *Let H be a subgroup of a free product $G = G_1 * G_2$.*

Then there exists a free group F and systems of representatives $X_i \subseteq G$ for the double cosets ${}_H\backslash G/_{G_i}$ such that

$$H \cong F * (\mathop{\ast}_{x\in X_1} H \cap xG_1x^{-1}) * (\mathop{\ast}_{x\in X_2} H \cap xG_2x^{-1}).$$

Proof. Let $(\mathcal{G}, Y)$ be the graph of groups where Y is a segment, the edge group is trivial, and the vertex groups are G_1 and G_2 respectively. Let $\widetilde{Y}$ be its universal cover, i.e. the tree of Theorem 6.5. Then the action of $\pi_1(\mathcal{G}, Y) = G$ on $\widetilde{Y}$ induces a non-inversive action of H, and $H \cong \pi_1({}_H\backslash\!\backslash^{\widetilde{Y}})$ by Corollary 7.45.

Now the edges of $\widetilde{Y}$ have trivial stabiliser under the action of G, since the edge group of $(\mathcal{G}, Y)$ is trivial. Hence the edge groups of ${}_H\backslash\!\backslash^{\widetilde{Y}}$ are trivial. So by Lemma 7.50,

$$H \cong \pi_1({}_H\backslash\!\backslash^{\widetilde{Y}}) \cong \pi_1({}_H\backslash^{\widetilde{Y}}) * \mathop{\Large *}_{z \in \left({}_H\backslash^{\widetilde{Y}}\right)^0} H_z,$$

where H_z is the vertex group in ${}_H\backslash\!\backslash^{\widetilde{Y}}$; by definition, $H_z = \mathrm{stab}^H(\widehat{z})$ where $\widehat{z} \in (\widetilde{Y})^0$ is a lift.

Now $(\widetilde{Y})^0 = G/G_1 \dot\cup G/G_2$, so each $\widehat{z}$ is of the form xG_i, and then

$$H_z = \mathrm{stab}^H(\widehat{z}) = H \cap \mathrm{stab}^G(\widehat{z}) = H \cap \mathrm{stab}^G(xG_i) = H \cap x\mathrm{stab}^G(G_i)x^{-1} = H \cap xG_ix^{-1}.$$

Finally, $\{\widehat{z} : z \in ({}_H\backslash^{\widetilde{Y}})^0\} = X_1/G_1 \dot\cup X_2/G_2$ where X_i/G_i contains one element in each orbit of the left action of H on G/G_i, and $X_i \to X_i/G_i$ is a bijection. By Lemma 7.51, X_i is a system of representatives for the double cosets ${}_H\backslash^G/_{G_i}$, as required. □

Remark 7.53. Essentially the same proof yields the following generalisation to amalgamated products: if $H \le G = G_1 *_A G_2$ and $H \cap gAg^{-1} = 1$ for each $g \in G$, then H has the same form as in Theorem 7.52.

8 Amalgams and fixed points

8.1 FA groups

Definition 8.1. Let $G \circlearrowleft X$ be an action of a group on a graph. Then X^G is the subgraph of X fixed by all elements of G, $(X^G)^i := \{x \in X^i : \forall g \in G.\ g * x = x\}$ $(i = 0, 1)$.

Remark 8.2. If X is a tree and X^G is non-empty, then X^G is a tree. Indeed, if $x, y \in (X^G)^0$ then the geodesic from x to y is contained in X^G.

Definition 8.3. Let G be a group.

- G is **FA** if for any non-inversive action of G on a tree X, X^G is non-empty.
- G is a **non-trivial amalgam** if $G = G_1 *_A G_2$ with $A \le G_1, G_2 \le G$ and $G_1, G_2 \ne G$.

Lemma 8.4. *Let T be a finite tree.*

(i) Suppose $|T^0| > 1$. Then there exists a subtree $T' \subseteq T$, $e \in T^1$, and $y \in T^0$, such that $T = T' \dot\cup \{e, \overline{e}, y\}$. (Such a y is called a "terminal vertex" of T.)

(ii) If $T' \subseteq T$ is a subtree, then there exists a chain of subtrees $T' = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n = T$ such that each T_{i+1} is of the form $T_i \dot\cup \{e_i, \overline{e}_i, y_i\}$.

Proof. Exercise. For (i), consider a geodesic of maximal length. For (ii), consider a geodesic from a vertex of T' to a vertex outside T'. □

Theorem 8.5. *Let G be a countable group. Then G is FA if and only if it satisfies the following three conditions:*

(i) G is not a non-trivial amalgam.

(ii) No quotient of G is isomorphic to $\mathbb{Z}$.

(iii) G is finitely generated.

Proof.

$\Rightarrow$ Suppose G is FA. We show (i)-(iii).

(i) Suppose $G \cong G_1 *_A G_2$, and let X be the corresponding Bass-Serre tree (as in Theorem 6.5). By FA, some vertex of X has stabiliser G. But the stabiliser of any vertex is a conjugate of either G_1 or G_2, so $G = G_1$ or $G = G_2$.

(ii) Suppose $\theta : G \twoheadrightarrow \mathbb{Z}$. Define an action of G on C_∞ by $g * n := \theta(g) + n$. Then $\mathbb{Z}^G = \emptyset$, contradicting FA.

(iii) Since G is countable, it is the union of a chain $G_0 \subseteq G_1 \subseteq \ldots$ of subgroups (indeed, if $G = \{g_i : i \in \omega\}$, we can take $G_i := \langle \{g_j : j \leq i\} \rangle \leq G$).
Let X be the graph:

$$\begin{aligned} X^0 &:= \dot{\bigcup_i} G/G_i \\ X^+ &:= \dot{\bigcup_i} G/G_i \\ \alpha(gG_i) &:= gG_i \\ \omega(gG_i) &:= gG_{i+1} \end{aligned}$$

Then X is a tree. Indeed, if C is a circuit in X, then say $c = gG_i \in C^0$ with i minimal, then the two edges from c in C must be in X^+ and so must both be equal to gG_i, contradicting the definition of a circuit. So X is acyclic by Lemma 6.4. It is connected since given two vertices gG_i and $g'G_{i'}$, there is $j > i, i'$ such that $g, g' \in G_j$, and then each of gG_i and $g'G_{i'}$ is connected by a path to $G_j \in G/G_j$.
Let $G \circlearrowleft X$ be the obvious left action. By FA, say $gG_i \in X^G$. Then $G_i = G$, so G is finitely generated.

$\Leftarrow$ Suppose G satisfies (i)-(iii) and acts non-inversively on a tree X, and suppose $X^G = \emptyset$.

By Corollary 7.49, $F := \pi_1({}_G\backslash^X)$ is a quotient of G, so by (ii) F has no quotient isomorphic to $\mathbb{Z}$. But F is free, so this implies F is trivial. So $T := {}_G\backslash^X$ is a tree.

Let $(\mathcal{G}, T) := {}_G\backslash\backslash^X$. For $T' \subseteq T$ a subtree, let $G_{T'} := \pi_1(\mathcal{G}|_{T'}, T')$ be the fundamental group of the subtree of groups, and identify G with G_T.

Claim 8.6.

(i) Let $T' \subseteq T$ be a finite subtree. The natural homomorphism $G_{T'} \to G_T$ is an inclusion $G_{T'} \leq G_T$.

(ii) $G = G_T = \bigcup_{T' \subseteq T \text{ finite subtree}} G_{T'}$.

Proof.

(i) It suffices to see this in the case that T is finite, since if $x \in G_{T'}$ satisfies a relation in $G_T = \langle X \mid R \rangle$, i.e. $x \in \left\langle R^{G_T} \right\rangle$, then $x \in \left\langle R_0^{G_0} \right\rangle$ for some finite R_0, G_0, which already appear in the presentation of $G_{T''}$ for some finite T''.
So by induction and Lemma 8.4(ii), it suffices to consider the case $T = T' \dot\cup \{e, \bar{e}, y\}$. But then $G_T = G_{T'} *_{G_e} G_y$, and the result follows from Theorem 5.19(i) (or Theorem 7.19).

(ii) This follows from (i), since G_T is certainly generated by the $G_{T'}$ since they contain all the vertex groups.

$\square_{8.6}$

So since G is finitely generated by (iii), there is a minimal finite subtree $T' \subseteq T$ such that $G = G_{T'}$.
If $|(T')^0| = 1$, then $G = G_y$ where $y \in (T')^0$, contradicting $X^G = \emptyset$.
So $|(T')^0| > 1$, and then by Lemma 8.4(i), $T' = T'' \dot\cup \{e, \bar{e}, y\}$ for some subtree $T'' \subseteq T'$, and then $G = G_{T'} = G_{T''} *_{G_e} G_y$. But $G \neq G_{T''}$ by the minimality of T', and $G \neq G_y$ since $X^G = \emptyset$, contradicting (i).

$\square$

Lemma 8.7. *Let H be FA, and suppose $H \leq G_1 *_A G_2 =: G$ (with $A \leq G_1, G_2 \leq G$). Then H is contained in some conjugate in G of G_1 or of G_2.*

Proof. The induced action of H on the Bass-Serre tree of $G = G_1 *_A G_2$ has a fixed point, i.e. H is contained in the stabiliser for the action of G of that point, which is a conjugate of G_1 or of G_2. $\square$

8.2 Automorphisms of trees

Lemma 8.8. *Let X be a tree, let $x \in X$, and let $T \subseteq X$ be a subtree. Then there is a unique path of minimal length from x to a vertex of T.*
*This path is called the **geodesic from** x **to** T.*

Proof. Exercise. $\square$

Let σ be a non-inversive automorphism of a tree X.

Definition 8.9.

- A **fixed point** of σ is an $x \in X^0$ with $\sigma(x) = x$.
- Define $X^\sigma := X^{\langle \sigma \rangle}$.

Remark 8.10. σ has a fixed point if and only if X^σ is non-empty, in which case it is a tree by Remark 8.2.

Definition 8.11.

- A **straight path** in a graph X is a subgraph isomorphic to C_∞.
- A **translation** by $m \in \mathbb{N}$ on a straight path T is an automorphism which is induced via an isomorphism $T \cong C_\infty$ by the automorphism $x \mapsto x+m$ of C_∞. It is **non-trivial** if $m \neq 0$.

Notation 8.12. If $p = (e_0, \ldots, e_{n-1})$ and $q = (f_0, \ldots, f_{m-1})$ are paths in a graph with $\omega(e_{n-1}) = \alpha(f_0)$, their **concatenation** is the path

$$p^\frown q := (e_0, \ldots, e_{n-1}, f_0, \ldots, f_{m-1}).$$

Lemma 8.13. *Suppose σ has a fixed point.*

(i) Let $x \in X^0$. Let $p = (e_0, \ldots, e_{n-1})$ be the geodesic from x to X^σ. Then the geodesic from x to $\sigma(x)$ is $p^\frown \overline{\sigma(p)} = (e_0, \ldots, e_{n-1}, \overline{\sigma(e_{n-1})}, \ldots, \overline{\sigma(e_0)})$.

(ii) σ does not act by non-trivial translation on any straight path in X.

Proof.

(i) If $n = 0$, then $x = \sigma(x)$ and the result is immediate. So suppose $n > 0$.

Suppose $\sigma(e_{n-1}) = \overline{e_{n-1}}$. Then $\omega(e_{n-2}) = \alpha(e_{n-1}) \in X^\sigma$, contradicting the minimality of p. So $p^\frown \overline{\sigma(p)}$ is reduced, so is the geodesic as required.

(ii) Suppose T is a straight path on which σ acts by translation by $m \neq 0$. Let $x \in T^0$. Then $\sigma(x) \in T^0$, so the geodesic from x to $\sigma(x)$ lies within T. But by (i), the geodesic passes through a vertex of X^σ. Hence $T^\sigma \neq \emptyset$, contradicting $m \neq 0$.

□

Lemma 8.14. *Suppose σ has no fixed point.*

Then there is a unique straight path $T \subseteq X$ on which σ acts by a non-trivial translation.

Proof.

- Existence: Let $m := \min_{x \in X^0} d(x, \sigma(x))$. Since σ has no fixed point, $m > 0$. Let $x \in X^0$ with $d(x, \sigma(x)) = m$. Let $p = (e_0, \ldots, e_{m-1})$ be the geodesic from x to $\sigma(x)$.

  Claim 8.15. $p^\frown \sigma(p)$ is a reduced path from x to $\sigma^2(x)$.

  Proof. Else, $e_{m-1} = \overline{\sigma(e_0)}$. Then $m \neq 1$ since σ acts non-inversively, so $m > 1$, and $\sigma(\omega(e_0)) = \alpha(e_{m-1})$, so $d(\omega(e_0), \sigma(\omega(e_0))) = m - 2 < m$, contradicting the minimality of m. $\square_{8.15}$

  Hence $\sigma^n(p)^\frown \sigma^{n+1}(p)$ is reduced for any $n \in \mathbb{Z}$, and it follows that

  $$\ldots^\frown \sigma^{-1}(p)^\frown p^\frown \sigma(p)^\frown \sigma^2(p)^\frown \ldots$$

  forms a straight path T on which σ acts as translation by m.

- Uniqueness: Suppose $T, T' \subseteq X$ are straight paths on which σ acts as translation by m, m' respectively, with $m, m' > 0$.

  Let $x \in T^0$, and let $p = (e_0, \dots, e_{n-1})$ be the geodesic from x to T'. Let q be the geodesic from $\omega(e_{n-1}) \in T'$ to $\sigma(\omega(e_{n-1}))$. Then q is a path within T' of length $m' > 0$. Therefore $p^\frown q^\frown \sigma(p)$ is a reduced path from x to $\sigma(x)$, so it is the geodesic from x to $\sigma(x)$. Hence $m = d(x, \sigma(x)) = 2\,\text{length}(p) + m' \geq m'$. Then by symmetry, $m = m'$, and p is trivial, so $x \in (T')^0$.

  So $T \subseteq T'$, and by symmetry, $T = T'$.

□

Putting the previous two lemmas together, we conclude:

Theorem 8.16. *Let σ be a non-inversive automorphism of a tree X. The following are equivalent:*

(i) $X^\sigma = \emptyset$.

(ii) There is a straight path $T \subseteq X$ on which σ acts by a non-trivial translation.

(iii) There is a unique straight path $T \subseteq X$ on which σ acts by a non-trivial translation.

Proof.

- (i) $\Rightarrow$ (iii): Lemma 8.14.
- (iii) $\Rightarrow$ (ii): Immediate.
- $\neg$(i) $\Rightarrow$ $\neg$(ii): Lemma 8.13(ii).

□

8.3 Nilpotent groups acting on trees

Recall that a group G is **nilpotent** if it has a finite central series, i.e. a sequence

$$1 = G_0 \trianglelefteq \dots \trianglelefteq G_n = G$$

with $G_i \trianglelefteq G$ and with each G_{i+1}/G_i central in G/G_i.

We omit the proof of the following fact; it will be obvious for the groups we apply it to in Theorem 8.23.

Fact 8.17. *Let G be a finitely generated nilpotent group.*

Then there exists a sequence $1 = G_0 \trianglelefteq \dots \trianglelefteq G_n = G$ with each G_{i+1}/G_i cyclic.

(A group with this property is called "polycyclic".)

Proof idea. First show that every subgroup of G is finitely generated (this is not so easy). Then proceed by induction on the length of a central series and use that any finitely generated abelian group is a product of cyclic groups.

A full proof can be found in [Rob96, 5.2.18] □

We will also use the easily verified fact that any subgroup and any homomorphic image of a nilpotent group is nilpotent.

Theorem 8.18. *Let G be a finitely generated nilpotent group acting non-inversively on a tree X. Then*

(i) $X^G = \emptyset$ if and only if there exists a unique straight path $T \subseteq X$ such that the action restricts to a non-trivial action of G on T by translations (i.e. $\sigma|_T$ is a translation for any $\sigma \in G$, and is a non-trivial translation for some $\sigma \in G$).

(ii) If $G = \langle g_1, \ldots, g_n \rangle$ and each g_i has a fixed point, then $X^G \neq \emptyset$.

(iii) Any element $g \in G'$ of the commutator subgroup of G has a fixed point.

Proof.

(i)

$\Leftarrow$: Say $\sigma|_T$ is a non-trivial translation. Then σ has no fixed point by Theorem 8.16, so $X^G = \emptyset$.

$\Rightarrow$: First, note that it suffices to show existence of T, since uniqueness follows from the uniqueness in Theorem 8.16; indeed, if T and T' are straight paths on which G acts non-trivially by translations, and if $g \in G$ acts non-trivially on T, then $X^g = \emptyset$ by Theorem 8.16(ii)⇒(i), so g also acts non-trivially on T', so $T = T'$ by Theorem 8.16(ii)⇒(iii).
By Fact 8.17, we have a sequence $1 = G_0 \trianglelefteq \ldots \trianglelefteq G_n = G$ with G_{i+1}/G_i cyclic. We have $n > 0$ since $X^G = \emptyset$. We prove the result by induction on n for any nilpotent group admitting such a sequence. So inductively, we may assume the result for $H := G_{n-1}$ (which is nilpotent, since it is a subgroup of the nilpotent group G).
First suppose $X^H \neq \emptyset$. Say $G/H = \langle \sigma H \rangle$. Then the action of σ restricts to an action on X^H: indeed, if $x \in X^H$ and $h \in H$, then $h\sigma x = \sigma h^\sigma x = \sigma x$ since $H \trianglelefteq G$, and similarly for σ^{-1}. So σ acts non-inversively on the tree X^H without fixed points (since $X^G = \emptyset$), so by Theorem 8.16, σ acts by non-trivial translations on a straight path $T \subseteq X^H \subseteq X$. Then G acts by translations on T, since if $g \in G$ then $g = \sigma^n h$ for some $n \in \mathbb{Z}$ and $h \in H$, and then for $t \in T$, $gt = \sigma^n ht = \sigma^n t$ since $t \in X^H$. So G acts non-trivially by translations on T, as required.
Finally, suppose $X^H = \emptyset$. By the inductive hypothesis, let $T \subseteq X$ be the unique straight path on which H acts non-trivially by translations.

Claim 8.19. T is G-invariant.

Proof. Let $g \in G$. Then H also acts non-trivially by translations on gT, since the action of $h \in H$ on gT is induced via the isomorphism $g : T \to gT$ by the action of $h^g \in H$ on T (since $hgt = gh^g t$). So $T = gT$ by uniqueness of T. $\square_{8.19}$

Then $\theta(g) := (g*)|_T$ defines a homomorphism $\theta : G \to \mathrm{Aut}(T)$. Identify $\mathrm{Aut}(T) \cong \mathrm{Aut}(C_\infty)$ with the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}$ acts by translations. Now $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is centreless and hence not nilpotent, so is not a homomorphic image of a nilpotent group, and any infinite subgroup of $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is either a subgroup of $\mathbb{Z}$ or is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. So $\theta(G) \leq \mathbb{Z} \leq \mathrm{Aut}(T)$ is a non-trivial group of translations, as required.

(ii) Suppose $X^G = \emptyset$ and let T be as in (i). Then some g_i acts by non-trivial translation on T, contradicting Theorem 8.16.

(iii) If $X^G \neq \emptyset$ then certainly g has a fixed point. Otherwise, let T be as in (i), identify the group of translations of T with $\mathbb{Z}$, and let $\theta : G \to \mathbb{Z}$ be the homomorphism $g \mapsto (g*)|_T$. Then $\theta(G') \leq \mathbb{Z}' = 1$, so G' acts trivially on T.

□

8.4 Intersecting subtrees

Lemma 8.20. *Let T_1, T_2 be subtrees of a tree X. Then there is a unique path p of minimal length amongst the paths from a vertex of T_1 to a vertex of T_2, called the* ***geodesic from*** *T_1* ***to*** *T_2.*

If $T_1 \cap T_2 = \emptyset$, then any subtree $T \subseteq X$ with $T \cap T_i \neq \emptyset$ for $i = 1, 2$ contains p.

Proof. Exercise. □

Lemma 8.21. *Let $X_1, \ldots, X_m$ be subtrees of a tree X. Suppose $X_i \cap X_j \neq \emptyset$ for all i, j. Then $X_1 \cap \ldots \cap X_m \neq \emptyset$.*

Proof. By induction, we have $Y := X_1 \cap \ldots \cap X_{m-1} \neq \emptyset$. Then Y is a subtree (as in the proof of Lemma 2.36). Suppose for a contradiction that $Y \cap X_m = \bigcap_{i \leq m} X_i = \emptyset$. Let p be the geodesic from Y to X_m. For $i < m$, $X_i \cap Y = Y \neq \emptyset \neq X_i \cap X_m$, so p is contained in X_i by Lemma 8.20. Hence p is contained in Y. But p contains a vertex of X_m, contradicting $Y \cap X_m = \emptyset$. □

8.5 $\mathrm{SL}_3(\mathbb{Z})$

For $1 \leq i, j \leq 3$, let $e_{ij} \in M_3(\mathbb{Z})$ be the elementary 3x3 matrix

$$(e_{ij})_{i'j'} = \delta_{(i,j)(i',j')} = \begin{cases} 1 & (i,j) = (i',j') \\ 0 & \text{else} \end{cases}$$

(so $e_{ij}e_{kl} = \delta_{jk}e_{il}$), and define the following elements $z_i \in \mathrm{SL}_3(\mathbb{Z})$ for $i \in \mathbb{Z}/6\mathbb{Z}$:

$$z_0 := 1 + e_{12} \qquad z_1 := 1 + e_{13} \qquad z_2 := 1 + e_{23}$$
$$z_3 := 1 + e_{21} \qquad z_4 := 1 + e_{31} \qquad z_5 := 1 + e_{32}.$$

Fact 8.22. $\mathrm{SL}_3(\mathbb{Z}) = \langle z_0, \ldots, z_5 \rangle$.

Theorem 8.23. $\mathrm{SL}_3(\mathbb{Z})$ *is FA.*

Proof. Let $i \in \mathbb{Z}/6\mathbb{Z}$ and let $B_i := \langle z_{i-1}, z_{i+1} \rangle \leq \mathrm{SL}_3(\mathbb{Z})$. Note $[z_{i-1}, z_{i+1}] \in \{z_i, z_i^{-1}\}$ and $[z_i, z_{i+1}] = 1$; e.g.

$$\begin{aligned}
[z_0, z_2] =& (1 - e_{12})(1 - e_{23})(1 + e_{12})(1 + e_{23}) = \\
& (1 - e_{12} - e_{23} + e_{13})(1 + e_{12} + e_{23} + e_{13}) = 1 - e_{13} + 2e_{13} = 1 + e_{13} = z_1 \\
[z_1, z_3] =& (1 - e_{13})(1 - e_{21})(1 + e_{13})(1 + e_{21}) = \\
& (1 - e_{13} - e_{21})(1 + e_{13} + e_{21}) = 1 - e_{23} = z_2^{-1} \\
[z_1, z_2] =& (1 - e_{13})(1 - e_{23})(1 + e_{13})(1 + e_{23}) = 1.
\end{aligned}$$

So B_i is nilpotent with central series $1 \trianglelefteq (B_i)' = Z(B_i) = \langle z_i \rangle \trianglelefteq B_i$. (Note that B_i is easily seen to be polycyclic, as a special case of Fact 8.17.) (B_i is isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z})$.)

Consider a non-inversive action of $\mathrm{SL}_3(\mathbb{Z})$ on a tree X.

By Theorem 8.18(iii) applied to each B_i, each z_i has a fixed point. So each B_i is generated by elements with fixed points, so $X^{B_i} \neq \emptyset$ by Theorem 8.18(ii).

Now the subtrees $X^{z_1}, X^{z_3}, X^{z_5} \subseteq X$ have non-trivial pairwise intersections, since $X^{z_1} \cap X^{z_3} \supseteq X^{B_2} \neq \emptyset$ and similarly for the other pairs. So $Y := X^{z_1} \cap X^{z_3} \cap X^{z_5} \neq \emptyset$ by Lemma 8.21. But $\langle z_1, z_3, z_5 \rangle = \langle z_0, z_1, z_2, z_3, z_4, z_5 \rangle = \mathrm{SL}_3(\mathbb{Z})$ by considering commutators, so $X^{\mathrm{SL}_3(\mathbb{Z})} = Y \neq \emptyset$ as required. □

Applying Theorem 8.5, we deduce:

Corollary 8.24. $\mathrm{SL}_3(\mathbb{Z})$ *is not a non-trivial amalgam.*

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