Dr. M. Bays
Dr. T. Zou

Universität Münster
Wintersemester 2022/23

Geometric Group Theory I
Exercise Sheet 2

Exercise 1.

a) Show that if all elements in a group G have order ≤ 2, then G is abelian and isomorphic to a vector space over $\mathbb{Z}/2\mathbb{Z}$.

b) Let G be a group acting primitively on a set X. Suppose N is a normal subgroup of G. Show that either N acts transitively on X or N lies in the kernel of the action.

c) Let G be a solvable group acting primitively on a finite set X. Show that $|X|$ is a prime power.
Hint: Show that if G acts faithfully, then every minimal non-trivial normal subgroup of G is elementary abelian, namely abelian and there is some prime p such that every non-trivial element has order p. Then use item b).

(4 Points)

A group action of G on a set X is called *2-transitive*, if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there is a $g \in G$ such that $gx = x'$ and $gy = y'$.

Exercise 2.

a) Let F be a field. Show that the action of

$$\mathrm{Aff}(F) = F \rtimes F^* \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F \setminus \{0\}, b \in F \right\} \leq \mathrm{GL}_2(F)$$

on F, defined by $(a, b)x := ax + b$, is 2-transitive.

b) Show that any 2-transitive group action is primitive.

c) Show that an action of a group G on a finite set X is 2-transitively if and only if for all $x, y \in X$ with $x \neq y$, we have

$$|G : G_x \cap G_y| = |X|(|X| - 1).$$

(4 Points)

Exercise 3. Let F be a field with at least 4 elements.

a) Let $\mathrm{SL}_2(F)$ be the subgroup of $\mathrm{GL}_2(F)$ generated by elements of the form: $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ with $t \in F$. Show that $\mathrm{SL}_2(F)$ is perfect.

b) Consider the natural action of $\mathrm{SL}_2(F)$ on $\mathbb{P}^1(F)$, i.e. the set of one-dimensional subspaces of the vector space F^2. Show that this action is 2-transitive. Describe the center Z of $\mathrm{SL}_2(F)$. Show that Z is also the kernel of this action.

c) Show that the group $\mathrm{PSL}_2(F) := \mathrm{SL}_2(F)/Z$ is simple.
Hint: Use Exercise 2 item b) and apply Iwasawa's Criterion.

(4 Points)

Exercise 4. Let A_n be the alternating group of degree n.

a) Find a proper non-trivial normal subgroup N of A_4.

b) Show that A_5 is perfect.
Hint: Use Exercise 3 item d) from the previous problem sheet.

c) Use Iwasawa's Criterion to show that A_5 is simple.
Hint: Use Exercise 3 item b) from the previous sheet.

(4 Points)

Submission by Thursday morning 10:00, 27.10.2020, in Briefkasten 161.
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.