Power functions and exponentials in o-minimal expansions of fields

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Abstract

This thesis aims to contribute to the model-theoretic study of expansions of ordered fields by exponential functions and power functions. We will mainly be interested in the special case that the ordered field is $\mathbb{R}$, but will also work in the context of arbitrary o-minimal expansions of ordered fields.

In chapter 1 we give an overview of the work contained in this thesis as well as providing motivation and background. In chapter 2 we set up some notation and conventions and then give a brief account of the parts of the theory of o-minimal structures which we will need. Finally we recall some basic results in model theory on quantifier elimination and model-completeness. In chapters 3 and 4 we prove our first main theorem: that given any first order formula ϕ in the language $L' = \{+, \cdot, <, (f_i)_{i \in I}, (c_i)_{i \in I}\}$, where the f_i are unary function symbols and the c_i are constants, one can find an existential formula ψ such that ϕ and ψ are equivalent in any L'-structure $\langle \mathbb{R}, +, \cdot, <, (x^{c_i})_{i \in I}, (c_i)_{i \in I} \rangle$. In chapter 5 we introduce a first order theory T_∞ which can be seen as the theory of certain real closed fields, each expanded by a power function with infinite exponent. We note that it follows from the first main theorem that T_∞ is model-complete, furthermore we prove that T_∞ is decidable if and only if the theory of the real field with the exponential function is decidable. In chapter 6 we consider the problem of expanding an arbitrary o-minimal expansion of a field by a non-trivial exponential function whilst preserving o-minimality. It is known that if $\tilde{\mathbb{R}}$ is an o-minimal expansion of the real field then $\langle \tilde{\mathbb{R}}, \exp \rangle$ is o-minimal. By different methods it is also known that if $\tilde{\mathbb{R}}$ defines $\exp \restriction_{[0,1]}$ then $\langle \tilde{\mathbb{R}}, \exp, \log \rangle$ admits quantifier elimination and a universal axiomatization relative to $\tilde{\mathbb{R}}$. We generalize this second result to o-minimal expansions of arbitrary ordered fields under certain assumptions. Finally, in chapter 7 we propose a future research project arising from the results in chapter 5.

Contents

Chapter 1

Introduction

The first part of this thesis is concerned with the study of first order definability in certain structures expanding the real field. Let us consider the simplest, yet certainly non-trivial, case: that of $\langle \mathbb{R}, +, \cdot \rangle$. The first thing to note about this structure is that the normal ordering on $\mathbb{R}$ is definable; the set of non-negative elements of $\mathbb{R}$ is precisely the set defined by the formula $\exists y(x = y^2)$. Consequently all semialgebraic sets are definable. Indeed, if we add a symbol for $<$ to our language then they are exactly the quantifier-free definable sets (of course, because the ordering is already definable using $+$ and $\cdot$, adding $<$ to our language does not add any new definable sets). Using quantifiers one readily sees that amongst other things we can define the interior and closure of any semialgebraic set. Remarkably, in 1930 [1] Tarski proved that the structure $\overline{\mathbb{R}} = \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$ has quantifier elimination so that in fact all definable sets in this structure are semialgebraic. In particular, the closure and interior of a semialgebraic set are semialgebraic.

Tarski-Seidenberg Theorem. [2] *There is an effective procedure which, given any formula $\phi(x_1, \ldots, x_n)$ in the language $L_{\text{ord}} = \{+, \cdot, <, 0, 1\}$ produces a quantifier-free formula $\psi(x_1, \ldots, x_n)$ such that*

$$\overline{\mathbb{R}} \models \phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n).$$

Let us make a couple of remarks about this theorem.

1. The quantifier elimination allows us to deduce many geometric and topological properties of sets definable in $\overline{\mathbb{R}}$. For instance, it follows immediately that every definable subset of $\mathbb{R}$ is a boolean combination of sets of the form $\{x : p(x) = 0\}$ and $\{x : p(x) > 0\}$ where $p(x)$ is a polynomial in 1-variable. Consequently every definable subset of $\mathbb{R}$ is a finite union of intervals and points; i.e. $\overline{\mathbb{R}}$ is o-minimal. Thus the theory of sets definable in o-minimal expansions of fields applies to $\overline{\mathbb{R}}$, so, amongst many other things, we get that definable families of sets have a uniform bound on the number of connected components and all definable sets have a definable triangulation[3]. It is worth commenting at this point that the quantifier elimination allows us to deduce properties of definable sets in $\overline{\mathbb{R}}$ which do not follow from the general theory of o-minimal expansions of fields. For instance, that every unary

[1] This result was actually not published until it appeared in [18] in 1948

[2] Although this theorem was first proved by Tarski it is commonly referred to as the Tarski-Seidenberg theorem

[3] This remark is slightly disingenuous; many of the properties which hold for definable sets in arbitrary o-minimal expansions of fields were proved for semialgebraic sets (and hence for all sets definable in $\overline{\mathbb{R}}$) before the notion of o-minimality was invented.

definable function $f : (r, \infty) \to \mathbb{R}$ is asymptotic to ax^q for some $q \in \mathbb{Q}$ and $a \in \mathbb{R}$ (see example 2.2.65).

2. The second remark is to highlight the use of the word 'effective' in the statement of the theorem. In Tarski's own words an effective procedure is one "which tells one what to do at each step so that no intelligence is required to follow it; and the method can be applied by anyone so long as he is able to read and follow instructions". In particular, given an L_{ord}-sentence one can effectively find a quantifier-free L_{ord}-sentence with the same truth value in $\overline{\mathbb{R}}$; this shows that $\mathrm{Th}(\overline{\mathbb{R}})$ is decidable since it is clear that one can effectively determine the truth-value of a quantifier-free L_{ord}-sentence.

Much of the subsequent work on structures expanding the real field has been in trying to find mathematically interesting structures whose definable sets share some of the good geometric and topological properties of the definable sets in $\overline{\mathbb{R}}$. Since the invention of o-minimality in the late 1980's this has taken the form of proving that certain expansions of the real field are o-minimal. Let us recall some significant successes in this direction. The first is the o-minimality of $\mathbb{R}_{\mathrm{an}}$, the real field expanded by predicates for all bounded semianalytic sets, which was seen by van den Dries [21] to be a consequence of theorems of Łojasiewicz and Gabrielov. Gabrielov's theorem says that the complement of a subanalytic set is subanalytic. One deduces from this that the structure $\mathbb{R}_{\mathrm{an}}$ is model-complete in the language described above. Now Łojasiewicz's theorem says that all bounded semianalytic sets, and hence all quantifier-free definable sets in $\mathbb{R}_{\mathrm{an}}$, have finitely many connected components. Since the number of connected components of a set cannot increase upon projection we see that $\mathbb{R}_{\mathrm{an}}$ is o-minimal. Using the fact that all subanalytic subsets of $\mathbb{R}^2$ are in fact semianalytic van den Dries proves that, as in $\overline{\mathbb{R}}$, every unary definable function $f : (r, \infty) \to \mathbb{R}$ is asymptotic to ax^q for some $q \in \mathbb{Q}$ and $a \in \mathbb{R}$. Notice that this implies that every such f is eventually bounded by the function $x \mapsto x^n$ for some n; i.e. $\mathbb{R}_{\mathrm{an}}$ is *polynomially bounded.*

The structure $\mathbb{R}_{\mathrm{an}}$ is often alternatively defined as $\langle \overline{\mathbb{R}}, (\mathfrak{F}_{\mathfrak{n}})_{n \geq 0} \rangle$ (see example 2.2.66), where $\mathfrak{F}_0$ contains a constant for every $r \in \mathbb{R}$, and, for $n \geq 0$, the set $\mathfrak{F}_n$ consists of those functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists a function g, analytic on a neighbourhood of $[0,1]^n$, such that on $[0,1]^n$ the function f is equal to g and outside $[0,1]^n$ the function f is identically zero; we call f a restricted analytic function. It is easy to see that both definitions of $\mathbb{R}_{\mathrm{an}}$ have the same quantifier-free definable sets, so model-completeness of the first definition implies the model-completeness of the second. A very natural question arising from the second formulation is the following: for which $\mathfrak{F}' \subset \bigcup_{n \geq 0} \mathfrak{F}_\mathfrak{n}$ is $\langle \overline{\mathbb{R}}, \mathfrak{F}' \rangle$ model-complete? Wilkie's theorem on restricted Pfaffian functions (see section 3.1 for definitions and the full statement of the theorem) provides a partial answer to this question. The theorem says that if the functions in $\mathfrak{F}'$ are the restrictions of a Pfaffian chain over a set C then $\langle \overline{\mathbb{R}}, \mathfrak{F}', (c)_{c \in C} \rangle$ is model-complete. Since (exp) is a Pfaffian chain of length 1, an application of this theorem tells us that the structure $\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle$ is model-complete. This is an important step in Wilkie's proof that $\mathbb{R}_{\exp} = \langle \overline{\mathbb{R}}, \exp \rangle$ is model-complete. Another key ingredient is the valuation inequality (theorem 2.2.59) which Wilkie proves for a special class of polynomially bounded o-minimal theories[4], so-called 'smooth o-minimal theories', of which $\mathrm{Th}(\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle)$ is one (note that since $\mathbb{R}_{\mathrm{an}}$ is polynomially bounded it follows of course that $\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle$ is polynomially bounded)[5]. Now Khovanskii's theorem tells us that all quantifier-free definable sets in $\mathbb{R}_{\exp}$ have finitely many connected components. Combining this with model-completeness we deduce that $\mathbb{R}_{\exp}$ is o-minimal just as we did for $\mathbb{R}_{\mathrm{an}}$.

Now let us begin to turn our attention to the first main result of this thesis. Let I be some fixed index set and let L' be the language expanding L_{ord} by a function symbol f_i and a constant symbol c_i for each $i \in I$. Now consider an L'-structure of the form $\tilde{\mathbb{R}} = \langle \overline{\mathbb{R}}, (x^{c_i})_{i \in I}, (c_i)_{i \in I} \rangle$, i.e. for each $i \in I$ we interpret f_i as a power function with exponent c_i. We will let $\mathfrak{C}'$ be the class of all such L'-structures. Note that because every real power function is definable in $\mathbb{R}_{\exp}$ the structure $\tilde{\mathbb{R}}$ is a reduct of $\mathbb{R}_{\exp}$ and hence is o-minimal. It is natural to ask whether the structure $\tilde{\mathbb{R}}$ is model-complete. It follows immediately from Wilkie's result on restricted Pfaffian functions that $\langle \overline{\mathbb{R}}, (x^{c_i} \restriction_{[1,2]})_{i \in I}, (c_i)_{i \in I} \rangle$ is model-complete[6]. Using a special case of the valuation inequality, Miller obtains a universal axiomatization of $\mathrm{Th}(\tilde{\mathbb{R}})$ over $\mathrm{Th}(\langle \overline{\mathbb{R}}, (x^{c_i} \restriction_{[1,2]})_{i \in I}, (c_i)_{i \in I} \rangle)$ in the style of Ressayre (albeit using different methods). From this Miller deduces that $\tilde{\mathbb{R}}$ is model-complete.

One way of stating Miller's result is to say that given any L'-formula ϕ there exists an existential L'-formula ψ such that $\tilde{\mathbb{R}} \models \phi \leftrightarrow \psi$. We ask the following question: must we choose different existential L-formulas for different members of $\mathfrak{C}'$? The first main theorem of this thesis tells us that in fact we can choose a single ψ which will work for all members of $\mathfrak{C}'$. In fact we prove the following theorem.

[4]Wilkie's special case predates the proof of the full result as stated in theorem 2.2.59

[5]It is worth remarking at this stage that in [15] Ressayre gives a different deduction of the model-completeness of $\mathbb{R}_{\exp}$ from that of $\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle$ by showing that the 'with parameters' theory of an arbitrary model of $\mathrm{Th}(\mathbb{R}_{\exp})$ is axiomatized by the 'with parameters' theory of the associated model of $\mathrm{Th}(\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle)$ and the 'without parameters' theory $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$.

[6]For any $r \in \mathbb{R}$ the sequence (x^{-1}, x^r) is a Pfaffian chain on $(0, \infty)$.

First Main Theorem. $\mathrm{Th}(\mathfrak{C}')$ *is model-complete.*

In order to prove this theorem we first generalize Wilkie's theorem on restricted Pfaffian functions. This is the content of chapter 3. We complete the proof of the 'First Main Theorem' in chapter 4. Making much use of the o-minimality of $\mathbb{R}_{\exp}$ we apply the generalization of Wilkie's theorem on restricted Pfaffian functions to prove a version of the 'First Main Theorem' for restricted power functions (this is the content of sections 4.4 and 4.5). In section 4.6 we generalize the work of Miller mentioned above, using the full strength of the valuation inequality, in order to obtain the full result.

Let us now return to our second remark about the Tarski-Seidenberg theorem. As already mentioned, it follows from the effective quantifier elimination for $\overline{\mathbb{R}}$ that $\mathrm{Th}(\overline{\mathbb{R}})$ is decidable. In the concluding remarks to his paper [18], Tarski asks whether $T_{\exp}$ is decidable. Now $\mathbb{R}_{\exp}$ does not admit quantifier elimination [20] so we cannot hope to reduce the decidability of $\mathbb{R}_{\exp}$ to the decidability of the quantifier-free theory of $\mathbb{R}_{\exp}$ à la Tarski. Furthermore, even if we could do this, the decidability of the quantifier-free theory of $\mathbb{R}_{\exp}$ is not trivial as it is for $\overline{\mathbb{R}}$. In [8], Wilkie and Macintyre are able to use and extend Wilkie's model-completeness for $\mathbb{R}_{\exp}$ to reduce the problem of proving the decidability of $T_{\exp}$ to that of proving that the existential part of $T_{\exp}$ is recursively enumerable (although they do not prove that $T_{\exp}$ is effectively model-complete). They then go on to prove that the existential part of $T_{\exp}$ *is* recursively enumerable (and hence $T_{\exp}$ is decidable) under the assumption of Schanuel's conjecture for $\mathbb{R}$, an established conjecture in transcendental number theory. However, this conjecture is generally thought to be out of reach of current techniques (even very special cases, such as the statement that e^e is irrational, are open).

In chapter 5 we introduce the theory T_∞ which may be defined as follows. Consider $\mathfrak{C}'$, constructed as above, in the special case that the index set I consists of one element. So $L' = L_{\mathrm{ord}} \cup \{f, c\}$, where f is unary function symbol and c is a constant symbol, and $\mathfrak{C}'$ is the class of all structures of the form $\langle \overline{\mathbb{R}}, x^r, r \rangle$ where $r \in \mathbb{R}$. Then $T_\infty = \mathrm{Th}(\mathfrak{C}') \cup \{c > n : n \in \mathbb{N}\}$. Note that by compactness T_∞ is consistent and, since it extends $\mathrm{Th}(\mathfrak{C}')$, it is model-complete. Each model of T_∞ is an expansion of a non-Archimedean ordered field by a power function with positive infinite exponent.

Second Main Theorem. $T_{\exp}$ *is decidable if and only if* T_∞ *is decidable.*

That the decidability of $T_{\exp}$ implies the decidability of T_∞ readily follows from the o-minimality of $T_{\exp}$. In order to prove the reverse implication, using the reduction of Wilkie and Macintyre, we assume that T_∞ is decidable and prove that the existential part of $T_{\exp}$ is recursively enumerable. To do this we prove a stronger version of the classical limit formula

$$\lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y = \exp(x).$$

This allows us to approximate the value of the exponential function on finite elements of models of $T_{\exp}$ by the definable function

$$x \mapsto \left(1+\frac{x}{\lambda}\right)^\lambda,$$

where λ is positive infinite. As a corollary to the Second Main Theorem we obtain the following.

Corollary 1.0.1. *$T_{\exp}$ is decidable if and only if there is an effective procedure which, given $n \geq 1$ and $p \in \mathbb{Z}[x_1, \ldots, x_{2n}, x_{2n+1}]$, terminates if and only if for all positive integers d the polynomial $p(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d, d)$ has a zero in the positive orthant of $\mathbb{R}^n$.*

For the final chapter we consider the problem of extending an arbitrary o-minimal expansion of a field by an exponential function whilst preserving o-minimality. Following Wilkie's proof that $\mathbb{R}_{\exp}$ is o-minimal, work of Speissegger shows that if $\tilde{\mathbb{R}}$ is any o-minimal expansion of the real field then $\langle \tilde{\mathbb{R}}, \exp\rangle$ is o-minimal [17]. When the o-minimal structure under consideration is not an expansion of the real field the situation is less clear; for a start, one does not have a ready made exponential function to append. Now, work of van den Dries and Speissegger in [28] (following the methods of Macintyre, Marker and van den Dries in [26]) shows that if $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of the real field which defines the exponential function on $[0,1]$ then $\mathrm{Th}(\langle \tilde{\mathbb{R}}, \exp, \log\rangle)$ admits quantifier elimination and a universal axiomatization over $\mathrm{Th}(\tilde{\mathbb{R}})$. We consider the situation where $\mathcal{R}$ is an o-minimal expansion of an ordered field which defines a restricted exponential function *and* has field of exponents cofinal in its prime model $\mathcal{P}$. We are then able to naturally construct an exponential function $\tilde{\mathrm{E}}$ on $\mathcal{P}$. Adapting the methods of [26] we prove that $\mathrm{Th}(\langle \mathcal{P}, \tilde{\mathrm{E}}, \mathrm{Log}\rangle)$ (where Log is the compositional inverse of $\tilde{\mathrm{E}}$) admits quantifier elimination and a universal axiomatization over $\mathrm{Th}(\mathcal{P})$. Finally we prove that $\langle \mathcal{P}, \tilde{\mathrm{E}}\rangle$ is o-minimal. From this we easily deduce the third main theorem.

Third Main Theorem. *Let $\mathcal{R}$ be an o-minimal expansion of a field which defines an exponential function* E *on* $[0,1]$ *and has field of exponents cofinal in its prime model. Then there is an elementary extension $\mathcal{S}$ of $\mathcal{R}$ which supports a (global) exponential function* $\tilde{\mathrm{E}}$ *which extends* E*. Furthermore the expansion of $\mathcal{S}$ by this exponential function is o-minimal.*

Chapter 2

Preliminaries

2.1 Notation and conventions

Let us briefly set out some of the notation and conventions that will be used throughout this thesis. We will typically denote model-theoretic structures by $\mathcal{A}, \mathcal{B}, \ldots$ and their domains by $A, B, \ldots$. If we say that a subset X of A^n is definable (in the structure $\mathcal{A}$) this will mean 'definable *with* parameters from A'. We will say that X is C-definable, where C is a subset of A, if X is definable with parameters taken only from C, and we will say that X is 0-definable if it is $\emptyset$-definable. If we simply say that X is definable in $\mathcal{A}$ we mean that X is a subset of A^n, for some $n \geq 1$, which is definable in the structure $\mathcal{A}$. If $\mathcal{A}$ is an L-structure and $\phi(x_1, \ldots, x_n)$ is an L-formula we will write $\phi(\mathcal{A})$ to denote the subset of A^n defined by $\phi(x_1, \ldots, x_n)$ in $\mathcal{A}$. We will not normally distinguish notationally between non-logical symbols in a first order language and their interpretations in a given structure; a notable exception to this is in our treatment of power functions in chapter 4. We will let $L_{\text{ord}} = \{+, \cdot, <, 0, 1\}$ denote the language of ordered rings. If $L \supset L_{\text{ord}}$ and $\mathcal{A}$ is an L-structure we will denote the L_{ord}-reduct of $\mathcal{A}$ by $\overline{A}$. In particular $\overline{\mathbb{R}} = \langle \mathbb{R}, +, \cdot, <, 0, 1\rangle$. If $\mathcal{A}$ expands an ordered field we will let $\text{Pos}(A)$ denote the (definable) set $\{x \in A : x > 0\}$. If $\mathcal{A}$ expands a real closed field and $\bar{r} = (r_1, \ldots, r_n) \in A^n$ we will let $\|\bar{r}\|$ denote the Euclidean norm of $\bar{r}$. If $\phi(\bar{y}, \bar{x})$ is a formula and we wish to consider the definable family given by $\phi(\bar{y}, \bar{x})$ by varying the interpretations of the variables $\bar{y}$ we may denote the formula by $\phi_{\bar{y}}(\bar{x})$. Finally, if we say that a structure $\mathcal{A}$ is model-complete or admits quantifier elimination we mean that its theory $\text{Th}(\mathcal{A})$ has this property.

2.2 o-minimality

Throughout this thesis o-minimality will play a large role. Here we recall some definitions and results from the theory of o-minimal structures.

Definition 2.2.1. Let $\mathcal{R} = \langle R, <, \ldots\rangle$ be a structure expanding $\langle R, <\rangle$, a dense linear order without endpoints. The structure $\mathcal{R}$ is said to be *o-minimal* if every definable subset of R is a finite union of intervals and points (i.e. singletons).

Remark 2.2.2. By an interval in $\langle R, <\rangle$ we mean a set of one of the following forms:

1. (a, b) where $a, b \in R \cup \{\pm\infty\}$ with $a < b$,

2. $[a, b)$ where $a \in R$ and $b \in R \cup \{+\infty\}$ and $a < b$,

3. $(a, b]$ where $a \in R \cup \{-\infty\}$ and $b \in R$ and $a < b$,

4. $[a, b]$ where $a, b \in R$ and $a < b$.

Example 2.2.3. Let $\mathcal{R} = \langle R, < \rangle$ be a dense linear order without endpoints. Since the theory of dense linear orders without endpoints has quantifier elimination $\mathcal{R}$ is o-minimal.

Example 2.2.4. Let $\overline{\mathbb{R}} = \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$. A theorem of Tarski [18] says that $\overline{\mathbb{R}}$ has quantifier elimination so definable subsets of $\mathbb{R}$ are boolean combinations of sets of the form $\{x \in \mathbb{R} : p(x) = 0\}$ and $\{x \in \mathbb{R} : p(x) > 0\}$ where $p \in \mathbb{R}[X]$, consequently $\overline{\mathbb{R}}$ is o-minimal.

Example 2.2.5. Let $\mathbb{R}_{\exp} = \langle \overline{\mathbb{R}}, \exp \rangle$. It is a theorem of Wilkie [29] that $\mathbb{R}_{\exp}$ is o-minimal. We will say more about this later (example 2.2.67).

2.2.1 Topological properties

Unless otherwise stated, all results in this section can be found in [24].

Let $\mathcal{R}$ be an o-minimal structure. Then $\mathcal{R}$ carries a natural topology on R induced by its ordering; i.e. the topology whose basic open sets are the open intervals of R with endpoints in $R \cup \{\pm\infty\}$. For each $n \geq 2$ we give the Cartesian power R^n the product topology. Notice that for any $n \geq 1$ the family of all open boxes of R^n forms a definable family and a basis for the topology on R^n.

Notice that if $\mathcal{R}$ expands $\langle \mathbb{R}, < \rangle$ then the topology on $\mathcal{R}$ is just the Euclidean topology and the definable subsets of $\mathbb{R}$ are precisely those with finitely many connected components. This is because the only connected subsets of $\mathbb{R}$ are the intervals and the singletons. Of course there are dense linear orders which are totally disconnected so it is *not* the case in general that if $\mathcal{R}$ is o-minimal then the definable subsets of R are those with finitely many connected components. However, consider the following weakening of the notion of connectedness.

Definition 2.2.6. Let $\mathcal{R}$ be o-minimal. A definable subset X of R^n is *definably connected* if it is not the disjoint union of two non-empty *definable*, relatively open subsets.

Now we introduce the notion of a *cell* and state the cell decomposition theorem. This will tell us that in fact, in an o-minimal structure *all* definable sets (i.e. in any Cartesian power) have finitely many *definably* connected components.

Definition 2.2.7. The cells of R^n are defined inductively on n as follows.

1. The cells of R are of the from (a, b) and $\{a\}$, where a and b are 0-definable.
2. The cells of R^{n+1} are of one of the following forms:
   (a) graph(f), where f is a 0-definable continuous map with domain C, a cell in R^n,
   (b) $(f, g)_C$, or $(f, \infty)_C$ or $(-\infty, f)_C$, where f and g are 0-definable continuous maps with domain C, a cell in R^n, and

$$\begin{aligned}(f, g)_C &= \{(\bar{x}, y) \in C \times R : f(\bar{x}) < y < g(\bar{x})\},\\ (f, \infty)_C &= \{(\bar{x}, y) \in C \times R : f(\bar{x}) < y\},\\ (-\infty, f)_C &= \{(\bar{x}, y) \in C \times R : y < f(\bar{x})\}.\end{aligned}$$

The following lemma is proved by induction.

Lemma 2.2.8. *Cells are definably connected.*

We say that a finite collection $\mathfrak{C}$ of cells in R^n is a *partition* of R^n if the cells in $\mathfrak{C}$ are pairwise disjoint and their union covers R^n. We will say that a partition $\mathfrak{C}$ is *compatible* with a definable set $X \subseteq R^n$ if for all $C \in \mathfrak{C}$ either $C \subseteq X$ or $C \cap X = \emptyset$.

Theorem 2.2.9. *[Cell decomposition theorem] Let X be a 0-definable subset of R^n. Then there exists a partition $\mathfrak{C}$ of R^n into cells, compatible with X. Furthermore, if $f : X \to R$ is 0-definable then the collection $\mathfrak{C}$ can be chosen so that if $C \in \mathfrak{C}$ and $C \subseteq X$ then f is continuous on C.*

It follows from lemma 2.2.8 and the cell decomposition theorem that in an o-minimal structure all 0-definable sets have finitely many definably connected components. In fact it is not difficult to deduce the following stronger result about sets definable with parameters.

Theorem 2.2.10. *Let $\mathcal{R}$ be an o-minimal structure and let $S \subseteq R^n \times R^m$ be definable. Then there exists a natural number M (depending on S) such that for any $\bar{a} \in R^n$ the set $S_{\bar{a}} = \{\bar{x} \in R^m : (\bar{a}, \bar{x}) \in S\}$ has at most M definably connected components.*

So in an o-minimal structure definable sets have finitely many definably connected components and furthermore, for a definable family of definable sets there is a uniform bound (i.e. a bound depending only on the family) on the number of definably connected components of each set in the family. We remarked above that for o-minimal structures expanding $\mathbb{R}$, if X is a definable subset of $\mathbb{R}$ then X is connected if and only if X is definably connected. In fact this holds for Cartesian powers of $\mathbb{R}$, i.e. for any $n \geq 1$, if X is a definable subset of $\mathbb{R}^n$ then X is connected if and only if X is definably connected. So for o-minimal expansions of the real field, all definable sets have finitely many connected components and indeed for every definable family there is a uniform bound on the number of connected components of each set in the family.

A number of topological properties of $\mathbb{R}$ generalize to arbitrary o-minimal structures when we insert the word 'definable' at appropriate points in their statement. For instance, it is easy to see that the usual intermediate value theorem holds for *definable* continuous functions. For another example of this consider the classical result that says: if X is a closed and bounded subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ is continuous then $f(X)$ is closed and bounded. For an arbitrary o-minimal structure $\mathcal{R}$ we have the following theorem.

Theorem 2.2.11. *Let X be a closed and bounded definable subset of R^n and let $f : X \to R^m$ be a definable continuous map. Then $f(X)$ is closed and bounded.*

Remark 2.2.12. Bounded means contained in $[-a, a]^n$ for some $a \in \mathrm{Pos}(R)$.

2.2.1.1 Dimension

Let $\mathcal{R}$ be an o-minimal structure and let X be a definable subset of R^n. We define $\dim(X)$, the dimension of X, to be the largest natural number $k \leq n$ such that for some projection $\pi : R^n \to R^k$ the set $\pi(X)$ has non-empty interior in the ambient space R^k (note that the projection of X onto $\mathbb{R}^0$ has non-empty interior if and only if X is non-empty). If X is empty we set $\dim(X) = -\infty$. Using the cell decomposition theorem one obtains a number of desirable properties for this dimension.

Theorem 2.2.13.

1. $\dim(R^n) = n$ *for all* $n \geq 1$.

2. *If X is definable then* $\dim(X) = -\infty$ *if and only if X is empty and* $\dim(X) = 0$ *if and only if X is finite.*

3. *If X and Y are definable and $f : X \to Y$ is a definable bijection then* $\dim(X) = \dim(Y)$.

4. *If* $X, Y \subseteq R^n$ *are definable then* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

5. *Let* S *be a definable subset of* $R^n \times R^m$. *For each* $d \in \{-\infty, 0, 1, \ldots, m\}$ *let* $S(d) = \{\bar{x} \in R^n : \dim(S_{\bar{x}}) = d\}$. *Then* $S(d)$ *is definable and*

$$\dim\{(\bar{x}, \bar{y}) \in S : \bar{x} \in S(d)\} = \dim(S(d)) + d.$$

6. *If* f *is a definable function on* X *then the set*

$$Y = \{\bar{x} \in X : f \textit{ is continuous at } \bar{x}\}$$

is definable and large in X*,i.e.* $\dim(X \setminus Y) < \dim(X)$.

2.2.2 Model-theoretic properties

We recall here some important model-theoretic properties of o-minimal structures which we shall make frequent use of throughout the course of this thesis. All the material in this section is standard and can be found in [9], [22].

Theorem 2.2.10 is listed as a topological property of o-minimal structures but it has the following very important corollary.

Theorem 2.2.14. *o-minimality is preserved under elementary equivalence - i.e. if* $\mathcal{R}$ *is o-minimal and* $\mathcal{R}'$ *is elementarily equivalent to* $\mathcal{R}$ *then* $\mathcal{R}'$ *is o-minimal.*

Remark 2.2.15. We will say that a complete theory T is o-minimal if one of its models is o-minimal. It follows from theorem 2.2.14 that *all* models of a complete o-minimal theory are o-minimal.

2.2.2.1 Definable closure and rank

Let $\mathcal{R}$ be an arbitrary o-minimal structure. Given $A \subseteq R$ we define the *definable closure* of A, written $\mathrm{dcl}(A)$, to be the union of all A-definable singletons; i.e. $b \in \mathrm{dcl}(A)$ if and only if there is a formula $\phi(x)$ with parameters from A such that $\phi(\mathcal{R}) = \{b\}$. One can show that dcl is a *pregeometry* on R, i.e. it satisfies the following properties:

1. if $A \subseteq R$ then $A \subseteq \mathrm{dcl}(A)$,

2. if $A \subseteq R$ then $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$,

3. if $A \subseteq R$ and $b \in R$ such that $b \in \mathrm{dcl}(A)$ then there exists A_0, a finite subset of A, such that $b \in \mathrm{dcl}(A_0)$,

4. if $A \subseteq R$ and $a, b \in R$ such that $b \in \mathrm{dcl}(A \cup \{a\}) \setminus \mathrm{dcl}(A)$ then $a \in \mathrm{dcl}(A \cup \{b\})$.

Remark 2.2.16. Properties (1)-(3) are satisfied by dcl in any first-order structure. Property (4) is a consequence of the monotonicity theorem for o-minimal structures (see for instance [24]).

Given $X \subseteq R$ we say that X is *definably closed* if $X = \mathrm{dcl}(X)$.

Now, because dcl is a pregeometry on $\mathcal{R}$ it determines a notion of independence for subsets of R. Namely, if $X \subseteq R$ we say that X is *independent* if for all $x \in X$ we have $x \notin \mathrm{dcl}(X \setminus \{x\})$. A subset B of a set Y is said to be a *basis* for Y if it is maximal amongst independent subsets of Y. It follows from the fact that dcl is a pregeometry that if B_1 and B_2 are bases for a set Y then B_1 and B_2 have the same cardinality; we define this cardinality to be the *rank* of the set Y, written $\mathrm{rk}(Y)$.

If $A \subseteq R$ we may write $\mathrm{dcl}_A(X)$ to denote $\mathrm{dcl}(A \cup X)$. Note that dcl_A is also a pregeometry (it is just definable closure in the o-minimal structure expanding $\mathcal{R}$ where we take constant symbols for elements of A). We will sometimes write $\mathrm{rk}(X|A)$ to denote the rank of X with respect to dcl_A.

The notions of rank and dimension are connected by the following lemma.

Lemma 2.2.17. *Let $\bar{r} \in R^m$ and suppose that $\mathrm{rk}(\bar{r}|A) = n$. Then there exists X an n-dimensional A-definable subset of R^m containing $\bar{r}$, and $\bar{r}$ is not contained in any A-definable sets of dimension strictly smaller than n. Conversely, if X is an n-dimensional A-definable set then, in an $(|A|+|L|)^+$-saturated elementary extension of $\mathcal{R}$ (where L is the language of $\mathcal{R}$), X contains a point $\bar{r}$ with $\mathrm{rk}(\bar{r}|A) = n$.*

If X is an A-definable set of dimension n then a point $\bar{a} \in X$ with $\mathrm{rk}(\bar{a}|A) = n$ is called a *generic* point of X over A. If we say that $\bar{a}$ is a generic point of X without mentioning a set of parameters we mean generic over $\emptyset$. The above lemma tells us that we can always find generic points by passing to sufficiently saturated elementary extensions. The following lemma tells us that in an important special case we do not need to pass to an elementary extension in order to find generic points.

Lemma 2.2.18. *Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of $\mathbb{R}$ in a countable language and let $A \subseteq \mathbb{R}$ be a countable set of parameters. If X is a non-empty A-definable set in $\tilde{\mathbb{R}}$, then X contains a point generic over A.*

Sketch of proof. Suppose $X \subseteq \mathbb{R}^n$. By taking a suitable projection of X if necessary, we may assume that X has interior in $\mathbb{R}^n$. It follows from lemma 2.2.17 that it is sufficient to find a point $\bar{r}$ in X such that $\bar{r}$ is not contained in any A-definable subsets of X of strictly smaller dimension, i.e. we must prove that X is not covered by its A-definable subsets of strictly smaller dimension. To see this we note that there only countably many A-definable subsets of X of strictly smaller dimension since $\tilde{\mathbb{R}}$ has a countable language, furthermore each such set has empty interior in $\mathbb{R}^n$. It follows from the Baire category theorem that their union has empty interior and hence does not cover X. $\square$

2.2.2.2 Definable Skolem functions and prime models

Let us now assume that our o-minimal structure $\mathcal{R}$ expands an ordered group. We must also assume that $\mathcal{R}$ has a 0-definable non-zero element, i.e. there exists $r \in R \setminus \{0\}$ such that $\{r\}$ is definable in $\mathcal{R}$ (this is required for the proof of lemma 2.2.19). Of course this assumption holds for all o-minimal expansions of fields. Both of these assumptions hold for all o-minimal structures considered in this thesis.

Lemma 2.2.19. *Let $A \subseteq R$ and let $S \subseteq R^n \times R^m$ be A-definable. Let $\pi(S)$ be the projection of S onto R^n. Then there exists an A-definable function $f : \pi(S) \to R^m$ such that* $\mathrm{graph}(f) \subseteq S$.

Lemma 2.2.19 says that $\mathcal{R}$ has definable Skolem functions. It follows that if $A \subseteq R$ then $\mathrm{dcl}(A)$ is the domain of an elementary substructure of $\mathcal{R}$. Consequently o-minimal theories have unique prime models. The following lemma makes this statement precise.

Lemma 2.2.20. *Let $A \subseteq R$. Then there exists $\mathcal{S}$, an elementary substructure of $\mathcal{R}$ containing A, with the property that if $\mathcal{M} \models \mathrm{Th}(\langle R, (a)_{a\in A}\rangle)$ then S embeds elementarily in $\mathcal{M}$ over A (and the domain of $\mathcal{S}$ is just $\mathrm{dcl}_{\mathcal{R}}(A)$). We call $\mathcal{S}$ the prime model of $\mathcal{R}$ over A.*

Notation 2.2.21. If $\mathcal{R}$ is an o-minimal expansion of a group and $\mathcal{S}$ is an elementary extension of $\mathcal{R}$ and $X \subseteq S$ then we write $\mathcal{R}\langle X\rangle$ to denote the prime model of $\mathcal{S}$ over $R \cup X$, i.e. the elementary substructure with domain $\mathrm{dcl}(R \cup X)$. If $X = \{x\}$ we write $\mathcal{R}\langle x\rangle$ instead of $\mathcal{R}\langle\{x\}\rangle$.

Let us record one more consequence of lemma 2.2.19 which we will use throughout this thesis. First we state a lemma which holds for all first-order theories.

Lemma 2.2.22. *Let T be a first-order theory and suppose that T has the property that for any formula $\phi(\bar{x}, y)$ such that $T \models \forall\bar{x}\exists y\phi(\bar{x}, y)$ there exists a function symbol f_ϕ such that $T \models \forall\phi(\bar{x}, f_\phi(\bar{x}))$. Then T admits quantifier elimination and a universal axiomatization.*

Proof. Model-complete universal theories admit quantifier elimination (see remark 2.3.5 below). Consequently it is sufficient to prove that if $\mathcal{B} \models T$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq \mathcal{B}$. This is immediate from the Tarski-Vaught test. $\square$

Remark 2.2.23. It follows from lemma 2.2.19 and lemma 2.2.22 that if we let $\mathcal{R}'$ be the expansion of $\mathcal{R}$ by a function symbol for each 0-definable function $R^n \to R$ (for varying $n \geq 0$) then $\mathrm{Th}(\mathcal{R})$ has quantifier elimination and a universal axiomatization.

2.2.2.3 Types and embeddings

Let $\mathcal{R}$ be an arbitrary o-minimal structure and let A be a definably closed subset of R. It follows immediately from the o-minimality of $\mathcal{R}$ that for any $a \in R$ the type of a over $\mathcal{A}$ is determined by its cut in A. Consequently, in the case that $\mathcal{R}$ expands an ordered group and has a 0-definable non-zero element (so that the results from section 2.2.2.2 hold) we have the following lemma.

Lemma 2.2.24. *Let T be an o-minimal theory and let $\mathcal{R}, \mathcal{R}' \models T$. Suppose that $\mathcal{A} \preccurlyeq \mathcal{R}$ and $\mathcal{A}' \preccurlyeq \mathcal{R}'$ and that $\phi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism. Suppose further that $a \in R$ and $a' \in R'$ and the image of the cut in A made by a under ϕ is exactly the cut in A' made by a'. Then ϕ extends uniquely to an isomorphism ϕ^+ between $\mathcal{A}\langle a\rangle$ and $\mathcal{A}'\langle a'\rangle$ such that $\phi^+(a) = a'$.*

2.2.3 Differentiability in o-minimal expansions of fields

A more detailed development of the material in this section, including proofs, can be found in [24].

Let $\mathcal{R}$ be an o-minimal structure and suppose that $\mathcal{R}$ expands an ordered field. In the presence of this ordered field structure, given a definable $f : (a,b) \to R$ and a point $c \in (a,b)$ we may consider the limit

$$\lim_{y \to c} \frac{f(y) - f(c)}{y - c}.$$

It follows from the monotonicity theorem for o-minimal structures that this limit takes a value in $R \cup \{\pm\infty\}$. Just as in $\mathbb{R}$ we say that f is differentiable at c if the limit exists in R (of course we don't actually need to assume o-minimality in order to make this definition). Note that it is immediate from the definition of the derivative that if f is differentiable on (a,b) then its derivative $f' : (a,b) \to R$ is definable.

It is an immediate consequence of o-minimality that the mean value theorem holds in $\mathcal{R}$ under the additional assumption that the function is definable.

Lemma 2.2.25 (Mean value theorem)**.** *Let $f : [a,b] \to R$ be definable, continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Of course we can also take partial derivatives of functions of many variables: let U be an open definable subset of R^n and let $f : U \to R$ be definable. The function f is said to be C^n at $\bar{x} \in U$, where $n \geq 1$, if all its partial derivatives upto order n exist and are continuous in a neighbourhood of $\bar{x}$. Again it is immediate from the definition of the derivative that for each $i = 1 \dots n$ the partial derivative $\frac{\partial f}{\partial x_i} : U \to R$ is also definable. A definable function $f = (f_1, \dots, f_m) : U \to R^m$ is said to be C^n at $\bar{x} \in U$ if $f_1, \dots, f_m$ are C^n at $\bar{x}$. We say that f is C^n on U if it is C^n at $\bar{x}$ for all $\bar{x} \in U$. We will say that a definable function is *smooth* or C^∞ if it is C^n for every $n \geq 1$. If X is an arbitrary definable set and $f : X \to R^m$ is a definable function, we will say that f is C^n (or C^∞) on X if there exists U, a definable open neighbourhood of X, and $g : U \to R^m$, a definable C^n (or C^∞) map on U, such that $g \restriction X = f$. If we are working in an o-minimal expansion of the real field then we say a definable function f is analytic on a definable set X if it is the restriction of a definable analytic function on a open neighbourhood of X.

Stronger versions of the cell decomposition theorem hold in arbitrary o-minimal expansions of fields. For each $n \geq 0$ we define the collection of C^n cells in the same way as we defined cells, with the additional requirement that all functions involved should be C^n.

Theorem 2.2.26 (C^n cell decomposition)**.** *Take $n \geq 1$ and let X be a 0-definable subset of R^m and $f : X \to R$ be a 0-definable map. Then there exists a partition $\mathfrak{C}$ of R^m into C^n cells, compatible with X, such that if $C \in \mathfrak{C}$ and $C \subseteq X$ then f is C^n on C.*

Remark 2.2.27. Theorem 2.2.26 is often stated by the phrase "arbitrary o-minimal expansion of fields admit C^n cell decomposition for every n". Given $\mathcal{R}$, an arbitrary o-minimal expansion of a field, one can ask whether it admits C^∞ cell decomposition, i.e. does theorem 2.2.26 hold in $\mathcal{R}$ when C^n is replaced by C^∞? In the same way, if working in an arbitrary o-minimal expansion of the real field, one can ask whether it has analytic cell decomposition. In [7], Rolin and Le Gal exhibit an o-minimal expansion of the real field which does not have C^∞ cell decomposition and so the answers to these questions are not always yes. However, smooth and analytic cell decomposition has been established in special cases. In this thesis we will use only the fact that $\mathbb{R}_{\exp}$ admits analytic cell decomposition [27].

Theorem 2.2.28. *Let $X \subseteq R^m$ be definable and let $f : X \to R^n$ be a definable function. For each $k \geq 1$ there exists a definable subset Y of X (depending on k) such that Y is open in X, the function f is C^k on Y, and Y is large in X (i.e. $\dim(X \setminus Y) < \dim(X)$).*

Remark 2.2.29. Theorem 2.2.28 is a strong version of part (6) of lemma 2.2.13. Just as part (6) of lemma 2.2.13 is proved using the cell decomposition theorem, lemma 2.2.28 is proved using theorem 2.2.26.

Notation 2.2.30. Let U be a definable open subset of R^n and let $f = (f_1, \ldots, f_m) : U \to R^m$ be a definable function. If f is C^1 at $\bar{x}$ we will use the notation

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_{i_1}, \ldots, x_{i_k})}(\bar{x})$$

to denote the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_{i_1}}(\bar{x}) & \ldots & \frac{\partial f_1}{\partial x_{i_k}}(\bar{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_{i_1}}(\bar{x}) & \ldots & \frac{\partial f_m}{\partial x_{i_k}}(\bar{x}) \end{pmatrix}.$$

We write $J_f(\bar{x})$ to denote $\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(\bar{x})$.

Many classical theorems of differential calculus go through for arbitrary o-minimal expansions of fields.

Theorem 2.2.31 (Inverse function theorem)**.** *Let U be a definable open subset of R^m and let $f = (f_1, \ldots, f_m) : U \to R^m$ be a definable function. Let $\bar{a} \in U$ and suppose that f is C^1 at U and that $J_f(\bar{a})$ is invertible. Then there exists a definable open neighbourhood V of $\bar{a}$ such that $f(V)$ is open and f is invertible on V with (definable) C^1 inverse. Furthermore if g denotes the inverse then for $\bar{x} \in V$ we have*

$$J_g(f(\bar{x})) = J_f(\bar{x})^{-1}.$$

From this one deduces the implicit function theorem in the usual way.

Theorem 2.2.32 (Implicit function theorem)**.** *Let U be a definable open subset of $R^n \times R^m$ and let $f = (f_1, \ldots, f_m) : U \to R^m$ be a definable C^1 function. Let $\bar{a} = (a_1, \ldots, a_{n+m}) \in U$ and suppose that $f(\bar{a}) = 0$ and the $m \times m$ matrix*

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_{n+1}, \ldots, x_{n+m})}(\bar{a})$$

is invertible. Then there exists V a definable open neighbourhood of $(a_1, \dots, a_n)$ and W a definable open neighbourhood of $(a_{n+1}, \dots, a_{n+m})$ and a definable C^1 function $\phi : V \to R^m$ such that

$$\{\bar{x} \in R^n \times R^m : f(\bar{x}) = 0\} \cap (V \times W) = \operatorname{graph}(\phi).$$

Furthermore, for $\bar{y} \in V$

$$J_\phi(\bar{y}) = -\left(\frac{\partial(f_1, \dots, f_m)}{\partial(x_{n+1}, \dots, x_{n+m})}(\bar{y}, \phi(\bar{y}))\right)^{-1} \left(\frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)}(\bar{y}, \phi(\bar{y}))\right).$$

The usual Taylor's theorem (with the Lagrange form for the remainder) goes through for definable functions. Here we state a version, which we will use in chapter 3, for functions defined on the closed unit box. Let $f : [0,1]^m \to R$ be a definable function. For the purposes of the statement of this lemma only (and contrary to our previous convention), we will say that f is C^n if f has continuous partial derivatives on $[0,1]^m$ upto order n, where limits are taken to be one-sided when necessary.

Theorem 2.2.33 (Taylor's Theorem on the closed unit box)**.** *Let $f : [0,1]^n \to R$ be C^{m+1} and definable. Then for all $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in [0,1]^n$ there exists $\bar{z}$ on the line segment between $\bar{x}$ and $\bar{y}$ such that*

$$f(\bar{y}) = \sum_{j=0}^{m} \left[\frac{1}{j!}\left(\sum_{i=1}^{n}(y_i - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](\bar{x}) + \left[\frac{1}{(m+1)!}\left(\sum_{i=1}^{n}(y_i - x_i)\frac{\partial}{\partial x_i}\right)^{m+1} f\right](\bar{z}).$$

2.2.4 Power functions and exponentials

The elementary results about power functions and exponentials in this section can be found in [13].

2.2.4.1 Power Functions

Let F be an ordered field. A *power function* on F is an endomorphism of the multiplicative group of positive elements of F.

Example 2.2.34. For each $q \in \mathbb{Q}$ the map $x \mapsto x^q : \operatorname{Pos}(\mathbb{R}) \to \operatorname{Pos}(\mathbb{R})$ is a power function. Indeed, for any $r \in \mathbb{R}$ the map $x \mapsto x^r : \operatorname{Pos}(\mathbb{R}) \to \operatorname{Pos}(\mathbb{R})$ (where $x^r = \exp(r \log(x))$) is a power function. Hence note that for any $r \in \mathbb{R}$, the power function $x \mapsto x^r$ is definable in the o-minimal structure $\mathbb{R}_{\exp}$.

Now let $\mathcal{R}$ be an o-minimal expansion of an ordered field and let $f : \mathrm{Pos}(R) \to \mathrm{Pos}(R)$ be a definable power function.

Lemma 2.2.35.

1. *f is identically equal to 1 or f is an automorphism of $\langle \mathrm{Pos}(R), \cdot \rangle$.*

2. *f is differentiable with $f'(r) = f'(1)\frac{f(r)}{r}$.*

3. *f is monotonic; if $f'(1) > 0$ then f is strictly monotone increasing, if $f'(1) < 0$ then f is strictly monotone decreasing and if $f'(1)$ then f is constantly 1.*

Proof.

1. $\langle \mathrm{Pos}(R), \cdot, f \rangle$ is an o-minimal expansion of a group and o-minimal expansions of groups have no proper definable subgroups. The result follows.

2. Note that if $r > 0$ and $h \in R$ is sufficiently small then
$$\frac{f(r+h) - f(r)}{h} = \frac{f(r)}{r}\left(\frac{f(1 + hr^{-1}) - 1}{hr^{-1}}.\right)$$
Consequently f is differentiable at r if and only if f is differentiable at 1. Since f must be differentiable at all but finitely many points (lemma 2.2.28) we see that f is everywhere differentiable with derivative as stated.

3. This is immediate from (2) and the mean value theorem for o-minimal structures (lemma 2.2.25).

□

Remark 2.2.36. Note that the map $x \mapsto \frac{f(x)}{x}$ is a definable power function. Consequently, it follows from (2) that f is infinitely differentiable.

Given a definable power function f in $\mathcal{R}$, we call $f'(1)$ its exponent and, following the convention in $\mathbb{R}$, if $f'(1) = r$ we denote f by x^r. The following lemma justifies this notation:

Lemma 2.2.37. *Let f and g be definable power functions in $\mathcal{R}$ and suppose that $f'(1) = g'(1)$. Then $f = g$.*

Proof. Note that $\frac{f}{g}$ is a power function with exponent 0. The result follows from lemma 2.2.35 (3). □

The next lemma states further elementary properties of power functions definable in an o-minimal expansion of an ordered field $\mathcal{R}$.

Lemma 2.2.38. *Let x^r, x^s be power functions definable in $\mathcal{R}$.*

1. *The product $x^r \cdot x^s$ is a (definable) power function with exponent $r + s$.*

2. *The quotient $\frac{1}{x^r}$ is a power function with exponent $-r$.*

3. *$x \mapsto 1$ is a power function with exponent 0.*

4. *The composition $(x^r)^s$ is a power function with exponent rs.*

5. *The compositional inverse of x^r is a power function with exponent r^{-1}.*

6. *The identity map $x \mapsto x$ is a power function with exponent 1.*

Proof. Immediate. □

It follows immediately from lemma 2.2.38 that for $\mathcal{R}$ an o-minimal expansion of a field, the set of $r \in R$ such that there exists a definable power function in $\mathcal{R}$ with exponent r is a subfield of $\mathcal{R}$. This field is called the *field of exponents* of $\mathcal{R}$.

2.2.4.2 Exponentials

Let F be an ordered field. An exponential on F is a homomorphism from the additive group of F to the multiplicative group of positive elements of F.

Example 2.2.39. The map $x \mapsto \exp(x) : \mathbb{R} \to \mathbb{R}$ is an exponential function on the ordered field of real numbers. Note that exp is of course definable in the o-minimal structure $\mathbb{R}_{\exp}$.

Now let $\mathcal{R}$ be an o-minimal expansion of a field and let f be a definable exponential function on $\mathcal{R}$.

Lemma 2.2.40.

1. *If f is not identically equal to 1 then f is an isomorphism between $\langle R, + \rangle$ and $\langle \mathrm{Pos}(R), \cdot \rangle$.*

2. *f is differentiable with $f'(x) = f'(0)f(x)$.*

3. *f is monotonic; if $f'(0) > 0$ then f strictly monotone increasing, if $f'(0) < 0$ then f is strictly monotone decreasing and if $f'(0) = 0$ then f is identically equal to 1.*

Proof. The proof is similar to that of lemma 2.2.35. $\square$

Remark 2.2.41. It follows from (2) that a definable exponential function in an o-minimal structure is in fact infinitely differentiable.

Lemma 2.2.42. *If f and g are definable exponential functions in the structure $\mathcal{R}$ and $f'(0) = g'(0)$ then $f = g$.*

Proof. Note that $\frac{f}{g}$ is an exponential. Now use lemma 2.2.40 part (3). $\square$

Remark 2.2.43. Note that if $\mathcal{R}$ defines a non-trivial exponential f then for any $r \in R$ one obtains a definable exponential g in $\mathcal{R}$ such that $g'(0) = r$ by setting

$$g(x) = f\left(\frac{rx}{f'(0)}\right).$$

Lemma 2.2.44. *If $\mathcal{R}$ defines a non-trivial exponential function then $\mathcal{R}$ has field of exponents R.*

Proof. Let f be a non-trivial exponential function definable in $\mathcal{R}$ and let $r \in R$. The map $x \mapsto f(rf^{-1}(x)) : \mathrm{Pos}(R) \to \mathrm{Pos}(R)$ is a power function with exponent r. $\square$

Lemma 2.2.45. *If f is a definable exponential function in the structure $\mathcal{R}$ and $f'(0) > 0$ then for all $r \in R$*

$$\frac{f(x)}{x^r} \to +\infty \quad as \quad x \to +\infty.$$

Proof. Let $r \in R$ and let $g(x) = \frac{f(x)}{x^{r+1}}$. Then

$$g'(x) = \frac{f(x)}{x^{r+2}}\left(xf'(0) - r + 1\right).$$

So g' is eventually positive. Since $g(x)$ is always positive we can find $R > 0$ such that $g(x) > R$ for large x. Therefore $\frac{f(x)}{x^r} = xg(x) \to +\infty$ as $x \to +\infty$. $\square$

We call an o-minimal expansion of a field *exponential* if it defines a non-trivial exponential function.

Lemma 2.2.46. *Let $\mathcal{R}$ be exponential o-minimal expansion of a field. Then $\mathcal{R}$ 0-defines the exponential function f satisfying $f'(0) = 1$.*

Proof. Suppose that $\mathcal{R}$ defines a non-trivial exponential function. By remark 2.2.43, $\mathcal{R}$ defines (with parameters) the unique exponential function f satisfying $f'(0) = 1$. Let $\phi(\bar{z}, x, y)$ be a formula (without parameters) and let $\bar{r}$ be a tuple in R such that $\phi(\bar{r}, x, y)$ defines the function $f(x) = y$. The set consisting of parameters $\bar{a}$ such that $\phi(\bar{a}, x, y)$ defines $f(x) = y$ is 0-definable by a formula (with free variables $\bar{z}$) saying

$\phi_{\bar{z}}(x, y)$ is the graph of an exponential function with derivative at 0 equal to 1.

Let $\theta(\bar{z})$ be such a formula. Then $\exists \bar{z}(\theta(\bar{z}) \wedge \phi(\bar{z}, x, y))$ defines $f(x) = y$ without parameters. $\square$

Corollary 2.2.47. *Let $\mathcal{R}$ be an exponential o-minimal structure and let $\mathcal{R}' \equiv \mathcal{R}$. Then $\mathcal{R}'$ is exponential.*

2.2.4.3 Miller's dichotomy and piecewise uniform asymptotics

We say that an o-minimal expansion of a field $\mathcal{R}$ is *power-bounded* if for all definable functions $f : R \to R$ there exists k in the field of exponents of R such $x^k > f(x)$ for all sufficiently large x. We say that $\mathcal{R}$ is *polynomially bounded* if $\mathcal{R}$ is power-bounded and has Archimedean field of exponents (so that for any definable function $f : R \to R$ there exists $n \in \mathbb{N}$ such that $x^n > f(x)$ for all sufficiently large x). By lemma 2.2.45 an exponential o-minimal structure is not power-bounded. Perhaps surprisingly, the converse holds.

Theorem 2.2.48 (Miller's dichotomy [13])**.** *Let $\mathcal{R}$ be an o-minimal expansion of a field. If $\mathcal{R}$ is not power bounded then $\mathcal{R}$ is exponential.*

Corollary 2.2.49. *If $\mathcal{R}$ is a power-bounded and $\mathcal{R}' \equiv \mathcal{R}$ then $\mathcal{R}'$ is power bounded.*

Proof. This follows from corollary 2.2.47 and theorem 2.2.48. $\square$

We now restrict our attention to power-bounded o-minimal expansions of fields. So let $\mathcal{R}$ be a power-bounded o-minimal expansion of a field with field of exponents K. The next theorem tells us that every definable function $f : R \to R$ is asymptotic to a constant multiple of a definable power function x^r (we call r the exponent at ∞ of f) and furthermore that a definable family of functions $R \to R$ can have only finitely many different exponents at ∞.

Theorem 2.2.50 (Piecewise uniform asymptotics [13])**.** *Let $f : A \times M \to M$ be definable, where $A \subseteq M^n$. Suppose that for all $\bar{a} \in A$ the map $x \mapsto f(\bar{a}, x)$ is ultimately non-zero. Then there exists $k_1, \ldots, k_m \in K$ and a definable function $b : A \to M$ such that for any fixed $\bar{a} \in M^n$ there exists $i \in \{1, \ldots, m\}$ such that the function $x \mapsto f(\bar{a}, x)$ is asymptotic to $b(a)x^{k_i}$.*

As a consequence of this theorem we see that if x^r is definable in $\mathcal{R}$, a power-bounded o-minimal expansion of a field with field of exponents K, then x^r is 0-definable in $\mathcal{R}$. Hence K is contained within the prime elementary submodel of $\mathcal{R}$ and if $\mathcal{R}' \equiv \mathcal{R}$ then the isomorphism between their prime models restricts to an isomorphism between their fields of exponents.

2.2.5 Valuation theory

Let F be any ordered field and let V be a convex subring of F (containing 1). One easily sees that V is a valuation ring for F, i.e. for any $x \in F \setminus \{0\}$ we have $x \in V$ or $x^{-1} \in V$. So the pair (F, V) forms a valued field. We form the residue field and value group in the usual way. Let us describe this process. The ring V has unique maximal ideal μ_V given by

$$\mu_V = \{r \in F^\times : |r|^{-1} > x \text{ for all } x \in V\} \cup \{0\};$$

note that μ_V is also convex in R. Since μ_V is a maximal ideal in V the quotient V/μ_V is a field, which is known as the *residue field* and denoted by $\overline{V}$. The image of $r \in F$ under the quotient map $V \to \overline{V}$ is denoted by $\bar{r}$. The group of units of V, denoted $\mathrm{Un}(V)$, is given by $V \setminus \mu_V$. The quotient $F^\times/\,\mathrm{Un}(V)$ is called the *value group* and is denoted by Γ_V. We let v denote the quotient map $F^\times \to \Gamma_V$. As is customary we write Γ_V as an additive group so we have

(v1) $v(xy) = v(x) + v(y)$.

We give Γ_V the structure of a totally ordered group by setting

$$v(x) \geq 0 \text{ if and only if } x \in V.$$

One checks easily that

(v2) $v(x + y) \geq \min\{v(x), v(y)\}$.

We will make use of the following lemma which holds for any valued field.

Lemma 2.2.51. *Let $x_1, \ldots, x_n \in F^\times$ be such that $v(x_i) \leq v(x_j)$ whenever $i \leq j$ and suppose that $x_1 + \ldots + x_n = 0$. Then $v(x_1) = v(x_2)$.*

The following statement describes the interaction between the ordering on the value group Γ_V and the ordering on the field F:

$v(x) \geq v(y)$ if and only if $|x| \leq r|y|$ for some $r \in V$.

Example 2.2.52. Let F be any ordered field and let $\mathrm{Fin}(F)$ be the convex hull of $\mathbb{Q}$ in F; i.e. $\mathrm{Fin}(F) = \{x \in F : |x| < q \text{ for some } q \in \mathbb{Q}\}$. Then $\mathrm{Fin}(F)$ is a convex subring of F containing 1. If $\mathrm{Fin}(F) = F$ we say that F is *Archimedean*, otherwise we say that F is *non-Archimedean*. We call the elements of $\mathrm{Fin}(F)$ *finite*, the elements of the maximal ideal *infinitesimals* and the elements of $F \setminus \mathrm{Fin}(F)$ *infinite*. Notice that we have $v(x) = v(y)$ if and only if there exists $n, m \in \mathbb{N}$ such that $n|x| \geq |y|$ and $m|y| \geq |x|$, i.e. if and only if x and y are in the same Archimedean class of F.

Now let $\mathcal{R}$ be a power-bounded o-minimal expansion of a field with field of exponents K and let V be a convex subring of $\mathcal{R}$. If $\mathrm{Un}(V)$ is closed under raising to powers from K then one can give Γ_V the structure of a K-vector space by setting $kv(x) = v(x^k)$ for each $x \in \mathcal{R}^\times$ and $k \in K$. That this is well-defined follows from the assumption that $\mathrm{Un}(V)$ is closed under raising to powers from K.

2.2.5.1 T-convexity

We will now introduce the notion of a T-convex subring on an o-minimal expansion of a field $\mathcal{R}$. These will be convex subrings of $\mathcal{R}$ which respect the additional structure on $\mathcal{R}$. For a full development of the theory of T-convex subrings see [25], [23].

Notation 2.2.53. The 'T' in the statement "V is a T-convex subring of $\mathcal{R}$" is tacitly assumed to be the theory of $\mathcal{R}$. If we have already named the theory of $\mathcal{R}$ to be something other than 'T', for instance T', we will refer to a T'-convex subring of $\mathcal{R}$.

So let $\mathcal{R}$ be an o-minimal expansion of a field and let T be its theory. A convex subring of V is said to be T-convex if for every 0-definable continuous function $f : R \to R$ we have that $f(V) \subseteq V$.

Example 2.2.54. Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of a field with an Archimedean prime model. Let T denote the theory of $\mathcal{R}$. We claim that *any* convex subring of $\mathcal{R}$ is T-convex. To see this let V be a convex subring of $\mathcal{R}$ and let $f : \mathcal{R} \to \mathcal{R}$ be a 0-definable continuous function. Since $\mathcal{R}$ is polynomially bounded and has Archimedean prime model there exists $n, m \in \mathbb{N}$ such that

$$T \models \forall x(|x| > m \to |f(x)| < |x|^n).$$

Furthermore by theorem 2.2.11 there exists $M \in \mathbb{N}$ such that

$$T \models \forall x(|x| \leq m \to |f(x)| \leq M).$$

So

$$T \models \forall x(|f(x)| \leq \max\{M, |x|^n\}).$$

Let $a \in V$ then $|f(a)| \leq \max\{M, |a|^n\} \in V$. So we are done.

There are convex subrings which are not T-convex.

Example 2.2.55. Let $\mathcal{R}$ be a non-Archimdean model of $T_{\exp}$. Let a be a positive infinite element of R and let V be the convex hull of the subring of $\mathcal{R}$ generated by a. Clearly this is a convex subring of $\mathcal{R}$ but it is not closed under exp so it is not a $T_{\exp}$-convex subring.

The following theorem characterizes T-convex subrings.

Theorem 2.2.56 ([25])**.** *Let $\mathcal{R}$ be an o-minimal expansion of a field and let V be a convex subring of $\mathcal{R}$. Then V is T-convex if and only if V is the convex hull of an elementary substructure of $\mathcal{R}$.*

That convex hulls of elementary substructures give T-convex subrings is straightforward. The reverse implication is a consequence of the following lemma.

Lemma 2.2.57. *Let $\mathcal{R}$ be an o-minimal expansion of a field and let V be a T-convex subring of $\mathcal{R}$. Let*

$$\mathfrak{C} = \{\mathcal{R}' : \mathcal{R}' \preccurlyeq \mathcal{R} \text{ and } \mathcal{R}' \subseteq V\}$$

and consider $\mathfrak{C}$ as partially ordered by inclusion (note that $\mathfrak{C}$ is non-empty because it contains the prime elementary submodel of $\mathcal{R}$). Then $\mathcal{S}$ is maximal in $\mathfrak{C}$ if and only if $\overline{\mathcal{S}} = \overline{V}$, i.e. the image of $\mathcal{S}$ under the residue map is $\overline{V}$. Furthermore if $\mathcal{S}_1$ and $\mathcal{S}_2$ are maximal in $\mathfrak{C}$ then there is a unique isomorphism $h : \mathcal{S}_1 \to \mathcal{S}_2$ such that $\overline{x} = \overline{h(x)}$.

Remark 2.2.58. Note that if $\mathcal{R}'$ is a maximal amongst elementary substructures of $\mathcal{R}$ contained in V then the map $x \mapsto \bar{x} : \mathcal{R}' \to \overline{V}$ is a ring isomorphism. Using this isomorphism we can make $\overline{V}$ into a model of T. Lemma 2.2.57 tells us that this process is in fact independent of our choice $\mathcal{R}'$. From now on whenever we refer to a residue field arising from a T-convex valuation ring we consider it as a model of T.

2.2.5.2 The Valuation Inequality and the Valuation Property

In this section we state the Valuation Inequality and the Valuation Property for power-bounded o-minimal theories.

Let T be a complete power-bounded o-minimal theory with field of exponents K. Note that for any model $\mathcal{R}$ of T, if V is any T-convex subring of $\mathcal{R}$ then $\mathrm{Un}(V)$ is closed under raising to powers from K so we may consider the corresponding value group as a K-vector space.

For the remainder of this section we fix the following notation. Let $\mathcal{R}, \mathcal{S} \models T$ such that $\mathcal{R} \preccurlyeq \mathcal{S}$. Let $V_{\mathcal{R}}$, $V_{\mathcal{S}}$ be T-convex subrings of $\mathcal{R}$ and $\mathcal{S}$ respectively and suppose that

$$V_{\mathcal{S}} \cap R = V_{\mathcal{R}} \tag{2.1}$$

(equivalently that $\langle \mathcal{R}, V_{\mathcal{R}} \rangle \subseteq \langle \mathcal{S}, V_{\mathcal{S}} \rangle$). Let $\Gamma_{\mathcal{R}}$ and $\overline{V}_{\mathcal{R}}$ denote the value group and residue field of $\mathcal{R}$ with respect to $V_{\mathcal{R}}$. Likewise let $\Gamma_{\mathcal{S}}$ and $\overline{V}_{\mathcal{S}}$ denote the value group and residue field of $\mathcal{S}$ with respect to $V_{\mathcal{S}}$. It follows from our assumption (2.1) that we can naturally consider $\Gamma_{\mathcal{R}}$ as K-vector subspace of $\Gamma_{\mathcal{S}}$; we denote the valuation map by v. By the same assumption we may consider $\overline{V}_{\mathcal{R}}$ as a T-elementary substructure of $\overline{V}_{\mathcal{S}}$.

Theorem 2.2.59 (The Valuation Inequality, [23])**.** *Suppose* $\mathrm{rk}(\mathcal{S}|\mathcal{R}) < \infty$. *Then*

$$\mathrm{rk}\left(\mathcal{S}|\mathcal{R}\right) \geq \mathrm{rk}\left(\overline{V}_{\mathcal{R}}|\overline{V}_{\mathcal{S}}\right) + \dim_K\left(\Gamma_{\mathcal{R}}/\Gamma_{\mathcal{S}}\right).$$

Remark 2.2.60. Consider the special case where $\mathcal{S} = \mathcal{R}\langle a \rangle$ where $a \notin R$, so that $\mathrm{rk}(\mathcal{S}|\mathcal{R}) = 1$. Then theorem 2.2.59 says that either $\overline{V}_{\mathcal{R}} = \overline{V}_{\mathcal{S}}$ or $\Gamma_{\mathcal{R}} = \Gamma_{\mathcal{S}}$, i.e. for a rank 1 extension either the value group doesn't extend or the residue field doesn't extend. Notice that if we also have that $v(a) \notin \Gamma_{\mathcal{R}}$ then

$$\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{R}} \oplus K v(a).$$

We next state the Valuation Property for power-bounded o-minimal theories.

Theorem 2.2.61 (The Valuation Property, [19])**.** *Suppose that* $\mathcal{S} = \mathcal{R}\langle a \rangle$ *where* $a \notin R$. *Suppose further that* $\Gamma_{\mathcal{S}} \neq \Gamma_{\mathcal{R}}$. *Then there exists* $r \in R$ *such that* $v(a - r) \notin \Gamma_{\mathcal{R}}$.

Let us give a proof in the case that $T = T_{\mathrm{RCF}}$. First we prove a preliminary lemma.

Lemma 2.2.62. *Let R, S be real-closed fields and suppose that $R \subset S$. Let V_R and V_S be convex subrings of R and S respectively and suppose that $V_S \cap R = V_R$. Let v denote the common valuation. Let $a \in S \setminus R$ and suppose that $v(R(a)^\times) \neq v(R^\times)$. Then there exists $r \in R$ such that $v(a - r) \notin v(R^\times)$.*

Proof. Since $v(R(a)^\times) \neq v(R^\times)$ we must have $q(X) \in R[X]$ such that $v(q(a)) \notin v(R^\times)$. Since R is real closed

$$q(X) = \prod_i (X - c_i) \prod_j ((X - d_j)^2 + h_j^2)$$

for some $c_i, d_j, h_j \in R$. Suppose that $v(a - c_i), v(a - d_j) \in v(R^\times)$ for all i, j. Then for some j we must have that $v((a - d_j)^2 + h_j^2) \notin v(R^\times)$. By our assumption, $v(a - d_j) \in v(R^\times)$ and of course $v(h_j) \in v(R^\times)$ therefore we must have that $v((a - d_j)^2 + h_j^2) > v(h_j^2)$ and hence $0 < (a - d_j)^2 + h_j^2 < h_j^2$, which is a contradiction. $\square$

Proof of theorem 2.2.61 for the special case where $T = T_{\mathrm{RCF}}$. Let R and S be real-closed fields such that $R \subseteq S$. Furthermore suppose that V_R is a T_{RCF}-convex subring of R and V_S is a T_{RCF}-convex subring of S such that $V_S \cap R = V_R$ (note that, by example 2.2.54, in fact *any* convex subring of a real closed field is T_{RCF}-convex). Let v denote the common valuation. Let $a \in S \setminus R$ and suppose that $v(R\langle a\rangle^\times) \neq v(R^\times)$. By lemma 2.2.62 it will be sufficient to prove that $v(R(a)^\times) \neq v(R^\times)$. Choose $b \in R\langle a\rangle^\times$ such that $v(b) \notin v(R^\times)$. Since $b \in R\langle a\rangle$ there exists $p(X, Y) \in R[X, Y] \setminus \{0\}$ such that $p(a, b) = 0$. Now

$$p(X, Y) = \sum_j q_j(X) Y^j$$

for some $q_j(X) \in R[X]$. Since $p(a, b) = 0$, by lemma 2.2.51, there exists distinct $l, m \geq 0$ such that $q_l(a), q_m(a) \neq 0$ and $v(q_l(a) b^l) = v(q_m(a) b^m)$. Since we are assuming that $v(b) \notin v(R^\times)$ one easily deduces that there exists $q(X) \in R[X]$ such that $v(q(a)) \notin v(R^\times)$. Hence $v(R(a)^\times) \neq v(R^\times)$. $\square$

2.2.6 Hardy fields

Let K be an ordered field and $\mathfrak{F}$ be a ring of functions $K \to K$ containing all constant functions and the identity function. We will say that a property $P(x)$ holds *ultimately* if $P(x)$ holds for all sufficiently large $x \in K$. Now let $\mathfrak{I}$ be the ideal of $\mathfrak{F}$ consisting of all those $f \in \mathfrak{F}$ that are *ultimately zero*. We call $\mathcal{H} = \mathfrak{F}/\mathfrak{I}$ the ring of germs at $+\infty$ of $\mathfrak{F}$. Notice that if $f, g \in \mathfrak{F}$ then $f + \mathfrak{I} = g + \mathfrak{I}$ if and only if f and g are ultimately equal. Where no confusion should arise we will often not distinguish notationally between $f \in \mathfrak{F}$ and its germ at $+\infty$ (i.e. its coset in $\mathcal{H}$). We consider K as a subfield of $\mathcal{H}$ under the embedding which takes $k \in K$ to the constant function $x \mapsto k$. Now suppose that $\mathcal{H}$ is a field. Notice this is the case if and only if for all $f \in \mathfrak{F} \setminus \mathfrak{I}$ there exists $g \in \mathfrak{F}$ such that fg is ultimately equal to 1; in particular we must have that f is ultimately non-zero. If we also know that for all $f \in \mathfrak{F}$ the function f has ultimately constant sign then $\mathcal{H}$ becomes an ordered field when we say that $f < g$ if and only if $g - f$ is ultimately positive. We say that $\mathcal{H}$ is a *Hardy Field* if all $f \in \mathfrak{F}$ are ultimately differentiable and $\mathfrak{F}$ is closed under differentiation (clearly this induces a derivation on $\mathcal{H}$). Notice that if $K = \mathbb{R}$ then assuming that $f \in \mathfrak{F}$ is ultimately non-zero and ultimately differentiable implies that f has ultimately constant sign.

Example 2.2.63. Let $\mathfrak{F} = \mathbb{R}(x)$, the ring of rational functions on $\mathbb{R}$. Clearly $\mathbb{R}(x)$ induces a Hardy field (and in this case $\mathfrak{I} = \{0\}$).

So let $\mathcal{H}$ be a Hardy field on the ordered field K and let V be the convex hull of K in $\mathcal{H}$. Clearly V is a convex subring of $\mathcal{H}$ and so induces a valuation on $\mathcal{H}$. This is called the canonical valuation on the Hardy field.

2.2.6.1 Hardy fields of definable functions in o-minimal expansions of fields

Let $\mathcal{R}$ be an o-minimal expansion of a field. It follows from the o-minimality of $\mathcal{R}$ that the ring of definable functions $R \to R$ induces a Hardy field which we will denote by $\mathcal{H}(\mathcal{R})$. Let v denote the canonical valuation on $\mathcal{H}(\mathcal{R})$. Notice that because all definable functions must have a limit at ∞ in $R \cup \{\pm\infty\}$, for all $f, g \in \mathcal{H}(\mathcal{R})$ we have that $v(f) = v(g)$ if and only if there exists $r \in R$ such that $f(x)/g(x) \to r$ as $x \to \infty$.

Let L be the language of $\mathcal{R}$ and let T be its L-theory. We next show that $\mathcal{H}(\mathcal{R})$ carries a natural L-structure under which it becomes a model of T with $\mathcal{R}$ as an elementary substructure. We give $\mathcal{H}(\mathcal{R})$ an L-structure as follows:

1. Given a constant symbol c of L we interpret as the constant function which takes value c (as interpreted in $\mathcal{R}$).

2. Given an n-ary function symbol F of L and $f_1, \dots, f_n \in \mathcal{H}(\mathcal{R})$ we interpret $F(f_1, \dots, f_n)$ as the germ at ∞ of the function $x \mapsto F(f_1(x), \dots, f_n(x))$.

3. Given an n-ary relation symbol P of L and $f_1, \dots, f_n \in \mathcal{H}(\mathcal{R})$ we say that $(f_1, \dots, f_n) \in P$ if and only if $(f_1(x), \dots, f_n(x)) \in P$ for all sufficiently large $x \in R$. That this is well-defined follows from the o-minimality of $\mathcal{R}$.

Lemma 2.2.64. *Considered as an L-structure, $\mathcal{H}(\mathcal{R}) \models T$.*

Proof. Let $\mathcal{S}$ be a proper elementary extension of $\mathcal{R}$ and suppose $\mathcal{S}$ contains elements which are larger than any element of $\mathcal{R}$ (one easily sees that such an $\mathcal{S}$ exists by a compactness argument). Choose such an element a; so for all $r \in R$ we have $a > r$. Note that

$$\mathcal{R}\langle a\rangle = \{f(a) : f : R \to R \text{ is an } R\text{-definable function in } \mathcal{R}\}. \tag{2.2}$$

Let $i_a : \mathcal{H}(\mathcal{R}) \to \mathcal{R}\langle a\rangle$ be given by $i_a(f) = f(a)$. To see that this map is well-defined and injective we note that if f, g, definable unary functions on $\mathcal{R}$, have the same germ at ∞ then there exists $r \in R$ such that

$$\mathcal{R} \models \forall x(x > r \to f(x) = g(x)), \tag{2.3}$$

and if f and g have different germs then there exists $r \in R$ such that

$$\mathcal{R} \models \forall x(x > r \to f(x) \neq g(x)). \tag{2.4}$$

Now $\mathcal{R} \preccurlyeq \mathcal{R}\langle a\rangle$ (since $\mathcal{R} \subseteq \mathcal{R}\langle a\rangle \preccurlyeq \mathcal{S}$ and $\mathcal{R} \preccurlyeq \mathcal{S}$ implies $\mathcal{R} \preccurlyeq \mathcal{R}\langle a\rangle$), therefore we see that (2.3) holds if and only if $f(a) = g(a)$. By (2.2) the map i_a is onto. A similar argument shows that the L-structure is preserved under the mapping i_a. So i_a is an L-isomorphism and hence $\mathcal{H}(\mathcal{R}) \models T$. Since i_a preserves the embeddings of $\mathcal{R}$ in $\mathcal{H}(\mathcal{R})$ and $\mathcal{R}\langle a\rangle$, and $\mathcal{R} \preccurlyeq \mathcal{R}\langle a\rangle$, the embedding of $\mathcal{R}$ in $\mathcal{H}(\mathcal{R})$ is elementary. $\square$

2.2.7 Some further examples of o-minimal structures

In this section we will give properties of some of the known o-minimal structures and theories.

Example 2.2.65. Let T_{RCF} be the theory of real-closed fields in the language

$$L_{\text{ord}} = \{+, \cdot, <, 0, 1\}$$

, i.e. T_{RCF} consists of those L_{ord} sentences which are true for every real-closed field. Since the property of being a real-closed field is first-order axiomatizable in L_{ord} all models of T_{RCF} are real-closed (see for instance [10]). Now $\overline{\mathbb{R}} \models T_{\text{RCF}}$. In example 2.2.4 we remarked that $\text{Th}(\overline{\mathbb{R}})$ has quantifier elimination. In fact T_{RCF} has quantifier elimination. From this one deduces that T_{RCF} is complete (so $T_{\text{RCF}} = \text{Th}(\overline{\mathbb{R}})$) and that all real-closed fields are o-minimal. We will now see that T_{RCF} is polynomially bounded with field of exponents $\mathbb{Q}$. Let $\mathcal{R} \models T_{RCF}$ and let $f : R \to R$ be definable and ultimately non-zero. It follows from the quantifier elimination that there exists a $p(X,Y) \in R[X,Y] \setminus \{0\}$ such that $p(x, f(x)) = 0$ for all sufficiently large x in R. Let $\mathcal{H}(\mathcal{R})$ be the Hardy field of definable functions of $\mathcal{R}$ (see section 2.2.6). Identifying f with its image in $\mathcal{H}(\mathcal{R})$ and denoting the identity function in $\mathcal{H}(\mathcal{R})$ by x we must have $p(x, f) = 0$. Now $p(X,Y)$ is of the form

$$\sum_{|\alpha| \leq N} a_\alpha X^{\alpha_1} Y^{\alpha_2},$$

where each $\alpha \in \mathbb{N}^2$ and $\alpha = (\alpha_1, \alpha_2)$ and $a_\alpha \in R$. Since $p(x, f) = 0$ there exists β, γ such that $v(a_\beta x^{\beta_1} f^{\beta_2}) = v(a_\gamma x^{\gamma_1} f^{\gamma_2})$, where $a_\beta, a_\gamma \neq 0$ and v is the canonical valuation on $\mathcal{H}(\mathcal{R})$. Using elementary properties of v one deduces that there exists $q \in \mathbb{Q}$ such that $v(f/x^q) = 0$. But this implies that there exists $r \in R$ such that $f(x)/rx^q \to 1$ as $x \to \infty$.

Example 2.2.66. For each $n \geq 0$ let $\mathfrak{F}_n$ consist of those functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists U an open neighbourhood of $[0,1]^n$ and $g : U \to \mathbb{R}$ such that g is real analytic and f satisfies

$$f(\bar{x}) = \begin{cases} g(\bar{x}) & \bar{x} \in [0,1]^n, \\ 0 & \bar{x} \in \mathbb{R}^n \setminus [0,1]^n, \end{cases}$$

(let $\mathfrak{F}_0 = \mathbb{R}$). Let $\mathbb{R}_{\text{an}} = \langle \overline{\mathbb{R}}, (\mathfrak{F}_n)_{n \geq 0} \rangle$. A theorem of Gabrielov says that the complement of a subanalytic set is subanalytic. Observations made by van den Dries in [21] show that Gabrielov's theorem implies that $\mathbb{R}_{\text{an}}$ is model-complete in the language described above and that the definable sets are precisely the finitely subanalytic sets. In the same paper van den Dries makes the following additional observations. By a theorem of Łojasiewicz, all finitely subanalytic sets have finitely many connected components, consequently $\mathbb{R}_{\text{an}}$ is o-minimal. Furthermore, using the Weierstrass preparation theorem, Puiseux's theorem and the fact that subanalytic sets in $\mathbb{R}^2$ are in fact seminanalytic one can show that $\mathbb{R}_{\text{an}}$ is polynomially bounded with field of exponents $\mathbb{Q}$. Furthermore, in [2] van den Dries and Denef prove the stronger result that $\mathbb{R}_{\text{an}}$ admits quantifier elimination when expanded by a function symbol for the map $x \mapsto x^{-1}$ (where we set $0^{-1} = 0$). In [26] it is shown that $\mathbb{R}_{\text{an}}$ also has a universal axiomatization when one expands the language by a function symbol for x^q for each $q \in \mathbb{Q}$. We denote the resulting structure by $\mathbb{R}_{\text{an}}^{\mathbb{Q}}$.

Example 2.2.67. Let $\mathbb{R}_{\exp} = \langle \overline{\mathbb{R}}, \exp \rangle$. In [29] Wilkie proves that $\mathbb{R}_{\exp}$ is model-complete. This is equivalent to the statement that all definable subsets in $\mathbb{R}_{\exp}$ are projections of zero-sets of exponentials polynomials (see theorem 2.3.1). Khovanskii's theorem [4] tells us that the zero-set of an exponential-polynomial must have finitely many connected components. This implies the o-minimality of $\mathbb{R}_{\exp}$. Of course $\mathbb{R}_{\exp}$ is not polynomially bounded and has field of exponents $\mathbb{R}$.

2.3 Model-completeness and quantifier elimination

In this section we give some standard results relating to model-completeness and quantifier elimination, all of which can be found in [16].

Throughout this section we will fix a theory T in a language L. Recall that we say that T is model-complete if for any $\mathcal{A}, \mathcal{B} \models T$, if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq \mathcal{B}$.

Theorem 2.3.1. *The following are equivalent:*

1. *T is model-complete.*
2. *If $\mathcal{A} \models T$ then $T \cup \mathrm{Diag}(\mathcal{A})$ is a complete theory.*
3. *If $\phi(\bar{x})$ is an L-formula then there exists an existential L-formula $\psi(\bar{x})$ such that $T \models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.*
4. *If $\mathcal{A}, \mathcal{B} \models T$ and $\mathcal{A} \subseteq \mathcal{B}$ then for any quantifier-free L-formula $\phi(\bar{x}, \bar{y})$ and any tuple $\bar{a}$ in A, if $\mathcal{B} \models \exists \bar{y} \phi(\bar{a}, \bar{y})$ then $\mathcal{A} \models \exists \bar{y} \phi(\bar{a}, \bar{y})$.*
5. *Every diagram of the following type can be completed*

$$\begin{tikzcd} & \mathcal{C} \\ \mathcal{A} \arrow[ur, "\subseteq"] \arrow[r, "\subseteq"'] & \mathcal{B} \arrow[u, dotted, "\subseteq"'] \end{tikzcd}$$

   where $\mathcal{A}, \mathcal{B}, \mathcal{C} \models T$ and $\mathcal{C}$ is $|\mathcal{B}|^+$-saturated.

6. *Every diagram of the following type can be completed*

$$\begin{tikzcd} & \mathcal{C} \\ \mathcal{A} \arrow[ur, "\preccurlyeq"] \arrow[r, "\subseteq"'] & \mathcal{B} \arrow[u, dotted, "\subseteq"'] \end{tikzcd}$$

   where $\mathcal{A}, \mathcal{B}, \mathcal{C} \models T$ and $\mathcal{C}$ is $|\mathcal{B}|^+$-saturated.

Remark 2.3.2. Statement (4) says that all models of T are existentially closed. This is known as Robinson's tests for model-completeness.

Theorem 2.3.3. *The following are equivalent:*

1. *T admits quantifier elimination.*

2. *T is substructure-complete; i.e. if $\mathcal{A} \models T$ and $\mathcal{B} \subseteq \mathcal{A}$ then $T \cup \mathrm{Diag}(\mathcal{B})$ is a complete theory.*

3. *Every diagram of the following type can be completed*

$$\begin{tikzcd} \mathcal{C} \arrow[r, dotted, "\preccurlyeq"] & \mathcal{D} \\ \mathcal{A} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"'] & \mathcal{B} \arrow[u, dotted, "\subseteq"'] \end{tikzcd}$$

where $\mathcal{B}, \mathcal{C} \models T$.

Corollary 2.3.4. *T admits quantifier elimination if every diagram of the following type can be completed*

$$\begin{tikzcd} & \mathcal{C} \\ \mathcal{A} \arrow[ur, "\subseteq"] \arrow[r, "\subseteq"'] & \mathcal{B} \arrow[u, dotted, "\subseteq"'] \end{tikzcd}$$

where $\mathcal{B}, \mathcal{C} \models T$ and $\mathcal{C}$ is $|\mathcal{B}|^+$-saturated.

Remark 2.3.5. Notice that it follows immediately from theorem 2.3.1 (2) and theorem 2.3.3 (2) that a model-complete universal theory admits quantifier elimination.

Chapter 3

A general model-completeness result

In this chapter we will recast a theorem of Wilkie from [29] in a more general situation. For the purposes of readability we reproduce Wilkie's argument here making the few necessary adjustments on the way but we stress that the majority of the proof is identical to Wilkie's original.

The results presented in this chapter and the next appear in a similar form in [3].

3.1 Wilkie's theorem on restricted Pfaffian functions

Let $n \geq 1$ and let U be an open subset of $\mathbb{R}^n$. For $i = 1, \ldots, m$ let $f_i : U \to \mathbb{R}$ be a C_1 function. We say that $(f_1, \ldots, f_m)$ is a *Pfaffian chain of length* m if for each $i = 1, \ldots, m$ and each $j = 1, \ldots, n$ there exists $p_{ij}(X_1, \ldots, X_{n+i}) \in \mathbb{R}[X_1, \ldots, X_{n+i}]$ such that for all $\bar{x} = (x_1, \ldots, x_n) \in U$,

$$\frac{\partial f_i}{\partial x_j}(\bar{x}) = p_{ij}(\bar{x}, f_1(\bar{x}), \ldots, f_i(\bar{x})).$$

If $C \subseteq \mathbb{R}$ we say that $(f_1, \ldots, f_n)$ is a Pfaffian chain over C if the p_{ij} can be chosen to have coefficients in C. We say that $f : U \to \mathbb{R}$ is a *Pfaffian function* if it is a member of a Pfaffian chain of length m for some $m \geq 1$.

Example 3.1.1. The function $\exp : \mathbb{R} \to \mathbb{R}$ satisfies $\exp' = \exp$ therefore $(\exp)$ is a Pfaffian chain (taking $n = 1$, $U = \mathbb{R}$, $m = 1$ and $p_{11}(X_1, X_2) = X_2$).

Example 3.1.2. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{aligned} f_1(x) &= \frac{1}{1+x^2}, \\ f_2(x) &= \arctan(x). \end{aligned}$$

Then $f_2'(x) = f_1(x)$ and $f_1'(x) = -2x f_1(x)$ for all $x \in \mathbb{R}$ so (f_1, f_2) is a Pfaffian chain.

See [27] for a proof of the following fact.

Lemma 3.1.3. *Let $n \geq 1$ and let U be a open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a Pfaffian function. Then f is analytic on U.*

Now let $n \geq 1$ and let U be an open neighbourhood of $[0,1]^n$. Let $(f_1, \ldots, f_m)$ be a Pfaffian chain on U over C. For each $i = 1 \ldots m$ let $g_i : \mathbb{R}^n \to \mathbb{R}$ be given by

$$g_i(\bar{x}) = \begin{cases} f_i(\bar{x}) & \bar{x} \in [0,1]^n, \\ 0 & \bar{x} \in \mathbb{R}^n \setminus [0,1]^n. \end{cases}$$

Theorem 3.1.4 (Wilkie's theorem on restricted Pfaffian functions [29]). *The structure*

$$\tilde{\mathbb{R}} = \langle \overline{\mathbb{R}}, g_1, \ldots, g_m, (c)_{c \in C} \rangle$$

is model-complete.

3.2 A generalization of Wilkie's theorem on restricted Pfaffian functions

We now state our generalization of Wilkie's theorem on restricted Pfaffian functions.

Theorem 3.2.1. *Let L be the first order language consisting of the language of ordered rings $L_{\text{ord}} = \{+, \cdot, <, 0, 1\}$, m-ary function symbols $f_1, \ldots, f_l$ and a set C of constants. Let T be a consistent (and possibly incomplete) L-theory satisfying:*

1. *T expands the theory of real closed fields.*
2. *For each $i = 1, \ldots, l$ we have that f_i is C^∞ on the unit box $[0,1]^m$ and identically zero outside it.*
3. *$(f_1, \ldots, f_m)$ is a Pfaffian chain over C (on $[0,1]^m$).*
4. *For each $i = 1, \ldots, l$ and each $\alpha \in \mathbb{N}^m$ there exists $c_{\alpha,i} \in C$ such that for all $\bar{x} \in [0,1]^m$ we have $|f_i^{(\alpha)}(\bar{x})| < c_{i,\alpha}$.*
5. *All models of T are o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$.*

Then T is model-complete.

Remark 3.2.2. The definition of Paffian chain for an ordered field is the obvious generalization of the definition of Pfaffian chain for $\mathbb{R}$.

Remark 3.2.3. Our notion of C^∞ on $[0,1]^m$ here is as defined before the statement of theorem 2.2.33. With this definition in place it is clear what is meant by (3).

Remark 3.2.4. Conditions (1) - (4) can each be expressed by a scheme of L-sentences.

Let us now observe that theorem 3.2.1 implies theorem 3.1.4. We must check that hypotheses (1)-(5) hold for the structure $\tilde{\mathbb{R}} = \langle \overline{\mathbb{R}}, g_1, \ldots, g_m, (c)_{c \in C} \rangle$. Hypothesis (1) is clear since $\overline{\mathbb{R}}$ is a real-closed field. To see that hypothesis (5) holds note that lemma 3.1.3 implies that $\tilde{\mathbb{R}}$ is a reduct of $\mathbb{R}_{\text{an}}$ (see example 2.2.66). Since $\mathbb{R}_{\text{an}}$ is o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$ the result follows. Clearly (2) and (3) are satisfied. Now (4) may not be satisfied by $\tilde{\mathbb{R}}$ but we may of course expand C by constant symbols for all natural numbers so that (4) is satisfied. Since we are just adding constants to name elements already interpreted by closed terms, model-completeness in the expanded language implies model-completeness in the original language. Thus $\tilde{\mathbb{R}}$ is model-complete.

3.3 The proof of theorem 3.2.1

We will use Robinson's test (see remark 2.3.2) to prove that T is model-complete. So we take $\mathcal{H}, \mathcal{K} \models T$ and suppose that $\mathcal{H} \subseteq \mathcal{K}$. We must show that any existential formula with parameters from H which is true in $\mathcal{K}$ is also true in $\mathcal{H}$.

Now, just as in section 2 of [29], we may assume that our existential formula is of the form

$$\exists x_1, \ldots, \exists x_r \bigwedge_{i=1}^{n} \chi(x_1, \ldots, x_r),$$

where each χ_i is of one of the following forms:

1. $p(x_1, \ldots, x_r) = 0$ where $p \in H[x_1, \ldots, x_r]$,
2. $\left(\bigwedge_{j \in S} 0 < x_{i_j} < 1\right) \wedge f_k(x'_{i_1}, \ldots, x'_{i_m}) - x_{i_{m+1}} = 0$, where $S \subseteq \{1, \ldots, m\}$ and $1 \leq i_1, \ldots, i_{m+1} \leq r$ and

$$x'_{i_j} = \begin{cases} x_{i_j} & j \in S, \\ 0 \text{ or } 1 & j \notin S. \end{cases}$$

We will break up the proof of theorem 3.2.1 into two lemmas (lemma 3.3.7 and lemma 3.3.8). First we introduce some terminology.

Definition 3.3.1. Let $n, r \in \mathbb{N}$. An (n, r)-*sequence* is a sequence $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of L-terms in variables $x_1, \ldots, x_r$ such that

1. for $s = 1, \ldots, n$, the term σ_s has the form $f_i(y_1, \ldots, y_m)$ for some $i = 1, \ldots, l$ and $y_1, \ldots, y_m \in \{0, 1, x_1, \ldots, x_r\}$,
2. if $1 \leq s \leq n$ and $1 < i \leq l$ and σ_s is $f_i(y_1 \ldots, y_m)$ then for some s' with $1 \leq s' < s$ the term $\sigma_{s'}$ is $f_{i-1}(y_1, \ldots, y_m)$.

Those variables which actually occur in some term of the (n, r)-sequence $\bar{\sigma}$ are called $\bar{\sigma}$-*bounded*.

Remark 3.3.2. An (n, r)-sequence is naturally an (n, s)-sequence for any $s \geq r$.

Definition 3.3.3. Let $\bar{\sigma}$ be an (n,r)-sequence. We define the *natural domain* of $\bar{\sigma}$, which we denote by $D^r(\bar{\sigma})$, as $\prod_{j=1}^r I_j$ where

$$I_j = \begin{cases} (0,1) & x_i \text{ is } \bar{\sigma}\text{-bounded}, \\ K & \text{otherwise}, \end{cases}$$

(where the interval $(0,1)$ is taken in K).

Note that $D^r(\bar{\sigma})$ is open in K^r.

Definition 3.3.4. Let $\bar{\sigma}$ be an (n,r)-sequence. We define $M^r(\bar{\sigma})$ to be the ring of functions $f : D^r(\bar{\sigma}) \to K$ such that there exists a polynomial

$$p \in H[x_1, \ldots, x_r, y_1, \ldots, y_n]$$

with

$$f(\bar{x}) = p(\bar{x}, \sigma_1(\bar{x}), \ldots, \sigma_n(\bar{x}))$$

for all $\bar{x} \in D^r(\bar{\sigma})$.

Remark 3.3.5.

1. If $g \in M^r(\bar{\sigma})$ then g is smooth on $D^r(\bar{\sigma})$.
2. By the definition of an (n,r)-sequence, $M^r(\bar{\sigma})$ is closed under partial differentiation.

Clearly it is sufficient for us to prove that if $\bar{\sigma}$ is an (n,r)-sequence, $g_1, \ldots, g_k \in M^r(\bar{\sigma})$ and $g_1, \ldots, g_k$ have a common zero in $D^r(\bar{\sigma})$ then they have one in $D^r(\bar{\sigma}) \cap H^r$.

Definition 3.3.6. Let $\bar{\sigma}$ be an (n,r)-sequence. A point $\bar{a} \in D^r(\bar{\sigma})$ is said to be $\bar{\sigma}$-*definable* if there exists $g_1, \ldots, g_r \in M^r(\bar{\sigma})$ such that

1. $g_1(\bar{a}) = \ldots = g_r(\bar{a}) = 0$,
2. $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(\bar{a}) \neq 0$.

It will be sufficient to prove the following two lemmas.

Lemma 3.3.7. *For any (n,r)-sequence $\bar{\sigma}$, every $\bar{\sigma}$-definable point of K^r lies in H^r.*

Lemma 3.3.8. *Let $\bar{\sigma}$ be an (n,r)-sequence and let $g \in M^r(\bar{\sigma})$ be such that there exists $\bar{a} \in D^r(\bar{\sigma})$ such that $g(\bar{a}) = 0$. Then there exists $s \geq r$ and $\bar{b} \in K^{s-r}$ and $\bar{b}' \in K^r$ such that $(\bar{b}', \bar{b})$ is $\bar{\sigma}$-definable and $g(\bar{b}') = 0$.*

3.3.1 Proof of lemma 3.3.8

First let us set up some notation. Given $\bar{\sigma}$ an (n, r)-sequence and $g_1, \ldots, g_k \in M^r(\bar{\sigma})$ let

$$\begin{aligned} V(g_1, \ldots, g_k) =& \{\bar{x} \in D^r(\bar{\sigma}) : g_1(\bar{x}) = \ldots = g_k(\bar{x}) = 0\}, \\ V_{\text{reg}}(g_1, \ldots, g_k) =& \{\bar{x} \in V(g_1, \ldots, g_k) : J_{g_1}(\bar{x}), \ldots, J_{g_k}(\bar{x}) \text{ are } K\text{-linearly independent}\}. \end{aligned}$$

The following lemma is a restatement of Theorem 4.9 in [29] but the proof given here is based on the proof of Theorem 2.5 of [5].

Lemma 3.3.9. *Let $\bar{\sigma}$ be an (n, r)-sequence and let $g_1, \ldots, g_k \in M^r(\bar{\sigma})$. Suppose that $\bar{a} \in V_{\text{reg}}(g_1, \ldots, g_k)$. Then one of the following holds:*

1. *$k = r$,*
2. *$k < r$ and for any $h \in M^r(\bar{\sigma})$, if h vanishes at $\bar{a}$ then h vanishes on $U \cap V_{\text{reg}}(g_1, \ldots, g_k)$ for some definable open neighbourhood U of $\bar{a}$,*
3. *$k < r$ and there exists $h \in M^r(\bar{\sigma})$ such that $\bar{a} \in V_{\text{reg}}(g_1, \ldots, g_k, h)$.*

Proof. Clearly if $k \neq r$ then $k < r$. So suppose that $k < r$ and that (2) does not hold. So there exists $h \in M^r(\bar{\sigma})$ such that h vanishes at $\bar{a}$ but h does not vanish on $U \cap V_{\text{reg}}(g_1, \ldots, g_k)$ for any definable open neighbourhood of $\bar{a}$. Now since $\bar{a} \in V_{\text{reg}}(g_1, \ldots, g_k)$ there is some $k \times k$ submatrix of

$$\frac{\partial(g_1, \ldots, g_k)}{\partial(x_1 \ldots, x_r)}(\bar{x})$$

whose determinant is non-zero when evaluated at $\bar{a}$. For ease of notation we assume that this submatrix consists of the last k columns. Let $\Delta(x_1, \ldots, x_r)$ be its determinant. Note that $\Delta \in M^r(\bar{\sigma})$. For $\bar{y} \in (y_1, \ldots, y_r)$, let $\tilde{y} = (y_1, \ldots, y_{r-k})$. Now, by the implicit function theorem for o-minimal expansions of fields (theorem 2.2.32) there is a definable open neighbourhood $U \subseteq K^{r-k}$ of $\tilde{a}$ and a smooth definable map $\phi : U \to K^k$ satisfying

(a) $(\tilde{a}, \phi(\tilde{a})) = \bar{a}$,

(b) $\{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\} = W \cap V_{\text{reg}}(g_1, \ldots, g_k)$ for some definable open neighbourhood $W \subseteq K^r$ of $\bar{a}$.

Furthermore, if we write $\phi = (\phi_1, \ldots, \phi_k)$ then for $\tilde{y} \in U$ and $j = 1, \ldots, r-k$

$$\begin{pmatrix} \frac{\partial \phi_1}{\partial x_j} \\ \vdots \\ \frac{\partial \phi_k}{\partial x_j} \end{pmatrix} = -\left(\frac{\partial(g_1, \ldots, g_k)}{\partial(x_{r-k+1}, \ldots, x_r)}\right)^{-1} \begin{pmatrix} \frac{\partial g_1}{\partial x_j} \\ \vdots \\ \frac{\partial g_k}{\partial x_j} \end{pmatrix}, \tag{3.1}$$

where the left-hand side is evaluated at $\tilde{y}$ and the righthand-side at $(\tilde{y}, \phi(\tilde{y}))$.

Now we may assume that Δ has no zeros in W. Now consider the definable function $G : U \to K$ given by

$$\tilde{y} \mapsto h(\tilde{y}, \phi(\tilde{y})).$$

Now G is not identically zero on U so, since K is polynomially bounded, by a theorem of Miller in [12], there exists $\alpha \in \mathbb{N}^{r-k}$ such that $G^* = \partial^\alpha G$ vanishes at $\tilde{a}$ but for some $j = 1, \ldots, m$ the partial derivative $\frac{\partial G^*}{\partial x_j}$ does not. Now by (3.1)

$$G^*(\tilde{y}) = \frac{h'(\tilde{y}, \phi(\tilde{y}))}{\Delta(\tilde{y}, \phi(\tilde{y}))^d}$$

for some $h' \in M^r(\bar{\sigma})$ and some $d \geq 0$. By the method of Lagrange multipliers

$$J_{g_1}(\bar{a}), \ldots, J_{g_k}(\bar{a}), J_{h'}(\bar{a}),$$

are linearly independent over K if and only if the map $\tilde{y} \mapsto h'(\tilde{y}, \phi(\tilde{y}))$ has non-zero differential at $\bar{a}$. Since $G^*(\tilde{a}) = 0$ and $\Delta(\bar{a}) \neq 0$ this follows from the fact that $\frac{\partial G^*}{\partial x_j}(\tilde{a}) \neq 0$. So the result follows by taking $h = h'$. $\square$

We are now in a position to prove the following theorem:

Theorem 3.3.10. *Let $\bar{\sigma}$ be an (n, r)-sequence. Let $g \in M^r(\bar{\sigma})$ be such that the set $V(g)$ is non-empty and closed in K^r. Then there exists $g_1, \ldots, g_r \in M^r(\bar{\sigma})$ such that $V_{\text{reg}}(g_1 \ldots, g_r) \cap V(g)$ is non-empty; i.e. $V(g)$ contains a $\bar{\sigma}$-definable point.*

Proof. For each $\bar{a} \in V(g)$ let $I_{\bar{a}} = \{h \in M^r(\bar{\sigma}) : h(\bar{a}) = 0\}$. Now $M^r(\bar{\sigma})$ is a finitely generated algebra over a field and hence it is Noetherian. Therefore we may choose $\bar{b} \in V(g)$ such that $I_{\bar{b}}$ is maximal in $\{I_{\bar{a}} : \bar{a} \in V(g)\}$. Now choose $g_1, \ldots, g_n$ to be generators for $I_{\bar{b}}$ and let $g' = \sum_{i=1}^n g_i^2$. So for $\bar{a} \in V(g)$ we have

$$I_{\bar{a}} = I_{\bar{b}} \text{ iff } g'(\bar{a}) = 0. \tag{3.2}$$

Now choose $s \leq r$ maximal such that there exists $f_1, \ldots, f_s \in M^r(\bar{\sigma})$ with $\bar{b} \in V_{\text{reg}}(f_1, \ldots, f_s)$. Let $f_1, \ldots, f_s$ witness this fact. If $s = r$ we are done. So assume for a contradiction that $s < r$. We claim that $V(g) \cap V(g') \subseteq V_{\text{reg}}(f_1, \ldots, f_s)$. Well if

$\bar{a} \in V(g) \cap V(g')$ then $I_{\bar{a}} = I_{\bar{b}}$ so in particular $\bar{a} \in V(f_1, \ldots, f_s)$. Furthermore, there exists an $s \times s$ submatrix of $\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots,x_r)}$ which has non-vanishing determinant at $\bar{b}$. Since this determinant is an element of $M^r(\bar{\sigma})$ and $I_{\bar{b}} = I_{\bar{a}}$ it must also be non-vanishing at $\bar{a}$. So $\bar{a} \in V_{\text{reg}}(f_1, \ldots, f_s)$. So the claim is proved.

Now let $\bar{a} \in V(g) \cap V(g')$ and let $h \in M^r(\bar{\sigma})$ and suppose that we have $\bar{a} \in V_{\text{reg}}(f_1, \ldots, f_s, h)$. Then arguing as above we see that $\bar{b} \in V_{\text{reg}}(f_1, \ldots, f_s, h)$ which contradicts the maximality of s. So, by applying lemma 3.3.9 we see that if $\bar{a} \in V(g) \cap V(g')$ then there exists a definable open neighbourhood U of $\bar{a}$ such that g and g' vanish identically on $U \cap V_{\text{reg}}(f_1, \ldots, f_s)$. So for any $\bar{a} \in V(g) \cap V(g')$ there exists U a definable open neighbourhood of a such that $V(g) \cap V(g') \cap U = V_{\text{reg}}(f_1, \ldots, f_s) \cap U$. Now $V(g) \cap V(g')$ is closed in K^r (since $V(g)$ is closed in K^r). So choose $\bar{\eta} = (\eta_1, \ldots, \eta_r) \in \mathbb{Q}^r$ and choose $\bar{a} \in V(g) \cap V(g')$ of minimum distance from $\bar{\eta}$. Let $h_{\bar{\eta}}(\bar{x}) = \sum_{i=1}^r (x_i - \eta_i)^2$. Since $\bar{a}$ is a local minimum of $h_{\bar{\eta}}$ on $V(g) \cap V(g')$ and $V(g) \cap V(g')$ and $V_{\text{reg}}(f_1, \ldots, f_s)$ coincide in a neighbourhood of $\bar{a}$, we must have that $\bar{a}$ is a local minimum of $h_{\bar{\eta}}$ on $V_{\text{reg}}(f_1, \ldots, f_s)$. This implies that $J_{h_{\bar{\eta}}}(\bar{a})$ is in the K-linear span of $J_{f_1}(\bar{a}), \ldots, J_{f_s}(\bar{a})$. Now this holds for all $\bar{\eta} \in \mathbb{Q}^r$. But $\frac{1}{2}(J_{h_{\bar{0}}}(\bar{a}) - J_{h_{\bar{\eta}}}(\bar{a})) = \bar{\eta}$ so $\mathbb{Q}^r$ is contained in the linear span of $J_{f_1}(\bar{a}), \ldots, J_{f_s}(\bar{a})$ which contradicts the fact that $s < r$. □

Proof of lemma 3.3.8. If $V(g)$ is closed in K^r then the result follows from lemma 3.3.10 (taking $s = r$ in the notation of lemma 3.3.8). Suppose that $V(g)$ is not closed in K^r. For each $i = 1, \ldots, r$ define $f_i, h_i : K^{3r} \to K$ by

$$f_i(x_1, \ldots, x_{3r}) = \begin{cases} x_i x_{r+i} - 1 & x_i \text{ is } \bar{\sigma}\text{ -bounded,} \\ x_{r+i} - x_i & \text{otherwise.} \end{cases}$$

$$h_i(x_1, \ldots, x_{3r}) = \begin{cases} (x_i - 1)x_{2r+i} - 1 & x_i \text{ is } \bar{\sigma}\text{ -bounded,} \\ x_{r+i} - x_i & \text{otherwise.} \end{cases}$$

Now let $\pi : K^{3r} \to K^r$ be the projection map onto the first r coordinates. Then

$$\pi(V(g, f_1, \ldots, f_r, h_1, \ldots, h_r)) = V(g), \tag{3.3}$$

(where on the left hand side we consider $\bar{\sigma}$ to be an $(n, 3r)$-sequence). Furthermore $V(g, f_1, \ldots, f_r, h_1, \ldots, h_r)$ is closed in K^{3r} (since all limit points of $V(g)$ have been 'pushed to ∞'). Let $f' = g^2 + \sum_{i=1}^r (f_i^2 + h_i^2)$ so $V(f') = V(g, f_1, \ldots, f_r, h_1, \ldots, h_r)$. Now apply lemma 3.3.10 to f'. The result follows by (3.3). □

3.4 Proof of lemma 3.3.7

It remains to prove lemma 3.3.7. We do this by induction on n (the length of the sequence $\bar{\sigma}$). Let us first consider the base case, $n = 0$. So let $\bar{a} = (a_1, \ldots, a_r) \in K^r$ and suppose we have $p_1, \ldots, p_r \in H[x_1, \ldots, x_r]$ such that

1. $p_1(\bar{a}) = \ldots = p_r(\bar{a}) = 0$,
2. $\det\left(\frac{\partial(p_1,\ldots,p_r)}{\partial(x_1,\ldots,x_r)}\right)(\bar{a}) \neq 0$.

We must prove that $\bar{a} \in H^r$. We first recall a standard characterization of algebraic closure for algebraically closed fields of characteristic 0 (see for instance [6]).

Fact 3.4.1. *Let F be an algebraically closed field of characteristic 0 and let L be a subfield of F. Let $a \in F$. Then $a \in \mathrm{acl}_F(L)$ if and only if every derivation of F that vanishes on L also vanishes on a.*

Now let $\overline{K}$ be the algebraic closure of K and let δ be a derivation on $\overline{K}$ which is zero on H. We must show that $\delta(a_i) = 0$ for $i = 1, \ldots, r$. Then, for $i = 1, \ldots, r$, we have $a_i \in \mathrm{acl}(H) \cap K = H$. Now, for each $j = 1, \ldots, r$ we have $p_j(\bar{a}) = 0$ so $\delta(p_j(\bar{a})) = 0$. Now a simple calculation reveals that

$$\delta(p_j(\bar{a})) = \sum_{i=1}^{r} \delta(a_i) \frac{\partial p_j}{\partial x_i}(\bar{a}).$$

Therefore

$$\begin{pmatrix} \frac{\partial p_1}{\partial x_1}(\bar{a}) & \ldots & \frac{\partial p_1}{\partial x_r}(\bar{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial p_r}{\partial x_1}(\bar{a}) & \ldots & \frac{\partial p_r}{\partial x_r}(\bar{a}) \end{pmatrix} \begin{pmatrix} \delta(a_1) \\ \vdots \\ \delta(a_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But by (2) this implies that $\delta(a_i) = 0$ for $i = 1, \ldots, r$.

We must now do the inductive step. So let $n \geq 0$ and suppose that for all $r \geq 1$ and all (n, r)-sequences $\bar{\sigma}$ every $\bar{\sigma}$-definable point of K^r lies in H^r. Now let $r \geq 1$ and let $\bar{\sigma}'$ be an $(n+1, r)$-sequence. We must prove that every $\bar{\sigma}'$-definable point of K^r lies in H^r. We do this in two steps. First we make a definition.

Definition 3.4.2. A point $a \in K$ will be called *H-bounded* is $|a| < r$ for some $r \in H$. A tuple $\bar{a} \in K^n$ will be called H-bounded if each of its coordinates is H-bounded.

(Part 1) Suppose that $\bar{b} \in K^r$ is $\bar{\sigma}'$-definable and H-bounded. Then $\bar{b} \in K^r$.

(Part 2) Suppose is $\bar{b} \in K^r$ is $\bar{\sigma}'$-definable. Then $\bar{b}$ is H-bounded.

3.4.1 Part 1

We first give two general results about smooth functions definable in o-minimal expansions of fields. So let $\mathcal{M}$ be an arbitrary o-minimal expansion of a field.

Theorem 3.4.3. *Let U be a definable open subset of M^n and let $f_1, \ldots, f_{n-1} : U \to M$ be definable and smooth. Suppose that for all $\bar{a} \in V = \{\bar{x} \in U : f_1(\bar{x}) = \ldots = f_{n-1}(\bar{x}) = 0\}$ we have*

$$\det\left(\frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_2, \ldots, x_n)}\right)(\bar{a}) \neq 0.$$

Then there exists $N \in \mathbb{N}$ such that for all $b \in \pi(U)$ (where π is the projection map onto the first coordinate) the set $V_b = \{\bar{a} \in M^{n-1} : (b, \bar{a}) \in V\}$ has at most N elements.

Proof. By the uniform finiteness property for definable families in o-minimal structures (theorem 2.2.10) and the fact that all discrete sets definable in o-minimal structures are finite, it is sufficient to prove that the sets V_b are discrete.

So choose $b \in \pi(U)$. Let $\bar{a} \in V_b$. By the implicit function theorem there is open interval I in M containing b, an open definable neighbourhood W of $\bar{a}$ and a smooth map $\phi : I \to W$ such that

1. $I \times W \subseteq U$,
2. $\phi(b) = \bar{a}$,
3. $(x, \bar{y}) \in V \cap (I \times W)$ iff $\bar{y} = \phi(x)$.

Now $W \cap V_b = \{\bar{a}\}$ so V_b is discrete.

□

Theorem 3.4.4. *Let $n \in \mathbb{N}$ with $n \geq 2$, U a definable open subset of M^n and let $f_1, \ldots, f_{n-1} : U \to M$ be smooth and definable. Suppose that*

1. *$V(f_1, \ldots, f_{n-1}) = \{\bar{x} \in U : f_1(\bar{x}) = \ldots = f_{n-1}(\bar{x}) = 0\}$ is closed in M^n,*
2. *for all $\bar{a} \in V(f_1, \ldots, f_{n-1})$*

$$\det\left(\frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_2, \ldots, x_n)}\right)(\bar{a}) \neq 0.$$

Then there exists a finite set $\mathfrak{P}$ of pairs (I, ϕ) such that *(i) for $(I, \phi) \in \mathfrak{P}$ we have that I is an open interval with endpoints in $M \cup \{\pm\infty\}$ and $\phi : I \to M^{n-1}$ is smooth and definable,*

(ii) for each $(I, \phi) \in \mathfrak{P}$, if $\sup I \neq \infty$ then $\|\phi(x)\| \to \infty$ as $x \to \sup I$ and similarly for $\inf I$,

(iii) $V(f_1, \ldots, f_{n-1}) = \bigcup\{\mathrm{graph}(\phi) : (I, \phi) \in \mathfrak{P}\}$ and the union is disjoint.

Proof. The proof is almost exactly the same as the proof of Theorem 6.2 in [29]. To obtain the uniform bound on the size of the fibres we use lemma 3.4.3. To replace use of transfer from a theory expanding $\bar{\mathbb{R}}$ we use the o-minimality of $\mathcal{M}$. □

We call $\mathfrak{P}$ as obtained in theorem 3.4.4 a *parameterization* of V. We now return to our specific setting, including the assumption that the inductive hypothesis holds upto n, and that $\bar{\sigma}' = (\bar{\sigma}, \sigma_{n+1})$ is an $(n + 1, r)$-sequence.

Lemma 3.4.5. *Let $f_1, \ldots, f_{r-1} \in M^r(\bar{\sigma})$ and suppose*

1. $V = V(f_1, \ldots, f_{r-1})$ is closed in $D^r(\sigma)$,

2. for all $\bar{a} \in V$ we have $\det\left(\frac{\partial(f_1,\ldots,f_{r-1})}{\partial(x_2,\ldots,x_r)}\right)(\bar{a}) \neq 0$.

Let $\bar{a} = (a_1, \ldots, a_r) \in K^r$ be H-bounded and suppose that $\bar{a} \in V$. Then there exists $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2 \in H$ such that $\gamma_2 < \gamma_1 < a_1 < \beta_1 < \beta_2$ and $\|(a_2, \ldots, a_r)\| < B_1 < B_2$ and there exists $n \in \mathbb{N}$ with $n \geq 1$ and definable (in the sense of K) smooth functions $\phi_j : (\gamma_2, \beta_2) \to K^{r-1}$, for $j = 1, \ldots, n$, such that

(i) for $j = 1, \ldots, n$ and $x \in (\gamma_2, \beta_2)$ we have $\|\phi_j(x)\| < B_1$,

(ii) $V \cap ((\gamma_2, \beta_2) \times \{\bar{c} \in K^{r-1} : \|\bar{c}\| < B_2\}) = \bigcup_{j=1}^{n} \mathrm{graph}(\phi_j)$ and the union is disjoint.

Furthermore, if $V \cap H^r$ is closed in H^r then there exists, for $j = 1, \ldots, n$, definable (in the sense of $\mathcal{H}$) smooth functions $\psi_j : (\gamma_2, \beta_2) \cap H \to H^{r-1}$ such that (i) and (ii) hold for ψ_j and $\mathcal{H}$ in place of ϕ_j and $\mathcal{K}$ when all quantifiers are interpreted in $\mathcal{H}$. So in particular if $\bar{a} = (a_1, \ldots, a_r) \in V$ is H-bounded and $a_1 \in H$ then $\bar{a} \in H^r$.

Proof. The proof is identical to that of lemma 6.3 in [29]. □

We are now in a position to perform part 3.4. So let $\bar{b} \in K^r$ be $\bar{\sigma}'$-definable and H-bounded. So, by definition,

$$\bar{b} \in D^r(\bar{\sigma}') \tag{3.4}$$

and there exists $g_1, \ldots, g_r \in M^r(\bar{\sigma}')$ such that

$$g_1(\bar{b}) = \ldots = g_r(\bar{b}) = 0, \tag{3.5}$$

$$\det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(\bar{b}) \neq 0. \tag{3.6}$$

We must prove that $\bar{b} \in H^r$. We do this under the following additional assumptions which we will justify afterwards. Set $V = V(g_1, \ldots, g_{r-1})$ (clearly we may assume that $r \geq 2$). We suppose

$$g_1, \ldots, g_{r-1} \in M^r(\bar{\sigma}), \tag{3.7}$$

$$V \subseteq D^r(\bar{\sigma}'), \tag{3.8}$$

$$V \text{ is closed in } K^r \text{ and } V \cap H^r \text{ is closed in } H^r, \tag{3.9}$$

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(\bar{x}) \neq 0 \text{ for all } \bar{x} \in V, \tag{3.10}$$

$$\text{for all } \bar{x} \in V, \text{ if } g_r(\bar{x}) = 0 \text{ then } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(\bar{x}) < 0. \tag{3.11}$$

Let $\bar{b}' = (b_2, \ldots, b_r)$. We apply lemma 3.4.5 to obtain $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2 \in H$ such that $\gamma_2 < \gamma_1 < b_1 < \beta_1 < \beta_2$, $\|\bar{b}'\| < B_1 < B_2$ and definable (in $\mathcal{K}$) smooth $\phi_1, \ldots, \phi_m : (\gamma_2, \beta_2) \to K^{r-1}$ such that

1. for $j = 1, \ldots, m$ and for $x \in (\gamma_2, \beta_2)$ we have $\|\phi_j(x)\| < B_1$,

2. $V \cap ((\gamma_2, \beta_2) \times \{\bar{c} \in K^{r-1} : \|\bar{c}\| < B_2\}) = \bigcup_{j=1}^{n} \mathrm{graph}(\phi_j)$ and the union is disjoint.

We also obtain definable (in $\mathcal{H}$) smooth $\psi_j : (\gamma_2, \beta_2) \cap H \to H^{r-1}$ satisfying (1) and (2) when $\mathcal{K}$ is replaced by $\mathcal{H}$.

Now let ϕ be one of the ϕ_j's. Let $f \in M^r(\bar{\sigma}')$. Since we assume that $V \subseteq D^r(\bar{\sigma}')$ we may define $\bar{f} : (\gamma_2, \beta_2) \to K^{r-1}$ by $\bar{f}(t) = f(t, \phi(t))$. Clearly $\bar{f}$ is smooth and definable in K. Furthermore, if we let

$$J(x_1, \ldots, x_r) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1}, f)}{\partial(x_1, \ldots, x_r)}\right), \tag{3.12}$$

$$J_1(x_1, \ldots, x_r) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right), \tag{3.13}$$

then $J, J_1 \in M^r(\bar{\sigma}')$ and we see that for all $t \in (\gamma_2, \beta_2)$ we have

$$\frac{d\bar{f}}{dt}(t) = (-1)^{r+1}\bar{J}(t)\bar{J}_1(t)^{-1}, \tag{3.14}$$

where $\bar{J}(t) = J(t, \phi(t))$ and $\bar{J}_1(t) = J_1(t, \phi(t))$; by (3.10) this formula makes sense.

Lemma 3.4.6.

1. *If $t \in (\gamma_2, \beta_2)$ and $\bar{g}_r(t) = 0$ then $\frac{d\bar{g}_r}{dt}(t)$ has the same sign as $\bar{J}_1(t)$ if r is even and the opposite sign if r is odd.*

2. *$\bar{g}_r(t)$ has at most one zero on (γ_2, β_2).*

Proof.

1. If $g_r(t) = 0$ then by (3.11) and (3.14) the result follows.

2. This immediate from part 1.

□

Since the assumptions (3.7)-(3.11) all imply their counterparts for $\mathcal{H}$ in place of $\mathcal{K}$, lemma 3.4.6 holds in $\mathcal{H}$ when ϕ is replaced by one of the ψ_j's. Now for $f \in M^r(\bar{\sigma}')$ we let $\bar{f}(\phi_i; \cdot)$ be the definable (in $\mathcal{K}$) function from (γ_2, β_2) to K^{r-1} obtained as above with ϕ_i in place of ϕ. Similarly we define $\bar{f}(\psi_i; \cdot)$. Now let i_0 be the unique number such that $1 \leq i_0 \leq m$ and $\phi_{i_0}(b_1) = (b_2, \ldots, b_r)$. Suppose that $\bar{J}_1(\phi_{i_0}; b_1) > 0$; the other case will be similar. Let $T = \{i : 1 \leq i \leq m \text{ and } \bar{J}_1(\phi_i; b_1) > 0\}$. By (3.10), $\bar{J}_1(\phi_i; t) > 0$ for all $i \in T$ and all $t \in (\gamma_2, \beta_2)$, and $\bar{J}_1(\phi_i; t) < 0$ for all $i \in \{1, \ldots, m\} \setminus T$ and all $t \in (\gamma_2, \beta_2)$. So in particular $\bar{J}_1(\phi_i; \gamma_1) > 0$ for all $i \in T$ and $\bar{J}_1(\phi_i; \gamma_1) < 0$ for all $i \in \{1, \ldots, m\} \setminus T$. Now it follows from lemma 3.4.5 that there is some $T' \subseteq \{1, \ldots, m\}$ such that $\{\psi_i(\gamma_1) : i \in T'\} = \{\phi_i(\gamma_1) : i \in T\}$. So $\bar{J}_1(\psi_i; t) > 0$ for all $i \in T'$ and all $t \in (\gamma_2, \beta_2) \cap H$. It follows that $\{\psi_i(t) : i \in T'\} = \{\phi_i(t) : i \in T\}$ for all $t \in (\gamma_2, \beta_2) \cap H$. Now for each $i = 1, \ldots, m$ the function $\bar{g}_r(\psi_i; \cdot)$ has only finitely many zeros on (γ_2, β_2), so we may choose $\gamma_3, \beta_3 \in H$ such that $\gamma_2 < \gamma_3 < \gamma_1$ and $\beta_1 < \beta_3 < \beta_2$ and such that for no $i = 1, \ldots, m$ does $\bar{g}_r(\phi_i; \cdot)$ have a zero at γ_3 or β_3. Now assume that r is even, the case where r is odd is similar. By lemma 3.4.6, if $i \in T$ then $\bar{g}_r(\phi_i; \cdot)$ has a zero in (γ_3, β_3) iff $\bar{g}_r(\phi_i; \gamma_3) < 0$ and $\bar{g}_r(\phi_i; \beta_3) > 0$. Similarly if $i \in T'$ then $\bar{g}_r(\psi_i; \cdot)$ has a zero in $(\gamma_3, \beta_3) \cap H$ iff $\bar{g}_r(\psi_i; \gamma_3) > 0$ and $\bar{g}_r(\psi_i; \beta_3) < 0$. Hence

$$\operatorname{card}\{i \in T : \bar{g}_r(\phi_i;\cdot) \text{ has a zero on } (\gamma_3, \beta_3)\}$$
$$= \operatorname{card}\{i \in T : \bar{g}_r(\phi_i;\gamma_3) < 0\} - \operatorname{card}\{i \in T : \bar{g}_r(\phi_i;\beta_3) < 0\}, \quad (3.15)$$

and

$$\operatorname{card}\{i \in T : \bar{g}_r(\psi_i;\cdot) \text{ has a zero on } (\gamma_3, \beta_3) \cap H\}$$
$$= \operatorname{card}\{i \in T : \bar{g}_r(\psi_i;\gamma_3) < 0\} - \operatorname{card}\{i \in T : \bar{g}_r(\psi_i;\beta_3) < 0\}. \quad (3.16)$$

By lemma 3.4.5 the two right-hand sides above are equal. Therefore if we have that $\bar{a} = (a_1, \ldots, a_r) \in V$ is such that $g_r(\bar{a}) = 0$, $J_1(\bar{a}) > 0$, $\gamma_3 < a_1 < \beta_3$ and $\|(a_2, \ldots, a_r)\| < B_1$ then $\bar{a} \in H^r$. Since $\bar{b}$ is such a point the result follows.

We must now justify the additional assumptions (3.7)-(3.11). We do this by modifying our functions $g_1, \ldots, g_r$ and our $\bar{b} \in K^r$ in a number of steps. Each modification will produce some $s \geq r$, new functions $f_1, \ldots, f_s \in D^s(\bar{\sigma}')$ and a new tuple $\bar{a} \in D^s(\bar{\sigma}')$ such that the properties (3.4),(3.5) and (3.6) are preserved and such that $\bar{b}$ is a subtuple of $\bar{a}$. After each step we will revert to the original notation. Once all the steps are complete the new $g_1, \ldots, g_r$ and $\bar{b}$ will satisfy the additional assumptions (3.7)-(3.11) and so by our above argument $\bar{b} \in H^r$. Since our original $\bar{b}$ will be a subtuple of this $\bar{b}$ the proof will be complete.

Step 1 *We may assume that for each $i = 1, \ldots, r$, if x_i is $\bar{\sigma}'$-bounded then there are variables y, z (which are not $\bar{\sigma}'$-bounded) amongst $x_1, \ldots, x_r$ such that the elements of $M^r(\bar{\sigma}')$ given by $x_i y^2 - 1, (1 - x_i)z^2 - 1$ occur amongst $g_1, \ldots, g_r$.*

To achieve this we take an i such that x_i is $\bar{\sigma}$-bounded. Define $g_{r+1}, g_{r+2} \in M^{r+2}(\bar{\sigma}')$ by

$$g_{r+1}(x_1, \ldots, x_{r+2}) = x_i x_{r+1}^2 - 1,$$
$$g_{r+2}(x_1, \ldots, x_{r+2}) = (1 - x_i)x_{r+2}^2 - 1.$$

Now let $b_{r+1} = b_i^{-\frac{1}{2}}$ and let $b_{r+2} = (1 - b_i)^{-\frac{1}{2}}$ (note that $0 < b_i < 1$ since x_i is $\bar{\sigma}'$-bounded and $\bar{b}$ is $\bar{\sigma}'$-definable). We must show that (3.4), (3.5) and (3.6) are satisfied by $g_1, \ldots, g_{r+2}$ and $\tilde{b} = (\bar{b}, b_{r+1}, b_{r+2})$. Clearly (3.4) and (3.5) are satisfied. A simple calculation shows that

$$\det\left(\frac{\partial(g_1, \ldots, g_{r+2})}{(x_1, \ldots, x_{r+2})}\right)(\tilde{b}) = 4 \det\left(\frac{\partial(g_1, \ldots, g_r)}{(x_1, \ldots, x_r)}\right)(\bar{b}) b_i^{\frac{1}{2}}(1 - b_i)^{\frac{1}{2}},$$

and so (3.6) is also satisfied. By repeated use of this process we see that the assumption of step 1 is justified.

Step 2 *We may assume that* $g_1, \dots, g_{r-1} \in M^r(\bar{\sigma})$ *and that* g_r *is of the form*

$$\sigma_{n+1}(x_1, \dots, x_r) - x_e,$$

where x_e *is not* $\bar{\sigma}'$*-bounded.*

To achieve this, for each $i = 1, \dots, r$, we replace $g_i \in M^r(\bar{\sigma}')$ by $\tilde{g}_i \in M^{r+1}(\bar{\sigma})$ where $\tilde{g}_i(x_1, \dots, x_{r+1})$ is obtained by replacing each occurrence of σ_{n+1} in g_i by x_{r+1}. Now let $\tilde{g}_{r+1}(x_1, \dots, x_{r+1}) = \sigma_{n+1}(x_1, \dots, x_r) - x_{r+1}$ and $\tilde{b} = (b_1, \dots, b_r, \sigma_{n+1}(b_1, \dots, b_r))$.

Cleary $\tilde{g}_1, \dots, \tilde{g}_r, g_{r+1}$ and $\tilde{b}$ satisfy (3.5) and (3.4). Furthermore, step 1 is preserved. We must show that (3.6) is satisfied, ie. that

$$\det\left(\frac{\partial(\tilde{g}_1, \dots, \tilde{g}_{r+1})}{(x_1, \dots, x_{r+1})}\right)(\tilde{b}) \neq 0.$$

Observe that

$$\frac{\partial(\tilde{g}_1, \dots, \tilde{g}_{r+1})}{\partial(x_1, \dots, x_{r+1})}(\tilde{b}) = \begin{pmatrix} \frac{\partial \tilde{g}_1}{\partial x_1}(\tilde{b}) & \dots & \frac{\partial \tilde{g}_1}{\partial x_r}(\tilde{b}) & \frac{\partial \tilde{g}_1}{\partial x_{r+1}}(\tilde{b}) \\ \vdots & & \vdots & \vdots \\ \frac{\partial \tilde{g}_r}{\partial x_1}(\tilde{b}) & \dots & \frac{\partial \tilde{g}_r}{\partial x_r}(\tilde{b}) & \frac{\partial \tilde{g}_r}{\partial x_{r+1}}(\tilde{b}) \\ \frac{\partial \sigma_{n+1}}{\partial x_1}(\bar{b}) & \dots & \frac{\partial \sigma_{n+1}}{\partial x_r}(\bar{b}) & -1 \end{pmatrix}.$$

We now note that by the chain rule

$$\frac{\partial g_i}{\partial x_j}(\bar{b}) = \frac{\partial \tilde{g}_i}{\partial x_j}(\tilde{b}) + \frac{\partial \tilde{g}_i}{\partial x_{r+1}}(\tilde{b})\frac{\partial \sigma_{n+1}}{\partial x_j}(\bar{b}),$$

for $j = 1, \dots, r$. Consequently, if for each $i = 1, \dots, r$ we multiply row $r+1$ by $\frac{\partial \tilde{g}_i}{\partial x_{r+1}}(\tilde{b})$ and add it to row i we obtain the matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\bar{b}) & \dots & \frac{\partial g_1}{\partial x_r}(\bar{b}) & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_r}{\partial x_1}(\bar{b}) & \dots & \frac{\partial g_r}{\partial x_r}(\bar{b}) & 0 \\ \frac{\partial \sigma_{n+1}}{\partial x_1}(\bar{b}) & \dots & \frac{\partial \sigma_{n+1}}{\partial x_r}(\bar{b}) & -1 \end{pmatrix},$$

which has determinant equal to $-\det\left(\frac{\partial(g_1,\dots,g_n)}{\partial(x_1,\dots,x_n)}\right)(\bar{b})$, which is non-zero. Hence (3.6) is satisfied.

Step 3 *We may assume that for all* $\bar{a} \in D^r(\bar{\sigma}')$*, if* $\bar{a} \in V$ *then*

$$\det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)}\right)(\bar{a}) \neq 0.$$

To achieve this we first observe the following: by (3.6) we must have that

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_r)}\right)(\bar{b}) \neq 0,$$

for some $i = 1,\ldots,r$. By a coordinate permutation we may assume that $i = 1$. (Strictly speaking we are now working with a new $(n+1,r)$-sequence, new functions $\tilde{g}_1,\ldots,\tilde{g}_r$ and a new tuple $\tilde{b}$. It is easy to see that these satisfy (3.5), (3.6), (3.4), step 1 and step 2 and furthermore $\tilde{b}$ is just a permutation of $\bar{b}$ so we revert to the original notation.) Now let

$$h(x_1,\ldots,x_{r+1}) = x_{r+1}\det\left(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_2,\ldots,x_r)}\right)(x_1,\ldots,x_r) - 1.$$

Clearly $h \in M^{r+1}(\bar{\sigma})$. Now set

$$b_{r+1} = \left(\det\left(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_2,\ldots,x_r)}\right)(b_1,\ldots,b_r)\right)^{-1}.$$

Clearly the sequence $g_1,\ldots,g_{r-1},h,g_r$, $(\bar{b},b_{r+1})$ satisfy (3.5) and (3.4) and steps 1 and 2 are preserved. A simple calculation shows that

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1},h,g_r)}{\partial(x_1,\ldots,x_{r+1})}\right)(b_1,\ldots,b_r,b_{r+1}) = \\ -\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(b_1,\ldots,b_r)b_{r+1}^{-1},$$

which is non-zero. Finally, suppose that $\bar{a} \in D^{r+1}(\bar{\sigma}')$ and $g_1(\bar{a}) = \ldots = g_{r-1}(\bar{a}) = h(\bar{a}) = 0$. Then $a_{r+1} \neq 0$ and

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1},h)}{\partial(x_2,\ldots,x_{r+1})}\right)(\bar{a}) = a_{r+1}^{-2},$$

so step 3 is satisfied.

Step 4 *We may assume that for all $\bar{a} \in D^r(\bar{\sigma}')$, if $\bar{a} \in V$ and $g_r(\bar{a}) = 0$ then*

$$\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(\bar{a}) < 0.$$

Let $h(x_1,\ldots,x_r) \in M^r(\bar{\sigma})$ be obtained from $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(x_1,\ldots,x_r)$ by replacing any occurrences of σ_{n+1} by x_e (as given by step 2). Now let

$$k(x_1,\ldots,x_{r+1}) = x_{r+1}h(x_1,\ldots,x_r) - 1.$$

Since $g_r(x_1, \dots, x_r) = \sigma_{n+1}(x_1, \dots, x_r) - x_e$ and $g_r(\bar{b}) = 0$, if we set $b_{r+1} = h(\bar{b})^{-1}$ then (3.5),(3.4), and steps 1 and 2 are satisfied by $g_1, \dots, g_{r-1}, k, g_r$ and $(\bar{b}, b_{n+1})$. Let us see that steps 3 and 4 are satisfied. So suppose that $\bar{a} \in D^{r+1}(\bar{\sigma}')$ and $g_1(\bar{a}) = \dots = g_{r-1}(\bar{a}) = k(\bar{a}) = g_r(\bar{a}) = 0$. By direct calculation we see that

$$\begin{aligned}\det\left(\frac{\partial(g_1, \dots, g_{r-1}, k, g_r)}{\partial(x_1, \dots, x_r, x_{r+1})}\right)(\bar{a}) &= -\det\left(\frac{\partial(g_1, \dots, g_{r-1}, g_r, k)}{\partial(x_1, \dots, x_r, x_{r+1})}\right)(\bar{a}) \\ &= -\det\left(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)}\right)(\bar{a})h(\bar{a}) \\ &= -h(\bar{a})^2.\end{aligned}$$

Furthermore, $h(a_1, \dots, a_r) \neq 0$ since $k(\bar{a}) = 0$. So step 4 is satisfied. Furthermore, if $\tilde{a} = (\bar{a}, a) \in D^{r+1}(\bar{\sigma}')$ and

$$g_1(\bar{a}) = \dots = g_{r-1}(\bar{a}) = k(\tilde{a}) = 0$$

then

$$\det\left(\frac{\partial(g_1, \dots, g_{r-1}, k)}{\partial(x_2, \dots, x_{r+1})}\right)(\tilde{a}) = \det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)}\right)(\bar{a})h(\bar{a}),$$

which is non-zero since step 3 holds for the $g_1, \dots, g_r$ and $k(\tilde{a}) = 0$. So step 3 is satisfied.

We now show that the assumptions of steps 1 - 4 imply (3.7) - (3.11). Cleary step 2 gives us (3.7). Together steps 1 and 2 give us (3.8). Condition (3.9) follows from step 1. Of course (3.10) follows from step 3 and (3.11) follows from step 4. So the proof of part 1 is complete.

3.4.2 Part 2

We must now prove part 3.4, i.e. we must prove that all $\bar{\sigma}'$-definable points are H-bounded (recall that $\bar{\sigma}' = (\bar{\sigma}, \sigma_{n+1})$ is an $(n+1, r)$-sequence and we are assuming that if $\bar{\mu}$ is an (n, s)-sequence for some $s \geq 1$ then all $\bar{\mu}$-definable points of K^s lie in H^s). So let $\bar{b} \in K^r$ be $\bar{\sigma}'$-definable. So

$$\bar{b} \in D^r(\bar{\sigma}') \tag{3.17}$$

and we have functions $g_1, \dots, g_r \in M^r(\bar{\sigma}')$ such that

$$g_1(\bar{b}) = \dots = g_r(\bar{b}) = 0, \tag{3.18}$$

$$\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(\bar{b})\neq 0. \tag{3.19}$$

As in the proof of part 3.4 we may additionally assume that

$$g_1,\ldots,g_{r-1}\in M^r(\bar{\sigma}) \tag{3.20}$$

$$g_r \text{ is of the form } \sigma_{n+1}(x_1,\ldots,x_r)-x_e \text{ where } x_e \text{ is not } \bar{\sigma}'\text{-bounded,} \tag{3.21}$$

and if we set $V=\{\bar{a}\in D^r(\bar{\sigma}):g_1(\bar{a})=\ldots=g_{r-1}(\bar{a})=0\}$ then

$$V\subseteq D^r(\bar{\sigma}') \text{ and } V \text{ is closed in } K^r \text{ and } V\cap H^r \text{ is closed in } H^r, \tag{3.22}$$

$$\text{if } \bar{a}\in V \text{ then } \det\left(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_2,\ldots,x_r)}\right)(\bar{a})\neq 0, \tag{3.23}$$

$$\text{if } \bar{a}\in V \text{ and } g_r(\bar{a})=0 \text{ then } \det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(\bar{a})\neq 0. \tag{3.24}$$

Lemma 3.4.7. *Let $\phi(x_1,\ldots,x_r)$ be a formula in the language of ordered rings with parameters from H. Suppose there exists $\bar{a}\in V$ such that $\mathcal{K}\models\phi(\bar{a})$. Then there exists $\bar{c}\in V\cap H^r$ such that $\mathcal{H}\models\phi(\bar{c})$.*

Proof. Using quantifier elimination for the theory of real-closed fields and standard logical equivalences we may assume that $\phi(x_1,\ldots,x_r)$ is of the form

$$\exists y_1,\ldots,y_m p(x_1,\ldots,x_r,y_1,\ldots,y_m)=0$$

where $p\in H[x_1,\ldots,x_r,y_1,\ldots,y_m]$. Now let $g=p^2+\sum_{i=1}^r g_i^2\in M^{r+m}(\bar{\sigma})$. By lemma 3.3.8 there exists $t\geq 0$ and $\bar{c}=(c_1,\ldots,c_{r+m+t})\in K^{r+m+t}$ such that $\bar{c}$ is $\bar{\sigma}$-definable and $g(c_1,\ldots,c_{r+m})=0$. By our inductive hypothesis $\bar{c}\in H^{r+m+t}$. Furthermore, since $g(c_1,\ldots,c_{r+m})=0$ we have $(c_1,\ldots,c_r)\in V$ and $\mathcal{H}\models\phi(c_1,\ldots,c_r)$. □

Now suppose for a contradiction that $\bar{b}$ is not H-bounded.

Claim 3.4.8. $b_1\notin H$.

Proof. Suppose that $b_1\in H$. Consider $h(x_1,\ldots,x_r)=x_1-b_1$. Then $h(\bar{b})=0$, $h\in M^r(\bar{\sigma})$ and

$$\det\left(\frac{\partial(h,g_1,\ldots,g_{r-1})}{\partial(x_1,\ldots,x_r)}\right)(\bar{b})\neq 0$$

so $\bar{b}$ is $\bar{\sigma}$-definable and hence, by our inductive hypothesis, we have $\bar{b}\in H^r$, which contradicts our assumption that $\bar{b}$ is not H-bounded. □

Now we may apply lemma 3.4.5 to obtain a parametrization $\{(I_j,\psi_j):1\leq j\leq m\}$ of $V\cap H^r$. For $j=1,\ldots,m$ let $I_j=(a_j,c_j)$ where $a_j\in H\cup\{-\infty\}$ and $c_j\in H\cup\{+\infty\}$.

Claim 3.4.9. If b_1 is H-bounded then there exists j such that $0 < b_1 - a_j < \alpha$ for all $\alpha \in \mathrm{Pos}(H)$ or $0 < c_j - b_1 < \alpha$ for all $\alpha \in \mathrm{Pos}(H)$.

Proof. Suppose that b_1 is H-bounded. We claim that there exists j such that $a_j < b_1 < c_j$. Suppose not, then, since $\{(I_j, \psi_j) : 1 \leq j \leq m\}$ is a parametrization of $V \cap H^r$, there exists $a, c \in H$ such that $c < b_1 < a$ but for no $(d_1, \ldots, d_r) \in V \cap H^r$ do we have $c < d_1 < a$. But this contradicts lemma 3.4.7. Now let $a = \max\{a_j : 1 \leq j \leq m,\ a_j < b_1\}$ and let $c = \min\{c_j : 1 \leq j \leq m,\ b_1 < c_j\}$ and suppose that there exists $\alpha \in H$ such that $\alpha > 0$ and $a + \alpha < b_1 < c - \alpha$. Clearly, if $a_j < b_1 < c_j$ then $[a + \alpha, c - \alpha] \subseteq (a_j, c_j)$. Recall that, for o-minimal expansions of fields, the image of a closed and bounded definable set under a continuous definable map is closed and bounded (see theorem 2.2.11). Consequently, by theorem 3.4.4 applied in $\mathcal{H}$, there exists $B \in H$ such that $\|\bar{d}\| \leq B$ for all $\bar{d} = (d_1, \ldots, d_r) \in V \cap H^r$ with $d_1 \in [a + \alpha, c - \alpha]$. Now let $a' = \max(\{a + \alpha\} \cup \{c_j : c_j < b_1\})$ and let $c' = \min(\{c - \alpha\} \cup \{a_j : a_j > c_1\})$. Then there is no $\bar{d} = (d_1, \ldots, d_r) \in V \cap H^r$ such that $a' < d_1 < c'$ and $\|\bar{d}\| > B$. But this contradicts lemma 3.4.7 since $\bar{b}$ is such a point in V. $\square$

We now claim that we may assume that $b_1 > \alpha$ for all $\alpha \in H$. By claim 3.4.9, if this is not already the case then we have one of the following

1. $b_1 < \alpha$ for all $\alpha \in H$,
2. there exists $a \in H$ such that $0 < b_1 - a < \alpha$ for all $\alpha \in \mathrm{Pos}(H)$,
3. there exists $c \in H$ such that $0 < c - b_1 < \alpha$ for all $\alpha \in \mathrm{Pos}(H)$.

Now define $h \in M^r(\bar{\sigma})$ by

$$h(x_1, \ldots, x_{r+1}) = \begin{cases} x_1 + x_{r+1} & \text{case (1)}, \\ x_{r+1}(x_1 - a) - 1 & \text{case (2)}, \\ x_{r+1}(c - x_1) - 1 & \text{case (3)}. \end{cases}$$

Clearly, in each case there is a unique b_{r+1} such that $(\bar{b}, b_{r+1})$ satisfies $g_1(\bar{b}, b_{r+1}) = \ldots = g_{r-1}(\bar{b}, b_{r+1}) = h(\bar{b}, b_{r+1}) = g_r(\bar{b}, b_{r+1}) = 0$ and $b_{r+1} > \alpha$ for all $\alpha \in H$. It is easy to see that (3.17), (3.18), (3.20), (3.21), (3.22) and (3.24) all hold for the system $g_1, \ldots, g_{r-1}, h, g_r, (\bar{b}, b_{r+1})$. Furthermore if $g_1(\bar{x}) = \ldots = g_{r-1}(\bar{x}) = h(\bar{x}) = 0$ then

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, h)}{\partial(x_1, \ldots, x_r)}\right)(\bar{x}) \neq 0.$$

So we relabel the variables such that x_1 becomes x_{r+1}. Then (3.17)-(3.24) are satisifed by the new system, and we revert to the original notation.

Lemma 3.4.10. *There exists S a finite subset of $H \cap [0,1]$, $B \in H$ and $\theta \in \mathbb{Q}$ with $\theta > 0$ such that for any $\bar{a} = (a_1, \ldots, a_r) \in V$, if $a_1 > B$ and j is such that x_j is $\bar{\sigma}'$-bounded then there exists $\alpha \in S$ such that $|a_j - \alpha| < a_1^{-\theta}$.*

Proof. By lemma 3.4.7 it is sufficient to prove this for all $\bar{a} \in V \cap H^r$. So let $\mathfrak{P}$ be a parametrization of $V \cap H^r$ in $\mathcal{H}$. Let $(I, \psi) \in \mathfrak{P}$ be such that $\sup I = +\infty$. Such must exist by lemma 3.4.7 and the fact that $b_1 > \alpha$ for all $\alpha \in H$. Suppose that x_l is $\bar{\sigma}'$-bounded. By (3.22) $0 < \psi_l(t) < 1$ for all $t \in I$ (where $\psi = (\psi_2, \ldots, \psi_m)$). Since $\mathcal{H}$ is o-minimal and polynomially bounded there exists $0 < \alpha < 1$ and there exists a positive rational θ and an element $B \in H$ such that $|\psi_l(t) - c| < t^{-\theta}$ for all $t \in I$ with $t > B$. Since there are only finitely many choices for (I, ψ) and for l the result follows. □

Lemma 3.4.11. *There exists a positive integer μ and an element $B' \in H$ such that if $\bar{a} = (a_1, \ldots, a_r) \in V \cap H^r$ with $a_1 > B'$ then $|g_r(\bar{a})| > a_1^{-\mu}$.*

Proof. As above let $\mathfrak{P}$ be a parametrization of $V \cap H^r$. Choose $(I, \psi) \in \mathfrak{P}$ such that $\sup I = +\infty$. Consider the definable map $\bar{g}_r : I \to H$ given by $t \mapsto g_r(t, \psi(t))$. By (3.24) and o-minimality $\bar{g}_r$ has only finitely many zeros, so by the fact that $\mathcal{H}$ is o-minimal and polynomially bounded there exists a positive integer μ and an element $B \in H$ such that $|\bar{g}_r(t)| > t^{-\mu}$ whenever $t > B$. Since there are only finitely many choices for (I, ψ) the result follows. □

Now $g_r(x_1, \ldots, x_r)$ is of the form $\sigma_{n+1}(x_1, \ldots, x_r) - x_e$, and $\sigma_{n+1}(x_1, \ldots, x_r)$ is of the form $f_i(y_1, \ldots, y_m)$ for some $i = 1, \ldots, l$ and $y_1, \ldots, y_m \in \{x_1, \ldots, x_r, 0, 1\}$. Now, working in $\mathcal{K}$ we apply theorem 2.2.33 to get that for each $\bar{x} \in [0,1]^m$, $\lambda \in \mathbb{N}$ with $\lambda \geq 1$ and $\bar{t} \in K^m$ such that $\bar{x} + \bar{t} \in [0,1]^m$

$$f_i(\bar{x} + \bar{t}) = \sum_{j=0}^{\lambda} \left[\frac{1}{j!} \left(\sum_{l=1}^{m} t_l \frac{\partial}{\partial x_l} \right)^j f_i \right] (\bar{x}) + R_\lambda, \tag{3.25}$$

where

$$R_\lambda = \left[\frac{1}{(\lambda+1)!} \left(\sum_{l=1}^{m} t_r \frac{\partial}{\partial x_l} \right)^{\lambda+1} f_i \right] (\bar{x} + \bar{t}'),$$

for some $\bar{t}'$ on the line segment between $\bar{x}$ and $\bar{x} + \bar{t}$. Using hypothesis (4) from theorem 3.2.1 we see that there exists C_λ a closed L-term (which does not depend on $\bar{x}$ or $\bar{t}$) such that

$$|R_\lambda| < C_\lambda \|\bar{t}\|^{\lambda+1}.$$

So, for all $\bar{x} \in [0,1]^m$ and all $\bar{t} \in K^m$ such that $\bar{x} + \bar{t} \in [0,1]^m$,

$$\left| f_i(\bar{x} + \bar{t}) - \sum_{j=0}^{\lambda} \left(\frac{1}{j!} \left(\sum_{l=1}^{m} t_l \frac{\partial}{\partial x_l} \right)^j f_i \right) (\bar{x}) \right| < C_\lambda \|\bar{t}\|^{\lambda+1}.$$

Now by hypothesis (3) for each $\lambda \geq 1$ and all monomials $\pi(x_1, \ldots, x_m)$ of degree at most λ there exists an L-term $\tau_\pi^\lambda(x_1, \ldots, x_m)$ such that

for all $\bar{x} \in [0,1]^m$ and all $\bar{t} \in K^m$ such that $\bar{x} + \bar{t} \in [0,1]^m$ we have

$$\left| \lambda! f_i(\bar{x} + \bar{t}) - \sum_{\text{monomials of degree} \leq \lambda} \tau_\pi^\lambda(\bar{x}) \pi(\bar{t}) \right| < \lambda! C_\lambda \|\bar{t}\|^{\lambda+1}. \tag{3.26}$$

Now recall that $\sigma_{n+1}(x_1, \ldots, x_r)$ is of the form $f_i(y_1, \ldots, y_m)$ where $y_1, \ldots, y_m \in \{x_1, \ldots, x_r, 0, 1\}$. For $\bar{w} = (w_1, \ldots, w_r) \in D^r(\bar{\sigma}')$ let

$$w_i' = \begin{cases} 0 & y_i = 0, \\ 1 & y_i = 1, \\ w_j & y_i = x_j \end{cases}$$

so that $\sigma_{n+1}(\bar{w}) = f_i(\bar{w}')$. Now let S, θ, B be as in lemma 3.4.10 and let μ, B' be as in lemma 3.4.11. Now let λ_0 be an integer larger than $\frac{\mu+1}{\theta}$. Now $b_1 > B$ so for each $i = 1, \ldots, m$ let $a_i \in S \cup \{0,1\}$ be such that $|b_i' - a_i| < b_1^{-\theta}$. So we have

$$\left| \lambda_0! b_e - \sum_{\text{monomials of degree} \leq \lambda_0} \tau_\pi^{\lambda_0}(\bar{a}) \pi(\bar{a} - \bar{b}') \right| < \lambda_0! C_{\lambda_0} b_1^{-\theta(\lambda_0+1)}, \tag{3.27}$$

$$b_1 > \max\{B', 2C_\lambda\} \tag{3.28}$$

$$|b_i' - a_i| < b_1^{-\theta}, \text{ for } i = 1, \ldots, m. \tag{3.29}$$

Now (3.27) $\wedge$ (3.28) $\wedge$ (3.29) may be expressed as $\chi(\bar{b})$ where $\chi(\bar{x})$ is a formula in the language of ordered rings with parameters from H. By lemma 3.4.7 there exists $\bar{c} \in V \cap H^r$ such that $\mathcal{H} \models \chi(\bar{c})$. We may also apply (3.26) in $\mathcal{H}$ to get

$$\left| \lambda_0! f_i(\bar{c}) - \sum_{\text{monomials of degree} \leq \lambda_0} \tau_\pi^{\lambda_0}(\bar{a}) \pi(\bar{a} - \bar{c}) \right| < \lambda_0! C_{\lambda_0} c_1^{-\theta(\lambda_0+1)} \tag{3.30}$$

So

$$|f_i(\bar{c}) - c_e| < 2C_{\lambda_0} c_1^{-\theta(\lambda_0+1)} \tag{3.31}$$

$$< 2C_{\lambda_0} c_1^{-(\mu+1)} \tag{3.32}$$

$$< c_1^{-\mu} \tag{3.33}$$

but this contradicts our choice of μ. So the proof is complete.

Chapter 4

‘Uniform’ model-completeness for o-minimal expansions of the real field by power functions

4.1 Overview

In this chapter we will use theorem 3.2.1 to prove a 'uniform' model-completeness result for raising to real powers. We begin by reviewing existing results by Miller.

4.2 Results by Miller on raising to real powers

The results summarised in this section are contained in the paper [11]. Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ and suppose that $\tilde{\mathbb{R}}$ is polynomially bounded with field of exponents $\mathbb{Q}$. Suppose further that $\tilde{\mathbb{R}}$ admits quantifier elimination and has a universal axiomatization (note that by remark 2.2.23 we can always ensure that this is the case by expanding our language). Let K be a subfield of $\mathbb{R}$ and suppose that for all $k \in K$ there is some closed bounded interval $I_k \subseteq \mathrm{Pos}(\mathbb{R})$ such that $x^k \restriction_{I_k}$ is definable in $\tilde{\mathbb{R}}$ by a quantifier-free formula $\phi_k(x, y)$. Let $\tilde{\mathbb{R}}^K = \langle \tilde{\mathbb{R}}, (x^k)_{k \in K} \rangle$, i.e. we expand the language of $\tilde{\mathbb{R}}$ by a function symbol f_k for each $k \in K$ and interpret f_k as the map

$$x \mapsto \begin{cases} x^k & x \in \mathrm{Pos}(\mathbb{R}), \\ 0 & x \in \mathbb{R} \setminus \mathrm{Pos}(\mathbb{R}). \end{cases}$$

Theorem 4.2.1. $\mathrm{Th}(\tilde{\mathbb{R}}^K)$ *admits quantifier elimination and is axiomatized by* $\mathrm{Th}(\tilde{\mathbb{R}})$ *together with the universal closures of*

(P1) $(x \leq 0 \to f_k(x) = 0) \wedge (x, y > 0 \to f_k(xy) = f_k(x) f_k(y))$; $k \in K$,

(P2) $(x > 1 \to f_k(x) > 1)$; $k \in K$ *with* $k > 1$,

(P3) $(f_{kl}(x) = f_k(f_l(x)) \wedge f_{k+l}(x) = f_k(x) f_l(x))$; $k, l \in K$,

(P4) $(x > 0 \to f_q(x) = x^q)$; $q \in \mathbb{Q}$,

(P5) $(x \in I_k \to \phi_k(x, f_k(x)))$; $k \in K$.

In particular $\mathrm{Th}(\tilde{\mathbb{R}}^K)$ *has a universal axiomatization.*

Remark 4.2.2. For theorem 4.2.1 we are assuming that $\tilde{\mathbb{R}}$ is polynomially bounded with field of exponents $\mathbb{Q}$. In Miller's paper he assumes instead that $\tilde{\mathbb{R}}$ is *of rational type.* It follows from theorem 2.2.59 that o-minimal theories which are polynomially bounded with field of exponents $\mathbb{Q}$ are of rational type. In fact it is not difficult to see that if T is the theory of an o-minimal expansion of the real field, then T is of rational type if and only if T is polynomially bounded with field of exponents $\mathbb{Q}$.

Corollary 4.2.3. *Let K be a subfield of $\mathbb{R}$. Then $\langle \mathbb{R}_{\text{an}}, (x^k)_{k\in K}\rangle$ admits quantifier elimination and has a universal axiomatization.*

Proof. This immediate from theorem 4.2.1 and the fact that $\mathbb{R}_{\text{an}}^{\mathbb{Q}}$ admits quantifier elimination, has a universal axiomatization and is polynomially bounded with field of exponents $\mathbb{Q}$ (see example 2.2.66). $\square$

Corollary 4.2.4. *Let S be a subset of $\mathbb{R}$. Then $\langle \overline{\mathbb{R}}, (x^s)_{s\in S}, (s)_{s\in S}\rangle$ is model-complete.*

Sketch of proof. Note that for any $r \in \mathbb{R}$, if we let $f_1, f_2 : \text{Pos}(\mathbb{R}) \to \mathbb{R}$ be given by $f_1(x) = x^{-1}$ and $f_2(x) = x^r$ then (f_1, f_2) is a Pfaffian chain. It follows from theorem 3.1.4 that $\langle \overline{\mathbb{R}}, (x^s \restriction_{[1,2]})_{s\in S}, (s)_{s\in S}\rangle$ is model-complete. Combining this with theorem 4.2.1 one can deduce that $\langle \overline{\mathbb{R}}, (x^s)_{s\in S}, (s)_{s\in S}\rangle$ is model-complete. $\square$

4.3 Statement of the theorem

For the statement of the main theorem of this chapter it will be helpful to clear about the first order language we are using. First we fix some index set I. Let

$$L' = L_{\text{ord}} \cup \{(f_i)_{i\in I}, (c_i)_{i\in I}\},$$

where the f_i are unary function symbols and the c_i are constant symbols. We will consider L'-structures of the form

$$\langle \overline{\mathbb{R}}, (x^{c_i})_{i\in I}, (c_i)_{i\in I}\rangle,$$

i.e. for each $i \in I$ we interpret the f_i as a real power functions with exponent (the interpretation of) c_i. Now let $\mathfrak{C}'$ be class of all such L'-structures and let $T' = \text{Th}(\mathfrak{C}')$.

Theorem 4.3.1. *T' is model-complete.*

Remark 4.3.2. Clearly theorem 4.3.1 implies corollary 4.2.4. Furthermore, it is in the following sense a 'uniform' version of theorem 4.2.4: let $\phi(\bar{x})$ be an L'-formula; theorem 4.2.4 says that for any $\mathcal{C}' \in \mathfrak{C}'$ there exists an existential L'-formula $\psi(\bar{x})$ such that $\text{Th}(\mathcal{C}') \models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Theorem 4.3.1 says that there exists an existential L'-formula such that $T \models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$, so in particular $\text{Th}(\mathcal{C}') \models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$ for all $\mathcal{C}' \in \mathfrak{C}'$ (i.e. the same $\psi(\bar{x})$ works for all $\mathcal{C}' \in \mathfrak{C}'$).

4.4 Preliminaries

Before proving that T' is model-complete, as an intermediate step, we introduce a theory T.

Let I be the same index set as above and let $L = L_{\text{ord}} \cup \{(g_i)_{i\in I}, (c_i)_{i\in I}, a, b\}$ where the g_i are new unary function symbols, the c_i are the same constant symbols as in L and a, b are new constant symbols (so $L \cap L' = L_{\text{ord}} \cup \{(c_i)_{i\in I}\}$). We will consider L-structures of the form

$$\langle \overline{\mathbb{R}}, (x^{c_i} \restriction_{[a,b]})_{i\in I}, (c_i)_{i\in I}, a, b\rangle,$$

i.e. for each $i \in I$ we interpret g_i as the power function with exponent (the interpretation of) c_i on the interval $[a, b]$ and 0 outside of $[a, b]$, and we also assume that $0 < a < b$. We let $\mathfrak{C}$ denote the class of all such L-structures and let $T = \text{Th}(\mathfrak{C})$.

Remark 4.4.1. The classes $\mathfrak{C}'$ and $\mathfrak{C}$ are of course not elementary classes, i.e. there are models of T' and T which are not in $\mathfrak{C}'$ and $\mathfrak{C}$ respectively. In chapter 5 we study a completion of T' (in the case that the index set I consists of one element) all of whose models are non-Archimedean and hence not in $\mathfrak{C}'$.

We now state and prove a number of lemmas which will allow us to deduce facts about the theories T, T' from properties of the theory T_{exp}.

Lemma 4.4.2. *Given an L'-formula $\phi(\bar{x})$ there is an L_{exp}-formula $\tilde{\phi}(\bar{x}, \bar{y})$ with the property that for any $\mathcal{C}' \in \mathfrak{C}'$ there exists a real tuple $\bar{a}$ such that for any real tuple $\bar{b}$ we have*

$$\mathcal{C}' \models \phi(\bar{b}) \text{ if and only if } \mathbb{R}_{\text{exp}} \models \tilde{\phi}(\bar{b}, \bar{a}).$$

Furthermore, if ϕ is an L'-sentence then

$$\phi \in T' \text{ if and only if } \forall \bar{y} \tilde{\phi}(\bar{y}) \in T_{\text{exp}}.$$

Proof. We note that the family of functions

$$\{x^r : \mathbb{R} \to \mathbb{R} : r \in \mathbb{R}\}$$

is a definable family in $\mathbb{R}_{\text{exp}}$ (since $x^r = \exp(r \log(x))$). Given the L'-formula $\phi(\bar{x})$ let $\{i_1, \ldots, i_n\}$ be the set of $i \in I$ such that the function symbol f_i or the constant symbol c_i occurs $\phi(\bar{x})$. Let $y_1, \ldots, y_n$ be free variables not already occurring in $\phi(\bar{x})$. Let $\tilde{\phi}(\bar{x}, y_1, \ldots, y_n)$ be the formula obtained from $\phi(\bar{x})$ by replacing occurrences of $f_{i_j}(x)$ by $x^{y_{i_j}}$ and occurrences of c_{i_j} by y_{i_j}, for each $j = 1, \ldots, n$. For a particular

$\mathcal{C}' \in \mathfrak{C}'$ we can clearly satisfy the first statement of the lemma by choosing $\bar{a}$ to be (the interpretations of) $(c_{i_1}, \ldots, c_{i_n})$ in $\mathcal{C}'$. The second statement now follows since $\phi \in T'$ if and only if $\mathcal{C}' \models \phi$ for all $\mathcal{C}' \in \mathfrak{C}'$ $\square$

We have a very similar lemma for the class of L-structures $\mathfrak{C}$.

Lemma 4.4.3. *Given an L-formula $\phi(\bar{x})$ there is an $L_{\exp}$-formula $\check{\phi}(\bar{x}, \bar{y}, u, v)$ with the property that for any $\mathcal{C} \in \mathfrak{C}$ there exists reals $0 < r < s$ and a real tuple $\bar{a}$ such that for any real tuple $\bar{b}$ we have*

$$\mathcal{C} \models \phi(\bar{b}) \text{ if and only if } \mathbb{R}_{\exp} \models \check{\phi}(\bar{b}, \bar{a}, r, s).$$

Furthermore, if ϕ is an L-sentence then

$$\phi \in T \text{ if and only if } \forall \bar{y}, u, v(0 < u < v \rightarrow \check{\phi}(\bar{y}, u, v).$$

Proof. The proof follows that of lemma 4.4.2. We note that the family of functions

$$\{x^r \restriction_{[a,b]} : \mathbb{R} \rightarrow \mathbb{R} : r, a, b \in \mathbb{R} \text{ with } 0 < a < b\}$$

is a definable family in $\mathbb{R}_{\exp}$. Given the L-formula $\phi(\bar{x})$ let $\{i_1, \ldots, i_n\}$ be the set of $i \in I$ such that the function symbol g_i or the constant symbol c_i occurs $\phi(\bar{x})$. Let $y_1, \ldots, y_n, u, v$ be free variables not already occurring in $\phi(\bar{x})$. Let $\check{\phi}(\bar{x}, y_1, \ldots, y_n, u, v)$ be the $L_{\exp}$-formula obtained by replacing occurrences of $g_{i_j}(x)$ by $x^{y_{i_j}} \restriction_{[u,v]}$ and occurrences of c_{i_j} by y_{i_j}, for each $j = 1, \ldots, n$. For a particular $\mathcal{C} \in \mathfrak{C}$ we can clearly satisfy the first statement of the lemma by choosing $\bar{a}$ to be (the interpretations of) $(c_{i_1}, \ldots, c_{i_n})$ and r, s to be (the interpretations of) a, b in $\mathfrak{C}$. The second statement now follows since $\phi \in T$ if and only if $\mathcal{C} \models \phi$ for all $\mathcal{C} \in \mathfrak{C}$ $\square$

We now prove a lemma which tells us that *all* models of T' naturally induce models of T.

Lemma 4.4.4. *Let $\mathcal{A}' \models T'$ and let $c, d \in A$ with $0 < c < d$. Let $\mathcal{A}$ be the natural L-structure induced by $\mathcal{A}$ with respect to c, d, i.e. for each $i \in I$ we interpret g_i as the restriction of x^{c_i} to the interval $[c, d]$. Then $\mathcal{A} \models T$.*

Proof. For any L-formula $\phi(\bar{x})$ there is an L'-formula $\hat{\phi}(\bar{x}, y, z)$ such that if $\mathcal{R}' \models T'$ and $r, s \in R$ with $0 < r < s$ and $\mathcal{R}$ is the natural L-structure induced by $\mathcal{R}'$ with respect to r, s then $\phi(\bar{x})$ and $\hat{\phi}(\bar{x}, r, s)$ define the same set when interpreted in $\mathcal{R}$ and $\mathcal{R}'$ respectively. It follows that if ϕ is an L-sentence then $T \models \phi$ if and only if $T' \models \forall x \forall y(0 < x < y \rightarrow \hat{\phi}(x, y))$. Now suppose $\phi \in T$. We must prove that $\mathcal{A} \models \phi$. It is sufficient to prove that $\mathcal{A}' \models \hat{\phi}(c, d)$. But $\mathcal{A}' \models T'$ so $\mathcal{A}' \models \forall x \forall y(0 < x < y \rightarrow \hat{\phi}(x, y))$. $\square$

As mentioned above, the aim of this chapter will be to prove that T' is model-complete. We will first prove that T is model-complete.

4.5 Model-completeness of T

4.5.1 Models of T are o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$

In this section we will use lemma 4.4.3 to prove that all models of T are polynomially bounded with field of exponents $\mathbb{Q}$.

In what follows we will sometimes denote the L_{exp}-formula $\hat{\phi}(\bar{x}, \bar{y}, u, v)$, produced as in lemma 4.4.3, from the L-formula $\phi(\bar{x})$ simply as $\hat{\phi}(\bar{x}, \bar{w})$ (i.e. we will not distinguish notationally between free variables representing exponents and free variables representing the interval of restriction).

Lemma 4.5.1. *All models of T are o-minimal.*

Proof. Let $\mathcal{R} \models T$. To prove that $\mathcal{R}$ is o-minimal it is sufficient to show that every definable subset of R which contains no intervals is finite. So let A be a definable subset of R given by an L-formula $\phi(\bar{a}, x)$, where $\bar{a}$ is some tuple in R, and suppose that A contains no intervals. For the L-formula $\phi(\bar{y}, x)$ we produce an L_{exp}-formula $\hat{\phi}(\bar{y}, x, \bar{z})$ as in lemma 4.4.3. By the o-minimality of T_{exp} and the uniform finiteness property of o-minimal structures (theorem 2.2.10) we can choose N such that

$$T_{\text{exp}} \models \forall \bar{z}, \bar{y} \left(\text{if } \phi'_{(\bar{z},\bar{y})}(x) \text{ contains no intervals then } \exists_{\leq N} x \phi'_{(\bar{z},\bar{y})}(x)\right).$$

So we have, for all $\mathcal{S} \in \mathfrak{C}$,

$$\mathcal{S} \models \forall \bar{y} \left(\text{if } \phi_{\bar{y}}(x) \text{ contains no intervals then } \exists_{\leq N} x \phi_{\bar{y}}(x)\right)$$

and so

$$\mathcal{R} \models \forall \bar{y} \left(\text{if } \phi_{\bar{y}}(x) \text{ contains no intervals then } \exists_{\leq N} x \phi_{\bar{y}}(x)\right)$$

Hence

$$\mathcal{R} \models \exists_{\leq N} x \phi(\bar{a}, x).$$

□

Remark 4.5.2. The same proof shows that all models of T' are o-minimal.

Lemma 4.5.3. *All models of T are polynomially bounded with field of exponents $\mathbb{Q}$.*

Proof. Note that for all $\mathcal{S} \in \mathfrak{C}$ we have that $\mathcal{S}$ is a reduct of $\mathbb{R}_{\mathrm{an}}$ and hence is polynomially bounded with field of exponents $\mathbb{Q}$ (example 2.2.66). Let $\phi(\bar{z}, x, y)$ be an L-formula. It will be sufficient to prove that there exists $n \geq 1$ and $q_1, \ldots, q_n \in \mathbb{Q}$ such that

$$T \models \forall \bar{z} \exists w \text{ (if } \phi_{\bar{z}}(x,y) \text{ defines a function } f(x) = y \text{ then } f(x) \text{ is asymptotic}$$
$$\text{to one of } wx^{q_1}, \ldots, wx^{q_n})$$

Let $\hat{\phi}(\bar{z}, x, y, \bar{w})$ be as given by lemma 4.4.3. Now for any real tuples $\bar{a}$, $\bar{b}$, the $\mathbb{R}_{\mathrm{exp}}$-definable set given by $\hat{\phi}(\bar{b}, x, y, \bar{a})$ is also definable in some $\mathcal{S} \in \mathfrak{C}$. Therefore, if $\hat{\phi}(\bar{b}, x, y, \bar{a})$ defines a function $f(x) = y$ then $f(x)$ must be asymptotic to cx^q for some $c \in \mathbb{R}$ and some $q \in \mathbb{Q}$. So we have an $\mathbb{R}_{\mathrm{exp}}$-definable function $P : \mathbb{R}^{l(\bar{w})+l(\bar{z})} \to \mathbb{R}$ which does the following: if $\hat{\phi}(\bar{b}, x, y, \bar{a})$ defines a function $f(x) = y$ then $P(\bar{a}, \bar{b})$ is the unique rational such that $f(x)$ is asymptotic to some real multiple of x^q and otherwise $P(\bar{a}, \bar{b}) = 0$. Since the image of P is contained in $\mathbb{Q}$, by the o-minimality of $\mathbb{R}_{\mathrm{exp}}$, it must be finite. Let $\{q_1, \ldots, q_n\}$ be the image of P. Then

$$T \models \forall \bar{z} \exists w \text{ (if } \phi_{\bar{z}}(x,y) \text{ defines a function } f(x) = y \text{ then } f(x) \text{ is asymptotic}$$
$$\text{to one of } wx^{q_1}, \ldots, wx^{q_n}).$$

□

4.5.2 Applying theorem 3.2.1 to obtain model-completeness of T

Recall that it is our intention to prove that T is model-complete. To do this we must modify our theory T so that theorem 3.2.1 is directly applicable. Let us first recall some standard facts about changing languages.

Let L_1, L_2 be first order languages such that $L_1 \subseteq L_2$. Let T_2 be an L_2-theory and let T_1 be the L_1-reduct of T_2. We say that T_2 is an *extension by definitions* of T_1 if for each L_2-formula ϕ there exists an L_1-formula ψ such that $T_2 \models \phi \leftrightarrow \psi$. Notice that in this case every model of T_1 naturally induces a model of T_2 and vice-versa and both have the same definable sets. Consequently all models of T_1 are o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$ if and only if all models of T_2 are o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$. Furthermore, it is straightforward to prove that if L_2 is an expansion of L_1 by constants and function symbols and T_2 is an extension by definitions of T_1 such that for each new function symbol f and each new constant symbol c the formulas $f(x) = y$ and $x = c$ are equivalent (modulo T_2) to existential L_1-formulas then T_2 is model-complete if and only if T_1 is.

We now perform the modifications. At each step it will follow from the above discussion that the new theory is model-complete if and only if the old one is. Furthermore at each stage we will preserve the property that all models are o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$.

Step 1 Expand L by a new function symbol h_i for each $i \in I$. Denote the extended language by L^*. For each $i \in I$ extend T by the L^*-sentence

$$\forall x \, (h_i(x) = f_i(a + (b-a)x))) \, .$$

Let T^* be the resulting extension of T.

Step 2 Let L^{**} be the reduct of L^* consisting of the language of ordered rings, the function symbols h_i, the constant symbols c_i and a, b. Let T^{**} be the L^{**}-reduct of L^*.

Step 3 Let L^{***} be the expansion of L^{**} by a single function symbol k. Let T^{***} be the extension of T^{**} by the sentence

$$\forall x \forall y (k(x) = y \leftrightarrow \\ ((0 \leq x \leq 1 \wedge y(a + (b-a)x) = 1) \vee (\neg(0 \leq x \leq 1) \wedge y = 0))).$$

Step 4 Finally we expand L^{***} by constant symbols for all existentially definable points and T^{***} by the appropriate sentences, ie. if $\phi(x)$ is an existential L^{***}-formula such that $T^{***} \models \exists! x \phi(x)$ then we add a constant symbol c_ϕ and a sentence $\phi(c_\phi)$. Let L^{****} and T^{****} be the resulting language and theory.

The language L^{****} consists of $+, \cdot, <$, function symbols h_i for each $i \in I$, a further function symbol k and a set of constants C. We wish to apply theorem 3.2.1 to show that for any language $\tilde{L} = \{+, \cdot, <, h_{i_1}, \ldots, h_{i_n}, k, (c)_{c \in C}\} \subseteq L^{****}$ the $\tilde{L}$-reduct $\tilde{T}$ of T^{****} is model-complete. The model-completeness of T^{****} (and hence T) would clearly follow from this. We must check that the conditions (1)-(4) of theorem 3.2.1 hold for $\tilde{T}$. Condition (1) follows from the definition of T and the fact that $\overline{\mathbb{R}}$ is real-closed. Condition (5) holds for T by lemma 4.5.1 and lemma 4.5.3 and hence also holds for T^{****} by the discussion at the beginning of section 4.5.2. Condition (2) follows from the fact that real power functions are smooth on $(0, \infty)$ and the fact that in an expansion of a field the statement 'f is differentiable at x' is first order. Condition (3) holds because for any $i \in I$ the interpretation of the sequence of functions (h_i, k) in a model of T^{****} is a Pfaffian chain. Finally, condition (4) holds because power functions are monotonic so their value on a closed bounded interval of the positive halfline is bounded by their value at one of the end points.

Hence T is model-complete.

4.6 Model-completeness of T'

In this section we will prove that T' is model-complete. First we prove a preliminary theorem.

4.6.1 A preliminary result

Let $\mathcal{R}$ be a power-bounded o-minimal expansion of an ordered field with field of exponents K. Suppose also that $\mathcal{R}$ has a constant symbol for each element of its domain. Let L be the language of $\mathcal{R}$. Now let S be a subfield of R containing K and suppose that for each $s \in S$ there exists a power function f_s on R with exponent s. Note that we do not assume f_s to be definable and indeed unless $s \in K$ it cannot be. Now assume that for each $s \in S$ and each I a closed bounded interval of $\mathrm{Pos}(R)$ the restriction of f_s to I *is* definable in $\mathcal{R}$. Denote this definable restriction by f_s^I. It is easy to check that the functions f_s satisfy the following (cf. theorem 4.2.1):

(P1) $(x \leq 0 \rightarrow f_s(x) = 0) \wedge (x, y > 0 \rightarrow f_s(xy) = f_s(x)f_s(y))$; $s \in S$,

(P2) $(x > 1 \rightarrow f_s(x) > 1)$; $s \in S$ and $s > 1$,

(P3) $(f_{rs}(x) = f_r(f_s(x)) \wedge f_{r+s}(x) = f_r(x)f_s(x))$; $r, s \in S$,

(P4) $(x > 0 \rightarrow f_k(x) = x^k)$; $k \in K$,

(P5) $(a \leq x \leq b \rightarrow f_s(x) = f_s^{[a,b]}(x))$; $s \in S$, $a, b \in R$ with $0 < a < b$.

Now let $\mathcal{R}^+$ be the expansion of $\mathcal{R}$ by f_s for each $s \in S$. Denote the language of $\mathcal{R}^+$ by L^+. The following theorem generalizes theorem 4.2.1 of Miller and we will prove it using the same methods.

Theorem 4.6.1. $\mathrm{Th}(\mathcal{R}^+)$ *is axiomatized by* $\mathrm{Th}(\mathcal{R})$ *and the universal closures of the axiom schemes (P1)-(P5). Furthermore, if* $\mathrm{Th}(\mathcal{R})$ *admits quantifier elimination then so does* $\mathrm{Th}(\mathcal{R}^+)$.

By taking an extension by definitions if necessary we may assume that $\mathrm{Th}(\mathcal{R})$ admits quantifier elimination and is axiomatized by its universal theory (see remark 2.2.23).

Let $T = \mathrm{Th}(\mathcal{R}) \cup \{(P1), \ldots, (P5)\}$. We must prove that T is complete and admits quantifier elimination. By remark 2.3.5 it will be sufficient to prove that T is model-complete. We will use theorem 2.3.1 (6) to establish the model-completeness of T.

Let $\mathcal{B} \models \mathrm{Th}(\mathcal{R})$. Since $\mathrm{Th}(\mathcal{R})$ admits quantifier elimination and we have a constant symbol in L for each element of R we must have that $\mathcal{R}$ occurs as an elementary substructure of $\mathcal{B}$ (upto isomorphism). Define $V_{\mathcal{B}}$ to be the convex hull of R in $\mathcal{B}$ so that $V_{\mathcal{B}}$ is a $\mathrm{Th}(\mathcal{R})$-convex subring of $\mathcal{B}$ (see section 2.2.5.1). Denote the corresponding valuation and value group by $v_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ respectively. In fact, we will often drop the subscript and denote the valuation map by v. In what follows in this section, if we talk about about a valuation on a structure $\mathcal{B} \models \mathrm{Th}(\mathcal{R})$ we will always mean the valuation with respect to the convex hull of R.

Now suppose that $\mathcal{B}$ expands to a model $\mathcal{B}^+$ of T. It follows from the axiom schemes (P1)-(P4) that each f_s is monotonic on $\mathrm{Pos}(B)$. Consequently, since $\mathcal{R}$ is closed under each f_s we have that $\mathrm{Un}(V_{\mathcal{B}})$ is closed under each f_s. Therefore the value group $\Gamma_{\mathcal{B}}$ has the structure of an S-vector space.

If we have $\mathcal{A}, \mathcal{B} \models \mathrm{Th}(\mathcal{R})$ such that $\mathcal{A} \subseteq \mathcal{B}$ then clearly $V_{\mathcal{B}} \cap A = V_{\mathcal{A}}$. Hence, we can consider $\mathcal{B}$ as a valued field-extension of $\mathcal{A}$. In this circumstance we will denote the common valuation map by v.

Given $\mathcal{A} \models T$ we will denote its L-reduct by $\mathcal{A}^-$. Clearly $\mathcal{A}^- \models \mathrm{Th}(\mathcal{R})$.

Lemma 4.6.2. *Let $\mathcal{B} \models T$ and let $\mathcal{A}$ be a substructure of $\mathcal{B}^-$ (so $\mathcal{A} \models \mathrm{Th}(\mathcal{R})$). Suppose further that there exists $P \subseteq \mathrm{Pos}(A)$ such that $v(P) = v(A)$ and $f_s(P) \subseteq A$ for all $s \in S$. Then $f_s(A) \subseteq A$ for all $s \in S$ and $\langle \mathcal{A}, (f_s \restriction_A)_{s \in S} \rangle \models T$.*

Proof. Since T has a universal axiomatization the second conclusion of the lemma will follow from the first. So take $a \in \mathrm{Pos}(A)$. There exists $p \in P$ with $v(a) = v(p)$, ie. $a = up$ where $u \in A$ and $v(u) = 0$. Since $v(u) = 0$ there exists $b, c \in R$ such that $b < u < c$. Now, for each $s \in S$

$$\begin{aligned} f_s(a) =& f_s(up) \\ =& f_s(u) f_s(p) \\ =& f_s^{[b,c]}(u) f_s(p) \in A. \end{aligned}$$

□

Lemma 4.6.3. *Let $\mathcal{A}, \mathcal{B} \models T$ and let $\phi : \mathcal{A}^- \to \mathcal{B}^-$ be an L-embedding. If there exists $P \subseteq \mathrm{Pos}(A)$ such that $v(P) = v(A)$ and $\phi(f_s(p)) = f_s(\phi(p))$ for all $s \in S$ and all $p \in P$. Then ϕ is an L^+-embedding.*

Proof. Let $a \in \mathrm{Pos}(A)$ and let $s \in S$. We must show that $\phi(f_s(a)) = f_s(\phi(a))$. Since $v(P) = v(A)$ there exists $u \in A$ with $v(u) = 0$ and $p \in P$ such that $a = up$. Choose $b, c \in R$ such that $u \in [b, c]$.

$$\begin{aligned}\phi(f_s(a)) =& \phi(f_s(up))\\ =& \phi(f_s(u))\phi(f_s(p))\\ =& \phi(f_s^{[b,c]}(u))\phi(f_s(p))\\ =& f_s^{[b,c]}(\phi(u))f_s(\phi(p))\\ =& f_s(\phi(u))f_s(\phi(p))\\ =& f_s(\phi(u)\phi(p))\\ =& f_s(\phi(a)).\end{aligned}$$

□

By theorem 2.3.1 (6) , in order to prove that T admits quantifier elimination it is sufficient to prove the following lemma:

Lemma 4.6.4. *Let $\mathcal{A}, \mathcal{B} \models T$ and suppose that $\mathcal{A}$ is a proper substructure of $\mathcal{B}$. Let $\mathcal{B}'$ be a $|B|^+$-saturated elementary extension of $\mathcal{A}$. Then there exists $\mathcal{C} \models T$ such that $\mathcal{A} \subsetneq \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ embeds into $\mathcal{B}'$ over $\mathcal{A}$.*

Proof. Since both $v_{\mathcal{B}}$ and $v'_{\mathcal{B}}$ extend $v_{\mathcal{A}}$ we may unambiguously denote the valuations on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}'$ by v. Suppose first that $v(A) = v(B)$. Choose $x \in B \backslash A$. By saturation we may choose $y \in B' \setminus A$ realizing the same cut in A as x. Let $C = \mathcal{A}^-\langle x \rangle$ and let $C' = \mathcal{A}^-\langle y \rangle$. Since $\mathrm{Th}(R)$ admits quantifier elimination it follows from lemma 2.2.24 that there exists an L-isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ that is the identity on A and sends x to y. Since $v(C) = v(A)$ we have $v(C') = v(A)$. So by lemma 4.6.2 and lemma 4.6.3 C and C' expand to models of T and ϕ is an L^+-embedding.

Now suppose that $v(A) \neq v(B)$. Choose $x \in \mathrm{Pos}(B)$ such that $v(x) \notin v(A)$. By saturation we may choose $y \in B'$ realizing the same cut in A as x. Now let κ be the dimension of S as a K-vector space and let $\langle e_\gamma : \gamma < \kappa \rangle$ be a K-basis for S. Define

$$\begin{aligned} x_\gamma =& x^{e_\gamma}, \\ y_\gamma =& y^{e_\gamma}. \end{aligned}$$

Now recursively define $\mathcal{C}_\gamma, \mathcal{C'}_\gamma \models \mathrm{Th}(R)$ as follows:

$$\mathcal{C}_\gamma = \begin{cases} \mathcal{A}^- & \gamma = 0 \\ \mathcal{C}_\beta\langle x_\beta \rangle & \gamma = \beta + 1 \\ \bigcup_{\beta<\gamma} \mathcal{C}_\beta & \gamma \text{ a limit ordinal.} \end{cases}$$

Similarly

$$\mathcal{C}'_\gamma = \begin{cases} \mathcal{A}^- & \gamma = 0 \\ \mathcal{C'}_\beta\langle y_\beta \rangle & \gamma = \beta + 1 \\ \bigcup_{\beta<\gamma} \mathcal{C'}_\beta & \gamma \text{ a limit ordinal.} \end{cases}$$

We must show the following:

1. $\mathcal{C}_\kappa$ expands to a model of T,

2. there is an L^+-isomorphism $\phi : \mathcal{C}_\kappa \to \mathcal{C'}_\kappa$ which is the identity on A.

We first show that $\{v(x_\gamma) : \gamma < \kappa\}$ is K-linearly independent over $v(A)$. For suppose that this is not the case. Then there exists $k_1, \ldots, k_n \in K$ and $\gamma_{i_1}, \ldots, \gamma_{i_n} < \kappa$ such that

$$\sum_{i=1}^{n} k_i v(x_{\gamma_i}) \in v(A).$$

Now

$$\begin{aligned} \sum_{i=1}^{n} k_i v(x_{\gamma_i}) &= \sum_{i=1}^{n} v(x_{\gamma_i}^{k_i}) \\ &= v\left(\prod_{i=1}^{n} x^{e_{\gamma_i} k_i}\right) \\ &= v\left(x^{\sum_{i=1}^{n} e_{\gamma_i} k_i}\right). \end{aligned}$$

Let $r = \sum_{i=1}^{n} e_{\gamma_i} k_i$. Then $v(x^r) \in v(A)$ so $v(x) \in v(A)$ which contradicts the choice of x. In the same way we see that $\{v(y_\gamma) : \gamma < \kappa\}$ is K-linearly independent over $v(A)$.

Since $\mathrm{Th}(\mathcal{R})$ is a power-bounded theory with field of exponents K, by the valuation inequality (theorem 2.2.59), for each $\gamma < \kappa$ we have $v(\mathcal{C}_{\gamma+1}^\times) = v(\mathcal{C}_\gamma^\times) \oplus K v(x_\gamma)$.

Now let

$$P = \{ax^s : a \in \mathrm{Pos}(A), s \in S\},$$
$$P' = \{ay^s : a \in \mathrm{Pos}(A), s \in S\}.$$

We claim that $v(P) = v(\mathcal{C}_\kappa)$ and $v(P') = v(\mathcal{C}'_\kappa)$. We will prove the former, the latter being proved in the same way.

It is clearly sufficient to prove that for all $\gamma < \kappa$ for any $\alpha \in \mathcal{C}_\gamma$ there exists $\beta \in P$ such that $v(\beta) = v(\alpha)$.

$\gamma = 0$ This is clear since $\mathrm{Pos}(A) \subseteq P$.

γ is a successor ordinal So $\gamma = \delta + 1$ for some ordinal δ, and for all $\alpha \in \mathcal{C}_\delta$ there exists $\beta \in P$ such that $v(\beta) = v(\alpha)$. Now let $\alpha \in \mathcal{C}_{\delta+1} = \mathcal{C}_\delta\langle x_\delta\rangle$. Now $v(\mathcal{C}^\times_{\delta+1}) = v(\mathcal{C}^\times_\delta) \oplus Kv(x_\delta)$. Therefore there exists $a \in A$, $r \in S$ and $k \in K$ such that $v(\alpha) = v(ax^r) + kv(x^{e_\delta})$. So $v(\alpha) = v(ax^{r+qe_\delta})$ and $ax^{r+ke_\delta} \in P$.

γ is a limit ordinal This is immediate.

Therefore, since $f_s(P) \subseteq C_\kappa$ and $f_s(P') \subseteq C'_\kappa$ for all $s \in S$, by lemma 4.6.2, both $\mathcal{C}_\kappa$ and $\mathcal{C}'_\kappa$ have expansions to models of T. It remains to prove (2). By lemma 4.6.3 it is sufficient to find an L-isomorphism $\phi : \mathcal{C}_\kappa \to \mathcal{C}_{\kappa'}$ such that ϕ is the identity on A and such that $\phi(p^r) = \phi(p)^r$ for all $p \in P$ and $r \in S$. We recursively construct L-isomorphisms $\langle \phi_\gamma : \mathcal{C}_\gamma \to \mathcal{C}'_\gamma : \gamma \leq \kappa\rangle$ such that ϕ_α extends ϕ_β whenever $\alpha > \beta$ and each ϕ_α is the identity on A.

$\gamma = 0$ Let ϕ be the identity map.

γ is a successor ordinal So $\gamma = \beta + 1$ for some ordinal β. We suppose we have an L-isomorphism $\phi_\beta : \mathcal{C}_\beta \to \mathcal{C}'_\beta$. By lemma 2.2.24 and the fact that $\mathrm{Th}(\mathcal{R})$ admits quantifier elimination, to show that ϕ_β extends to an L-isomorphism from $\mathcal{C}_{\beta+1}$ to $\mathcal{C}'_{\beta+1}$ it is sufficient to prove that x_β and y_β realize corresponding (under ϕ_β) cuts in $\mathcal{C}_\beta$ and $\mathcal{C}'_\beta$ respectively. So let $c \in \mathcal{C}_\beta$ and suppose that $x_\gamma < c$ (note that since $x > 0$ we must have $c > 0$). Choose $a \in A$ and $r \in S$ such that $v(c) = v(ax^r)$. Since $0 < x_\gamma < c$ and $v(x_\gamma) \neq v(c)$ we have that $v(x_\gamma) > v(ax^r)$, ie. $e_\gamma v(x) > v(a) + rv(x)$. So $v(a) < (e_\gamma - r)v(x)$. Now suppose that $e_\gamma > r$ (the other case is similar). Then $v(x) > \frac{1}{e_\gamma - r}v(a)$. Now since y realizes the same cut in A as x we have that $v(y) > \frac{1}{e_\gamma - r}v(a)$. Now reversing the steps above we get $v(y_\gamma) > v(ay^r) = v(\phi(c))$. So $y_\gamma < \phi(c)$.

γ is a limit ordinal Take ϕ_γ to be the union of the ϕ_α's for $\alpha < \gamma$.

So we have constructed ϕ an L-isomorphism from $\mathcal{C}_\kappa$ to $\mathcal{C}'_\kappa$ which is the identity on A. By lemma 4.6.3 it remains to prove that for all $c \in P$ and all $s \in S$ we have $\phi(c^s) = \phi(c)^s$. Take such a c and write it as ax^r where $a \in A$ and $r \in S$.

We first note the following: if $r \in S$ then $\phi(x^r) = y^r$. To see this we write r as $r = \sum_{i=1}^n k_i e_{\gamma_i}$ for some $k_i \in K$ and $\gamma_i < \kappa$.

Now

$$\begin{aligned}
\phi(x^r) &= \phi(x^{\sum_{i=1}^n k_i e_{\gamma_i}}) \\
&= \phi(\prod_{i=1}^n x^{k_i e_{\gamma_i}}) \\
&= \prod_{i=1}^n \phi(x^{e_{\gamma_i}})^{k_i} \\
&= \prod_{i=1}^n (y^{e_{\gamma_i}})^{k_i} \\
&= y^r.
\end{aligned}$$

So

$$\begin{aligned}
\phi(c^s) &= \phi(a^s x^{rs}) \\
&= \phi(a^s)\phi(x^{rs}) \\
&= a^s y^{rs},
\end{aligned}$$

and

$$\begin{aligned}
\phi(c)^s &= \phi(ax^r)^s \\
&= \phi(a)^s \phi(x^r)^s \\
&= a^s x^{rs}.
\end{aligned}$$

□

So T admits quantifier elimination. It remains to show that T is complete. To see this we let $\mathcal{A} \models T$. Now $\mathcal{R}^+$ embeds in $\mathcal{A}$. By quantifier elimination for T this embedding must be elementary. So in particular $\mathcal{A}$ and $\mathcal{R}^+$ are elementarily equivalent. So T is complete.

4.6.2 Model-completeness of T'

Let L, $\mathfrak{C}$ and T and L', $\mathfrak{C}'$ and T' be as in sections 4.3 and 4.4. We are now in a position to prove theorem 4.3.1

Proof of theorem 4.3.1. Let $\mathcal{H}, \mathcal{K} \models T'$ and suppose that $\mathcal{H} \subseteq \mathcal{K}$. Let $\mathcal{H}^+$ and $\mathcal{K}^+$ be the expansions of $\mathcal{H}$ and $\mathcal{K}$ by constant symbols for elements of H. To prove the model-completeness of T it will of course be sufficient to prove that $\mathcal{H}^+$ and $\mathcal{K}^+$ are elementarily equivalent. Let

$$\mathcal{H}^+_{res} = \langle \overline{H}, (x^{c_i} \restriction_J)_{i \in I, J \in \mathfrak{J}}, (r)_{r \in H} \rangle,$$
$$\mathcal{K}^+_{res} = \langle \overline{K}, (x^{c_i} \restriction_J)_{i \in I, J \in \mathfrak{J}}, (r)_{r \in H} \rangle,$$

where $\mathfrak{J}$ is the set of all closed bounded intervals of $\mathrm{Pos}(H)$.

By lemma 4.4.4, for each $J = [a_J, b_J] \in \mathfrak{J}$ we have

$$\langle \overline{H}, (x^{c_i} \restriction_J)_{i \in I}, a_J, b_J, (c_i)_{i \in I} \rangle \models T$$

and

$$\langle \overline{K}, (x^{c_i} \restriction_J)_{i \in I}, a_J, b_J, (c_i)_{i \in I} \rangle \models T.$$

So, since T is model-complete,

$$\mathcal{H}^+_{res} \equiv \mathcal{K}^+_{res}.$$

Now let

$$\mathcal{H}^{+*}_{res} = \langle \mathcal{H}^+_{res}, (x^{c_i})_{i \in I} \rangle,$$
$$\mathcal{K}^{+*}_{res} = \langle \mathcal{K}^+_{res}, (x^{c_i})_{i \in I} \rangle.$$

Clearly it is sufficient to prove that $\mathcal{H}^{+*}_{res} \equiv \mathcal{K}^{+*}_{res}$. This follows from the complete axiomatization of $\mathrm{Th}(\mathcal{H}^{+*}_{res})$ over $\mathrm{Th}(\mathcal{H}^+_{res})$ provided by theorem 4.6.1. $\square$

Chapter 5

Raising to an infinite power

5.1 Overview

In this chapter we will define the theory T_∞. This will be a completion of T' (see section 4.3). We will then establish some properties of T_∞ - that it is model-complete and decidable if and only if $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$ is decidable. Finally we find the field of exponents of T_∞.

5.2 Defining T_∞

Let $\mathcal{R}$ be a non-Archimedean model of $T_{\exp}$ and let λ be a positive infinite element of R (see example 2.2.67 for the definition of $T_{\exp}$ and example 2.2.52 for the definitions of Archimedean, infinite etc.). Since $\mathcal{R}$ is exponential it defines a power function with exponent λ (see lemma 2.2.44). Let

$$\mathcal{R}_\lambda = \langle \overline{R}, x^\lambda, \lambda \rangle,$$

and let L denote the language of $\mathcal{R}_\lambda$. Note that $\mathcal{R}_\lambda$ is a reduct of an o-minimal structure and is therefore o-minimal.

We now show that the first order theory of $\mathcal{R}_\lambda$ does not depend on $\mathcal{R}$, the choice of non-Archimedean model of $T_{\exp}$, or λ, the choice of positive infinite element of R. To see this take an L-formula $\phi(\bar{x})$. By replacing all occurrences of x^λ by the $\mathcal{R}$-definable map $(x, y) \mapsto x^y$ (where y is a new variable) and all occurrences of λ by y we may (effectively) obtain an $L_{\exp}$-formula $\phi'(\bar{x}, y)$ such that for any tuple $\bar{a}$ in R we have

$$\mathcal{R} \models \phi'(\bar{a}, \lambda) \text{ if and only if } \mathcal{R}_\lambda \models \phi(\bar{a}).$$

Now if ϕ is an L-sentence it follows from the o-minimality of $T_{\exp}$ that $\mathcal{R} \models \phi'(\lambda)$ if and only if $T_{\exp} \models \exists x \forall y (y > x \rightarrow \phi'(y))$. So the theory of $\mathcal{R}_\lambda$ depends only upon the theory $T_{\exp}$. Consequently we denote the theory of $\mathcal{R}_\lambda$ by T_∞.

As an aside, note that for any $r \in \mathbb{R}$ and any L-sentence ϕ we have

$$\mathbb{R}_{\exp} \models \phi'(r) \text{ if and only if } \langle \overline{\mathbb{R}}, x^r, r \rangle \models \phi.$$

So $\phi \in T_\infty$ if and only if $\langle \overline{\mathbb{R}}, x^r, r \rangle \models \phi$ for all sufficiently large $r \in \mathbb{R}$.

5.3 Particular constructions of models of T_∞

In this section we describe a number of different methods of producing models of T_∞.

Example 5.3.1. For each $r \in \mathbb{R}$ let $\mathbb{R}_r$ be the L-structure

$$\langle \overline{\mathbb{R}}, x^r, r \rangle.$$

Choose $(a_i)_{i\in\mathbb{N}}$, a sequence of real numbers which tends to infinity and $\mathfrak{U}$, a non-principal ultrafilter over $\mathbb{N}$. Now consider the ultraproduct

$$\prod_{i\in\mathbb{N}} \mathbb{R}_{a_i}/\mathfrak{U}.$$

Since $\prod_{i\in\mathbb{N}} \mathbb{R}_{a_i}/\mathfrak{U}$ is a reduct of the ultrapower

$$\mathbb{R}_{\exp}^{\mathbb{N}}/\mathfrak{U}$$

it is a model of T_∞.

Example 5.3.2. Let $\mathcal{S}$ be an arbitrary real-closed field. For each $n \in \mathbb{N}$ let

$$\mathcal{S}_n = \langle S, +, \cdot, <, x^n, n \rangle.$$

Since T_{RCF} is complete (see example 2.2.65) we have $\mathcal{S}_n \equiv \mathbb{R}_n$ for each $n \in \mathbb{N}$. Therefore, for any $\mathfrak{U}$ a non-principal ultrafilter on $\mathbb{N}$, we have

$$\prod_{n\in\mathbb{N}} \mathcal{S}_n/\mathfrak{U} \equiv \prod_{n\in\mathbb{N}} \mathbb{R}_n/\mathfrak{U},$$

and so

$$\prod_{n\in\mathbb{N}} \mathcal{S}_n/\mathfrak{U} \models T_\infty.$$

Example 5.3.3. Let $\mathcal{H} = \mathcal{H}(\mathbb{R}_{\exp})$, the Hardy field of germs at $+\infty$ of unary functions definable in $\mathbb{R}_{\exp}$. We consider $\mathcal{H}$ as an $L_{\exp}$-structure as in section 2.2.6.1. By lemma 2.2.64, when considered as an $L_{\exp}$-structure we have that $\mathbb{R}_{\exp} \preccurlyeq \mathcal{H}$. Consequently, if g is any positive infinite element of $H = \mathrm{dom}(\mathcal{H})$ then

$$\langle H, +.\cdot, <, x^g, g \rangle$$

is a model of T_∞. Note that g is just the germ at $+\infty$ of an $\mathbb{R}_{\exp}$-definable function which tends to $+\infty$, and given any eventually positive $\mathbb{R}_{\exp}$-definable function f, the map x^g acts on f by sending f to the germ at $+\infty$ of the function $x \mapsto f(x)^{g(x)}$. Note that any model of T_∞ obtained in this way has a natural derivation coming from its Hardy field structure.

5.4 T_∞ is model-complete

Let $T' = \bigcap_{r\in\mathbb{R}} \mathrm{Th}(\mathbb{R}_r)$ (recall $\mathbb{R}_r = \langle \overline{\mathbb{R}}, x^r, r\rangle$). It follows from theorem 4.3.1 that T' is model-complete (note that the T' considered here coincides with the T' of section 4.3 in the case where the index set I in 4.3 is taken to be a one element set). For the purposes of the following lemma we will let c denote the constant symbol in the language L which we typically interpret as the exponent of the power function.

Lemma 5.4.1. *$T' \subset T_\infty$ so T_∞ is model-complete. Furthermore,*

$$T' \cup \{c > n : n \in \mathbb{N}\} \vdash T_\infty.$$

Proof. It follows from the discussion at the end of section 5.2 that $\phi \in T_\infty$ if and only if $\phi \in \mathrm{Th}(\mathbb{R}_r)$ for all sufficiently large $r \in \mathbb{R}$. Consequently $T' \subset T_\infty$. Furthermore, if $\phi \in T_\infty$ then for some $N \in \mathbb{N}$ we have

$$(c > N \to \phi) \in T'.$$

The second conclusion of the lemma follows immediately. $\square$

5.5 Regarding decidability of T_∞

In this section we will prove the following theorem.

Theorem 5.5.1. *T_∞ is decidable if and only if $T_{\exp}$ is decidable.*

Before proving theorem 5.5.1 let us put it in context by recalling the work of Wilkie and Macintyre on the decidability of $T_{\exp}$

5.5.1 On the decidability of $T_{\exp}$

In [8] Wilkie and Macintyre (unconditionally) reduce the problem of proving the decidability of $T_{\exp}$ to that of obtaining a procedure P which does the following:

> P terminates on input $p(x_1, \dots, x_{2n}) \in \mathbb{Z}[x_1, \dots, x_{2n}]$ if and only if the function $\mathbb{R}^n \to \mathbb{R}$ given by
>
> $$(x_1, \dots, x_n) \mapsto p(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n))$$
>
> has a zero.

They obtain such a procedure under the assumption of Schanuel's Conjecture for $\mathbb{R}$.

Schanuel's Conjecture for $\mathbb{R}$. *Let $a_1, \ldots, a_n \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Then the field*

$$\mathbb{Q}\left(a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_n)\right)$$

has transcendence degree at least n.

So they establish the following theorem.

Theorem 5.5.2. *If Schanuel's Conjecture is true then $T_{\exp}$ is decidable.*

They also make a so-called 'Weak Schanuel's Conjecture'.

Weak Schanuel's Conjecture (WSC). *There exists an effective procedure which given $n \geq 1$ and $f_1, \ldots, f_n, g \in \mathbb{Z}[x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)]$ produces $N \in \mathbb{N} \setminus \{0\}$ such that for any $\bar{r} \in \mathbb{R}^n$, if $\bar{r} = (r_1, \ldots, r_n)$ is a non-singular zero of the system*

$$\begin{gathered} f_1(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) = 0, \\ \vdots \\ f_n(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) = 0, \end{gathered}$$

then $g(\bar{r}) = 0$ or $|g(\bar{r})| > N^{-1}$.

They prove that (WSC) is in fact *equivalent* to the decidability of $T_{\exp}$.

Theorem 5.5.3. *$T_{\exp}$ is decidable if and only if Weak Schanuel's Conjecture is true.*

Remark 5.5.4. It follows from theorem 5.5.2 and theorem 5.5.3 that Schanuel's Conjecture implies Weak Schanuel's Conjecture.

5.5.2 The proof of theorem 5.5.1

One direction of Theorem 5.5.1 follows from the o-minimality of $T_{\exp}$.

Lemma 5.5.5. *T_∞ is decidable if $T_{\exp}$ is decidable.*

Proof. As remarked in section 5.1, given an L-sentence ϕ we can effectively obtain an $L_{\exp}$-formula $\phi'(y)$ such that $T_\infty \models \phi$ if and only if $T_{\exp} \models \exists z \forall y (y > z \to \phi'(y))$. Since we are assuming that $T_{\exp}$ is decidable the result follows. $\square$

We must now prove the converse. So assume that T_∞ is decidable. Using the reduction of Wilkie and Macintyre, we will prove that T_{exp} is decidable by exhibiting a procedure P as described at the beginning of section 5.5.1.

Let us recall some terminology and standard results from non-standard analysis. So let $\tilde{\mathbb{R}}$ be an expansion of $\overline{\mathbb{R}}$ and let $\mathcal{M}$ be a proper elementary extension of $\tilde{\mathbb{R}}$. It follows from the completeness of $\mathbb{R}$ that given $x \in \text{Fin}(M)$ there exists a unique $r \in \mathbb{R}$ such that $x - r$ is infinitesimal; we denote this r by $\text{st}(x)$. If $\bar{r} = (r_1, \ldots, r_n) \in \text{Fin}(M)^n$ then we write $\text{st}(\bar{r})$ to denote $(\text{st}(r_1), \ldots, \text{st}(r_n))$.

We now give two standard lemmas. Since the proofs are short we reproduce them here.

Lemma 5.5.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be an definable function and suppose that $f(x) \to 0$ as $x \to \infty$. Let r be a positive infinite element of $\mathcal{M}$. Then $f(r)$ is infinitesimal.*

Proof. Let $\epsilon \in \text{Pos}(\mathbb{R})$. We must prove that $|f(r)| < \epsilon$. Since $f(r) \to 0$ as $x \to \infty$ there exists $R \in \mathbb{R}$ such that

$$\tilde{\mathbb{R}} \models \forall x(x > R \to |f(x)| < \epsilon).$$

Since $\mathcal{M} \succcurlyeq \tilde{\mathbb{R}}$ we have

$$\mathcal{M} \models \forall x(x > R \to |f(x)| < \epsilon).$$

Hence $|f(r)| < \epsilon$. ☐

Lemma 5.5.7. *Let C be a definable subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ a definable continuous function. Let $\bar{x} \in C_{\mathcal{M}}$ (the interpretation of C in $\mathcal{M}$) and suppose that $\bar{x}$ is finite and* $\text{st}(\bar{x}) \in C$. *Then $f(\bar{x})$ is finite and* $\text{st}\,(f(\bar{x})) = f\,(\text{st}(\bar{x}))$.

Proof. It will be sufficient to prove that $\text{st}(f(\bar{x})) = f(\text{st}(\bar{x}))$. Let $\epsilon \in \text{Pos}(\mathbb{R})$. We must show that $|f(\bar{x}) - f(\text{st}(\bar{x}))| < \epsilon$. Since f is continuous at $\text{st}(\bar{x})$ there exists $\delta \in \text{Pos}(\mathbb{R})$ such that

$$\tilde{\mathbb{R}} \models \forall \bar{y}(\|\bar{y} - \text{st}(\bar{x})\| < \delta \to |f(\bar{y}) - f(\text{st}(\bar{x}))| < \epsilon).$$

Since $\mathcal{M} \succcurlyeq \tilde{\mathbb{R}}$ we have

$$\mathcal{M} \models \forall \bar{y}(\|\bar{y} - \text{st}(\bar{x})\| < \delta \to |f(\bar{y}) - f(\text{st}(\bar{x}))| < \epsilon).$$

Hence $|f(\bar{x}) - f(\text{st}(\bar{x}))| < \epsilon$. ☐

Now let $\mathcal{R}$ be a proper elementary extension of $\mathbb{R}_{\exp}$. Choose $\lambda \in R$ to be positive and infinite. Let

$$\mathcal{R}' = \langle R, +, \cdot, <, x^\lambda, \lambda \rangle$$

so that $\mathcal{R}' \models T_\infty$. Let $\mu : R \to R$ be the $\mathcal{R}'$-definable function given by $x \mapsto (1 + \frac{x}{\lambda})^\lambda$. It follows from lemma 5.5.6 and the classical limit formula

$$\lim_{y \to \infty} \left(1 + \frac{x}{y}\right)^y = \exp(x),$$

where x, y are real and the limit is interpreted in $\mathbb{R}$, that $|\exp(x) - \mu(x)|$ is infinitesimal for real values of x.

In what follows we will actually find it more convenient to work with the $\mathcal{R}'$-definable function given by

$$\epsilon(x) = \begin{cases} \left(1 + \frac{x}{\lambda}\right)^\lambda & x \geq 0, \\ \left(1 - \frac{x}{\lambda}\right)^{-\lambda} & x < 0. \end{cases}$$

Note that $\epsilon(-x) = \epsilon(x)^{-1}$ for $|x| < \lambda$. It follows that $|\epsilon(x) - \exp(x)|$ is infinitesimal for real values of x. In fact we can prove the stronger statement that

$$|\exp(x) - \epsilon(x)| \leq \frac{x^2 \exp(x)}{2\lambda}, \tag{5.1}$$

for all finite values of x. We defer the proof of this inequality to section 5.6 at the end of this chapter.

Now for $p(x_1, \ldots, x_{2n}) \in \mathbb{Z}[x_1, \ldots, x_{2n}]$ let F_p, G_p be the maps given by

$$\begin{aligned} F_p(x_1, \ldots, x_n) &= p(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)), \\ G_p(x_1, \ldots, x_n) &= p(x_1, \ldots, x_n, \epsilon(x_1), \ldots, \epsilon(x_n)). \end{aligned}$$

We wish to find an upper bound on the difference $|F_p(\bar{r}) - G_p(\bar{r})|$ which holds for finite values of $\bar{r}$. We will use the following standard identities which hold for any commutative ring R.

Lemma 5.5.8. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$. Then*

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{j=1}^n \left((a_j - b_j) \prod_{i=1}^{j-1} a_i \prod_{i=j+1}^n b_i \right).$$

Corollary 5.5.9. *Let $a, b \in R$. Then*

$$a^n - b^n = (a - b) \sum_{j=1}^n a^{j-1} b^{n-j+1}.$$

Lemma 5.5.10. *Let $p(x_1, \ldots, x_{2n}) \in \mathbb{Z}[x_1, \ldots, x_{2n}]$. For any $\bar{r} \in \mathrm{Fin}(R)^n$ there exists $M \in \mathbb{R}$ (depending on $\bar{r}$ and p) such that*

$$|F_p(\bar{r}) - G_p(\bar{r})| \leq \frac{M}{\lambda}.$$

Notation 5.5.11. In the proof below we will use M in a series of inequalities. In each case M stands for a positive real depending on $\bar{r}$ and p, but not necessarily the same positive real in each inequality where it occurs.

Proof of lemma 5.5.10. Using multi-index notation,

$$p(x_1, \ldots, x_{2n}) = \sum_{|\alpha|,|\beta|<N} a_{\alpha,\beta}\bar{x}^{\alpha}\bar{y}^{\beta},$$

where N is some positive integer, $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (x_{n+1}, \ldots, x_{2n})$, $\alpha, \beta \in \mathbb{N}^n$ and $a_{\alpha,\beta} \in \mathbb{Z}$. So

$$F_p(\bar{x}) = \sum_{|\alpha|,|\beta|<N} a_{\alpha,\beta}\bar{x}^{\alpha}\exp(\bar{x})^{\beta},$$
$$G_p(\bar{x}) = \sum_{|\alpha|,|\beta|<N} a_{\alpha,\beta}\bar{x}^{\alpha}\epsilon(\bar{x})^{\beta}.$$

Now fix $\bar{r} \in \mathrm{Fin}(R)^n$.

$$|F_p(\bar{r}) - G_p(\bar{r})| \leq \sum_{|\alpha|,|\beta|<N} |a_{\alpha,\beta}|\,|\bar{r}|^{\alpha}\left|\exp(\bar{r})^{\beta} - \epsilon(\bar{r})^{\beta}\right| \tag{5.2}$$
$$\leq M \sum_{|\alpha|,|\beta|<N} \left|\exp(\bar{r})^{\beta} - \epsilon(\bar{r})^{\beta}\right| \tag{5.3}$$

By corollary 5.5.9 and inequality (5.1), for each $i = 1, \ldots, n$

$$\left|\exp(r_i)^{\beta_i} - \epsilon(r_i)^{\beta_i}\right| \leq \frac{M}{\lambda}. \tag{5.4}$$

By lemma 5.5.8 and (5.4), for each β

$$\left|\exp(\bar{r})^{\beta} - \epsilon(\bar{r})^{\beta}\right| \leq \frac{M}{\lambda}. \tag{5.5}$$

It follows from (5.3) and (5.5) that

$$|F_p(\bar{r}) - G_p(\bar{r})| \leq \frac{M}{\lambda}.$$

□

Corollary 5.5.12. *For any $n \geq 1$, any $p(x_1, \ldots, x_{2n}) \in \mathbb{Z}[x_1, \ldots, x_{2n}]$ and any $\bar{r} \in \mathrm{Fin}(R)^n$*

$$|F_p(\bar{r}) - G_p(\bar{r})| < \frac{1}{\sqrt{\lambda}}.$$

Proof. Note that if $M \in \mathrm{Fin}(R)$ then $M < \sqrt{\lambda}$. □

Corollary 5.5.13. *Let $n \geq 1$ and $p(x_1, \ldots, x_{2n}) \in \mathbb{Z}[x_1, \ldots, x_{2n}]$. Then F_p has a zero in $\mathbb{R}^n$ if and only if $|G_p(\bar{r})| < \frac{1}{\sqrt{\lambda}}$ for some $\bar{r} \in \mathrm{Fin}(R)^n$.*

Proof. Suppose that $F_p(\bar{a}) = 0$, where $\bar{a} \in \mathbb{R}^n$. By corollary 5.5.12, $|G_p(\bar{a})| < \frac{1}{\sqrt{\lambda}}$.

Now suppose that $|G_p(\bar{r})| < \frac{1}{\sqrt{\lambda}}$, where $\bar{r} \in \mathrm{Fin}(R)^n$. By lemma 5.5.7, $F_p(\bar{r}) - F_p(\mathrm{st}(\bar{r}))$ is equal to some infinitesimal μ. Now

$$\begin{aligned} |F_p(\mathrm{st}(\bar{r}))| &\leq |F_p(\mathrm{st}(\bar{r})) - F_p(\bar{r})| + |F_p(\bar{r}) - G_p(\bar{r})| + |G_p(\bar{r})| \\ &\leq \mu + \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}. \end{aligned}$$

So $F_p(\mathrm{st}(\bar{r}))$ is both real and infinitesimal and hence zero. □

We are now in a position to give our effective procedure P which will terminate if and only if F_p has a real zero.

For $N \in \mathbb{N}$ let ϕ_N be the L-sentence:

$\exists \bar{x}$ s.t. $|\bar{x}| \leq N$ and $|G_p(\bar{x})| < \frac{1}{\sqrt{\lambda}}$.

Our procedure P is as follows: at stage N run the decision procedure for T_∞ on ϕ_N. If it halts with output TRUE then terminate. If it halts with output FALSE then go to stage $N+1$. By corollary 5.5.13, this procedure terminates if and only if F_p has a real zero.

Remark 5.5.14. In fact we have shown $T_{\exp}$ is decidable if the existential theory of T_∞ is decidable. Furthermore we can modify the above procedure to show that $T_{\exp}$ is decidable if the existential theory of T_∞ is recursively enumerable. To see this we first note that the set Λ of L-sentences of the form

$\exists \bar{x}$ s.t. $|\bar{x}| \leq N$ and $|G_p(\bar{x})| < \frac{1}{\sqrt{\lambda}}$

for fixed p and varying $N \in \mathbb{N}$ is recursive. Now given p we know that F_p has a real zero if and only if Λ has non-empty intersection with the existential theory of T_∞. Let $\theta_1, \theta_2, \ldots$ be our recursive enumeration of the existential theory of T_∞. Our new procedure is as follows: at stage N we check whether θ_N is in Λ, if YES then terminate, if NO then proceed to stage $N+1$.

Theorem 5.5.15. *The following are equivalent.*

1. *$T_{\exp}$ is decidable.*

2. *T_∞ is decidable.*

3. *The existential theory of T_∞ is recursively enumerable.*

4. *The existential theory of $T_{\exp}$ is recursively enumerable.*

5. *There is an effective procedure which, given $n \geq 1$ and $p \in \mathbb{Z}[x_1, \ldots, x_{2n+1}]$, terminates if and only if for all sufficiently large real numbers r the function $p(x_1, \ldots, x_n, x_1^r, \ldots, x_n^r, r)$ has a zero in the positive orthant of $\mathbb{R}^n$.*

6. *There is an effective procedure which, given $n \geq 1$ and $p \in \mathbb{Z}[x_1, \ldots, x_{2n+1}]$, terminates if and only if for all sufficiently large integers d the polynomial function $p(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d, d)$ has a zero in the positive orthant of $\mathbb{R}^n$.*

7. *There is an effective procedure which, given $n \geq 1$ and $p \in \mathbb{Z}[x_1, \ldots, x_{2n+1}]$, terminates if and only if for all positive integers d the polynomial function $p(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d, d)$ has a zero in the positive orthant of $\mathbb{R}^n$.*

Proof. (1) implies (2) is lemma 5.5.5. Clearly (2) implies (3). By the remark following theorem 5.5.1 we have (3) implies (1). So (1), (2) and (3) are equivalent. Clearly (1) implies (4) and the converse follows from the reduction of Wilkie and Macintyre mentioned at the beginning of section 5.5.1. Now (5) and (6) are equivalent because, by the o-minimality of $\mathbb{R}_{\exp}$, the set of $r \in \mathbb{R}$ for which $p(x_1, \ldots, x_n, x_1^r, \ldots, x_n^r, r)$ has a zero in the positive orthant of $\mathbb{R}^n$ is a finite union of intervals and points. Since an L-sentence is true in T_∞ if and only if it is true in T_r for all sufficiently large real r we see that (3) implies (5). To see that (5) implies (3) we first note that any existential L-sentence can effectively be put in the form $\exists \bar{x} p(\bar{x}, \bar{x}^\lambda, \lambda) = 0$. Now we again use the fact that an L-sentence is true in T_∞ if and only if it is true for all sufficiently large r. Additionally we note that given $p \in \mathbb{Z}[x_1, \ldots, x_{2n}, x_{2n+1}]$ we can effectively produce $p' \in \mathbb{Z}[x_1, \ldots, x_{2n}, x_{2n+1}]$ such that $T_\infty \models \exists \bar{x} p(\bar{x}, \bar{x}^\lambda, \lambda) = 0$ if and only if $T_\infty \models \exists \bar{x} \left(\bigwedge_i x_i > 0 \wedge p'(\bar{x}, \bar{x}^\lambda, \lambda) = 0 \right)$. Let us see now that (1) implies (7). So assume that $T_{\exp}$ is decidable. Given $p(x_1, \ldots, x_{2n+1}) \in \mathbb{Z}[x_1, \ldots, x_{2n+1}]$ and N a positive integer, let ϕ_N be the $L_{\exp}$-sentence

$$\bigwedge_{d=1}^{N}\left(\exists\bar{x}\left(\bigwedge_{i=1}^{n}x_i>0\wedge p(\bar{x},\bar{x}^d,d)=0\right)\right)$$
$$\wedge\,\forall y>N\left(\exists\bar{x}\left(\bigwedge_{i=1}^{n}x_i>0\wedge p(\bar{x},\bar{x}^y,y)=0\right)\right)$$

Clearly if $T_{\exp}\models\phi_N$ then for all positive integers d the function $p(\bar{x},\bar{x}^d,d)$ has a zero in $\mathrm{Pos}(\mathbb{R})^n$. Furthermore, it follows from o-minimality that if for all positive integers d the function $p(\bar{x},\bar{x}^d,d)$ has a zero in $\mathrm{Pos}(\mathbb{R})^n$ then for some positive integer N we have $T_{\exp}\models\phi_N$. Thus the following procedure is sufficient: at stage N run the decision procedure for $T_{\exp}$ on ϕ_N; if FALSE proceed to stage $N+1$, if TRUE then terminate. Finally, we will prove that (7) implies (6). So assume that we have a procedure P as described in (7). Given $p(x_1,\ldots,x_{2n+1})$ let

$$q_N(x_1,\ldots,x_{2n+1})=\left(\prod_{k=1}^{N}(k-x_{2n+1})\right)p(x_1,\ldots,x_{2n+1}).$$

So $q_N(\bar{x},\bar{x}^d,d)=0$ if and only if $d\in\{1,\ldots,N\}$ or $p(\bar{x},\bar{x}^d,d)=0$. It follows that given a positive integer N, the function $p(\bar{x},\bar{x}^d,d)$ has a zero in $\mathrm{Pos}(\mathbb{R})^n$ for all integers $d>N$ if and only if $q_N(\bar{x},\bar{x}^d,d)$ has a zero in $\mathrm{Pos}(\mathbb{R})^n$ for all integers d. Thus our procedure is as follows: at stage N run the first N stages of the procedure P on the polynomials $q_1,\ldots,q_N$. This terminates if and only if P terminates on q_N for some N and therefore is as required.

□

5.6 The proof of inequality (5.1)

We must prove that $|\epsilon(x)-\exp(x)|\leq\frac{x^2\exp(x)}{2\lambda}$ whenever x is finite. We first prove two inequalities over $\mathbb{R}$.

Lemma 5.6.1. *Let $x,y\in\mathbb{R}$ with $y>|x|$. Then*

$$\exp(x)\geq\left(1+\frac{x}{y}\right)^y.$$

Proof. By taking logs we see that it is sufficient to prove that

$$\frac{x}{y}\geq\log\left(1+\frac{x}{y}\right).$$

This follows from the fact that $z\geq\log(1+z)$ whenever $z>-1$. □

Lemma 5.6.2. *Let $x, y \in \mathbb{R}$ and suppose that $x > 0$ and $y > \max\{x, 2\}$. Then*

$$\exp(x) - \left(1 + \frac{x}{y}\right)^y \leq \frac{x^2 \exp(x)}{2y}.$$

Proof. Take x, y as in the statement of the lemma. We must prove that

$$\exp(x)\left(1 - \frac{x^2}{2y}\right) \leq \left(1 + \frac{x}{y}\right)^y. \tag{5.6}$$

If $\frac{x^2}{2y} \geq 1$ then the result is clear so we assume in addition that $\frac{x^2}{2y} < 1$. Taking logs we see that it is sufficient to prove that

$$x + \log\left(1 - \frac{x^2}{2y}\right) \leq y \log\left(1 + \frac{x}{y}\right). \tag{5.7}$$

We recall that for $|a| < 1$,

$$\log(1 - a) = -\sum_{k=1}^{\infty} \frac{a^k}{k}. \tag{5.8}$$

So (5.7) becomes

$$x - \sum_{k=1}^{\infty} \frac{x^{2k}}{k 2^k y^k} \leq y \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k y^k}. \tag{5.9}$$

We may rewrite this as

$$0 \leq \sum_{k=1}^{\infty} \frac{x^{2k}}{k 2^k y^k} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1} x^k}{k y^{k-1}} = \sum_{k=1}^{\infty} \left(\frac{x^{2k}}{k 2^k y^k} + \frac{x^{2k+1}}{(2k+1) y^{2k}} - \frac{x^{2k}}{2k y^{2k-1}}\right). \tag{5.10}$$

So it is sufficient to prove that

$$\frac{x^{2k}}{k 2^k y^k} + \frac{x^{2k+1}}{(2k+1) y^{2k}} - \frac{x^{2k}}{2k y^{2k-1}} \geq 0 \tag{5.11}$$

for $k \geq 1$. So let $k \geq 1$. Rearranging (5.11) we see that it is sufficient to prove that

$$(2k+1) y^k + k 2^k x - (2k+1) 2^{k-1} y \geq 0, \tag{5.12}$$

i.e.

$$(2k+1) y (y^{k-1} - 2^{k-1}) + k 2^k x \geq 0. \tag{5.13}$$

Now since $y > 2$ we have

$$y^{k-1} - 2^{k-1} \geq 0.$$

Furthermore, since $y > \frac{x^2}{2}$ we have

$$y^{k-1} - 2^{k-1} \geq \frac{x^{2k-2}}{2^{k-1}} - 2^{k-1}.$$

So

$$y(y^{k-1} - 2^{k-1}) \geq \frac{x^2}{2}\left(\frac{x^{2k-2}}{2^{k-1}} - 2^{k-1}\right).$$

Therefore, it is sufficient to prove that

$$(2k+1)\frac{x^2}{2}\left(\frac{x^{2k-2}}{2^{k-1}} - 2^{k-1}\right) + k2^k x \geq 0. \tag{5.14}$$

Rearranging we see that it is sufficient to prove that

$$x^{2k+1} - 2^{2k}x + \frac{(k+1)2^{2k+2}}{2k+3} \geq 0,$$

where $k \geq 0$. So let

$$g_k(x) = x^{2k+1} - 2^{2k}x + \frac{(k+1)2^{2k+2}}{2k+3}.$$

We must prove that $g_k(x) \geq 0$ for $x > 0$. If $k = 0$ this is immediate, so assume $k \geq 1$. We note that $g_k(0) > 0$. Furthermore by differentiating we note that g_k has just one turning point for positive x at $x_k = \frac{2}{(2k+1)^{\frac{1}{2k}}}$. Since $g_k(x) \to \infty$ as $x \to \infty$ it remains to prove that $g_k(x_k) \geq 0$. Rearranging and pulling out factors we see that this reduces to proving

$$(k+1)(2k+1)^{\frac{2k+1}{2k}} \geq 2k^2 + 3k,$$

which follows since

$$(k+1)(2k+1)^{\frac{2k+1}{2k}} \geq (k+1)(2k+1) = 2k^2 + 3k + 1.$$

□

We now return to our structure $\mathcal{R}$ and deduce the inequality (5.1) on page 75.

Lemma 5.6.3. *Let $x \in R$ and suppose that x is finite. Then*

$$|\exp(x) - \epsilon(x)| \leq \frac{x^2 \exp(x)}{2\lambda}.$$

Proof. If $x = 0$ then both sides of the inequality are equal to 0. Suppose $x > 0$. This is immediate from lemmas 5.6.1 and 5.6.2 since they are both expressible as sentences in $L_{\exp}$. Finally, suppose that $x < 0$. Let $z = -x$. Now

$$\begin{aligned} |\exp(x) - \epsilon(x)| &= |\exp(-z) - \epsilon(-z)| \\ &= \frac{|\exp(z) - \epsilon(z)|}{\exp(z)\epsilon(z)} \\ &= \frac{\exp(z) - \epsilon(z)}{\exp(z)\epsilon(z)} \\ &\leq \frac{z^2 \exp(z)}{2\lambda \exp(z)\epsilon(z)} \\ &= \frac{z^2 \epsilon(-z)}{2\lambda} \\ &\leq \frac{z^2 \exp(-z)}{2\lambda} \\ &= \frac{x^2 \exp(x)}{2\lambda}. \end{aligned}$$

□

5.7 The field of exponents of T_∞

We prove that in any model $\langle \overline{R}, x^\lambda, \lambda \rangle$ of T_∞ the field of exponents is $\mathbb{Q}(\lambda)$.[1] This will be a consequence of the following theorem about o-minimal expansions of the real field with analytic cell decomposition.

Theorem 5.7.1. *Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of the real field with analytic cell decomposition. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a definable function with the property that for all $\bar{r} \in \mathbb{R}^n$ we have $F(\bar{r}) \in \mathbb{Q}(\bar{r})$. Then we can decompose $\mathbb{R}^n$ into analytic cells such that on each cell F is given by a rational function with coefficients from $\mathbb{Q}$.*

Proof. We first note that if X is a definable set in $\tilde{\mathbb{R}}$ we can find a reduct of $\tilde{\mathbb{R}}$ which has a countable language, analytic cell decomposition and in which X is 0-definable. Therefore we may assume that $\tilde{\mathbb{R}}$ has a countable language and F is 0-definable.

Let $\mathfrak{C}$ be a cell decomposition of $\mathbb{R}^n$ such that for each $C \in \mathfrak{C}$ the function F is analytic on C. Take $C \in \mathfrak{C}$ of positive dimension. We will prove that the restriction of F to C is given by a rational function with coefficients from $\mathbb{Q}$. By lemma 2.2.18, C has a generic point, $\bar{a}$ say. By our assumption, there exists $p(\bar{x}) \in \mathbb{Q}(\bar{x})$ such that $F(\bar{a}) = p(\bar{a})$. Since $\bar{a}$ is a generic point of C, the set of $\bar{x} \in C$ such that $F(\bar{x}) = p(\bar{x})$ has the same dimension as C. Note that since C is a cell it is definably connected and hence connected (since we are working over $\mathbb{R}$). Therefore, because F is analytic on C we must have that F and p are identically equal on C. □

[1] The argument given in this section is in large part due to Wilkie.

Corollary 5.7.2. *If* $\langle \overline{R}, x^\lambda, \lambda \rangle \models T_\infty$ *then* $\langle \overline{R}, x^\lambda, \lambda \rangle$ *has field of exponents* $\mathbb{Q}(\lambda)$.

Proof. Recall that L denotes the language of T_∞. Let $\phi(x, y)$ be an L-formula and suppose that $\phi(x, y)$ defines a function in models of T_∞. Let $\phi'(x, y, z)$ be the $L_{\exp}$-formula corresponding to $\phi(x, y)$ (as in section 5.2). Note that for each $r \in \mathbb{R}$, if $\phi'(x, y, r)$ defines a function in $\mathbb{R}_{\exp}$ then this function is also definable in $\mathbb{R}_r = \langle \overline{\mathbb{R}}, x^r, r \rangle$ and hence has some exponent at ∞. Since $\mathbb{R}_r$ has field of exponents $\mathbb{Q}(r)$ [14], the exponent at ∞ of the function defined by $\phi'(x, y, r)$ must lie in $\mathbb{Q}(r)$. Let $P : \mathbb{R} \to \mathbb{R}$ be the function, definable in $\mathbb{R}_{\exp}$, which does the following: if $\phi'(x, y, r)$ defines a function $f(x) = y$ then $P(r)$ is the exponent of f at ∞, otherwise $P(r) = 0$. By theorem 5.7.1 and the fact that $\mathbb{R}_{\exp}$ has analytic cell decomposition, $P(r)$ is piecewise (with finitely many pieces) a rational function with coefficients from $\mathbb{Q}$. In particular, there exists $s(x) \in \mathbb{Q}(x)$ such that for all sufficiently large $r \in \mathbb{R}$, the formula $\phi'(x, y, r)$ defines a function with exponent at ∞ given by $s(r)$. Therefore if $\mathcal{R} = \langle \overline{R}, x^\lambda, \lambda \rangle \models T_\infty$, the L-formula $\phi(x, y)$ defines a function in $\mathcal{R}$ with exponent at ∞ given by $s(\lambda)$. $\square$

Chapter 6

Extending exponentials

6.1 Overview

Let $\mathcal{R}$ be an arbitrary power-bounded o-minimal expansion of a field by functions and suppose that $\mathcal{R}$ defines a restricted exponential, i.e. a non-constant map $\mathrm{E} : P \to P$ satisfying

1. $\mathrm{E}(x) = 0$ for $x \notin [0, 1]$,

2. $\mathrm{E}(x + y) = \mathrm{E}(x)\,\mathrm{E}(y)$ for $x, y \in [0, 1]$ such that $x + y \leq 1$.

We will also assume that

3. the right hand derivative of E at 0 is 1.

Note that, by the same argument as in lemma 2.2.46, E is in fact 0-definable in $\mathcal{R}$. In this section we will consider the following question. Under what circumstances can we expand $\mathcal{R}$ to an exponential o-minimal structure?

Let K be the field of exponents of $\mathcal{R}$ and let $\mathcal{P}$ be the prime model of $\mathrm{Th}(\mathcal{R})$ considered with its canonical embedding in $\mathcal{R}$. Let V denote the convex hull of $\mathcal{P}$ in $\mathcal{R}$ and let Γ denote the value group of $\mathcal{R}$ with respect to V. Clearly, if $\mathcal{R}$ expands to an exponential o-minimal structure with exponential $\tilde{E}$, then, if Γ is non-trivial, it must be infinite dimensional when considered as a K-vector space (if $v(a) < 0$ then $v(\tilde{E}(a)) < kv(a)$ for all $k \in K$). On the other hand, if $\mathcal{R}$ is a finite rank extension of $\mathcal{P}$ then it follows from theorem 2.2.59 that Γ is finite-dimensional when considered as a K-vector space. In light of this observation we instead ask the following question.

Question 6.1.1. Under what circumstances can we expand an elementary extension of $\mathcal{R}$ to an exponential o-minimal structure?

Theorem 6.3.2 below will provide a partial solution to this question.

6.2 Background

One can consider the question 6.1.1 in the special case that $R = \mathbb{R}$. In this case the answer is 'always' (and indeed here we do not need to pass to an elementary extension). This is corollary 6.2.6 below. In fact, more is true. Let $\tilde{\mathbb{R}}$ be *any* o-minimal expansion of the real field (i.e. we do not assume that $\tilde{\mathbb{R}}$ defines a restricted exponential). Then the structure $\langle \tilde{\mathbb{R}}, \exp \rangle$ is o-minimal. This is because exp is definable in the Pfaffian closure of $\tilde{\mathbb{R}}$, which is known to be o-minimal [17]. In contrast, if we are given $\mathcal{R}$, an arbitrary power-bounded o-minimal expansion of a field, and we wish to expand (an elementary extension of) $\mathcal{R}$ to an exponential o-minimal structure we must first construct an exponential function. To do this we will need the additional assumption that $\mathcal{R}$ already defines a restricted exponential. Our methods will follow those developed for the proof of theorem 6.2.5 below and so we recall them here.

Let $E : \mathbb{R} \to \mathbb{R}$ be given by

$$E(x) = \begin{cases} \exp(x) & x \in [0,1] \\ 0 & x \in \mathbb{R} \setminus [0,1] \end{cases}$$

Now consider the following two structures.

$$\langle \overline{\mathbb{R}}, E \rangle,$$
$$\langle \overline{\mathbb{R}}, E, \exp \rangle.$$

The following axioms clearly hold for the structure $\langle \overline{\mathbb{R}}, E, \exp \rangle$.

(E1) $\forall x \forall y (\exp(x+y) = \exp(x)\exp(y))$,

(E2) $\forall x \forall y (x < y \to \exp(x) < \exp(y))$,

(E3) $\forall x (x > n^2 \to \exp(x) > x^n)$;for all $n \in \mathbb{N}$,

(E4) $\forall x > 0 \exists y (\exp(y) = x)$,

(E5) $\forall x (0 \leq x \leq 1 \to \exp(x) = E(x))$.

In fact, Ressayre proves that they are enough to axiomatize $\mathrm{Th}(\langle \overline{\mathbb{R}}, E, \exp \rangle)$ over $\mathrm{Th}(\langle \overline{\mathbb{R}}, E \rangle)$.

Theorem 6.2.1 ([15])**.** $\mathrm{Th}(\langle \overline{\mathbb{R}}, E, \exp \rangle)$ *is axiomatized by the axioms (E1)-(E5) and* $\mathrm{Th}(\langle \overline{\mathbb{R}}, E \rangle)$[1] .

[1] It is immediate from Wilkie's theorem on restricted Pfaffian functions (theorem 3.1.4) that $\mathrm{Th}(\langle \overline{\mathbb{R}}, E \rangle)$ is model-complete. In the same paper Wilkie uses this result to prove that $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$ is model-complete. One can also deduce the model-completeness of $T_{\exp}$ from that of $\mathrm{Th}(\langle \overline{\mathbb{R}}, E \rangle)$ by using a 'with parameters' version of theorem 6.2.1.

Theorem 6.2.2 ([15])**.** *Let* $\mathcal{A} \models \mathrm{Th}(\langle \overline{\mathbb{R}}, E, \exp \rangle)$ *and let* $\mathcal{A}'$ *denote its reduct to the language* $L_{\mathrm{ord}} \cup \{E\}$*. Then* $\mathrm{Th}(\langle \mathcal{A}, (a)_{a \in A} \rangle)$ *is axiomatized by* $\mathrm{Th}(\langle \mathcal{A}', (a)_{a \in A} \rangle)$ *and (E1)-(E5).*

To show that $T_{\exp}$ is model-complete it is of course sufficient to prove that $\mathrm{Th}(\langle \overline{\mathbb{R}}, E, \exp \rangle)$ is model-complete. So choose $\mathcal{A}, \mathcal{B} \models \mathrm{Th}(\langle \overline{\mathbb{R}}, E, \exp \rangle)$ and suppose that $\mathcal{A} \subseteq \mathcal{B}$. We must prove that

$$\langle \mathcal{A}, (a)_{a \in A} \rangle \equiv \langle \mathcal{B}, (a)_{a \in A} \rangle.$$

But this is immediate from the model-completeness of $\mathrm{Th}(\langle \overline{\mathbb{R}}, E \rangle)$ and theorem 6.2.2.

Now consider the structures $\langle \mathbb{R}_{\text{an}}, \exp \rangle$ and $\langle \mathbb{R}_{\text{an}}, \exp, \log \rangle$. Let $T_{\text{an}}(\exp)$ and $T_{\text{an}}(\exp, \log)$ denote their respective theories. In [26], van Den Dries, Macintyre and Marker prove the following theorem.

Theorem 6.2.3 ([26]). *$T_{\text{an}}(\exp, \log)$ admits quantifier elimination. Furthermore, $T_{\text{an}}(\exp)$ is axiomatized by T_{an} together with (E1)-(E5).*

Using this they are able to deduce the following theorem.

Theorem 6.2.4 ([26]). *$T_{\text{an}}(\exp)$ is o-minimal.*

A key step in proof of this theorem is to establish the valuation property (theorem 2.2.61) for models of the theory T_{an} with respect to the valuation induced by the ring of finite elements. In [28], van den Dries and Speissegger prove the valuation property for (theories of) polynomially bounded o-minimal expansions of the real field. Using this they are able to extend theorems 6.2.3 and 6.2.4.

Theorem 6.2.5. *Let $\tilde{\mathbb{R}}$ be a polynomially bounded o-minimal expansion of the real field with field of exponents K and suppose that $\tilde{\mathbb{R}}$ defines $E = \exp \restriction_{[0,1]}$. Then $\mathrm{Th}(\langle \tilde{\mathbb{R}}, \exp \rangle)$ is axiomatized by $\mathrm{Th}(\tilde{\mathbb{R}})$, (E1)-(E5) and the axiom scheme*

(E6) $\forall x(\exp(kx) = \exp(x)^k)$; for all $k \in K$.

Furthermore if $\mathrm{Th}(\tilde{\mathbb{R}})$ admits quantifier elimination then $\mathrm{Th}(\langle \tilde{\mathbb{R}}, \exp, \log \rangle)$ admits quantifier elimination.

Theorem 6.2.6. *$\mathrm{Th}(\langle \tilde{\mathbb{R}}, \exp \rangle)$ is o-minimal.*

6.3 Preliminaries

Let T be a complete power-bounded o-minimal theory expanding the theory of real closed fields by functions. Let K be the field of exponents of T. For any $\mathcal{A} \models T$ let $V_{\mathcal{A}}$ be the convex hull of K. Note that $V_{\mathcal{A}}$ is a convex subring of $\mathcal{A}$. Let $\Gamma_{\mathcal{A}}$ denote the value group corresponding to $V_{\mathcal{A}}$ and let $v_{\mathcal{A}}$ denote the valuation map. Where it should not cause any confusion, we will drop the subscripts.

Lemma 6.3.1. *The following are equivalent:*

1. *There exists a model $\mathcal{A}$ of T such that the convex subring $V_{\mathcal{A}}$ is T-convex in $\mathcal{A}$.*

2. *For any model $\mathcal{B}$ of T the convex subring $V_{\mathcal{B}}$ is T-convex in $\mathcal{B}$.*

3. K is cofinal in the prime model of T.

Proof. **(1)** $\Rightarrow$ **(2)** Let $f : B \to B$ be a 0-definable continuous function in $\mathcal{B}$. Since $V_{\mathcal{A}}$ is T-convex for $\mathcal{A}$ and continuous functions definable in o-minimal expansions of ordered fields map closed bounded sets to closed bounded sets (theorem 2.2.11), for each $k \in \mathrm{Pos}(K)$ there exists $l \in \mathrm{Pos}(K)$ such that

$$\mathcal{A} \models \forall x(|x| < k \to |f(x)| < l).$$

All points of K are 0-definable and so

$$\mathcal{B} \models \forall x(|x| < k \to |f(x)| < l).$$

Therefore $V_{\mathcal{B}}$ is T-convex.

(2) $\Rightarrow$ **(3)** Let $\mathcal{P}$ be the prime model of T. Let $a \in P$, then a is 0-definable and so the constant function which takes value a is 0-definable. Since $V_{\mathcal{P}}$ is T-convex in $\mathcal{P}$, there exists $k \in K$ such that $a < k$.

(3) $\Rightarrow$ **(1)** Clearly $V_{\mathcal{P}}$ is T-convex for $\mathcal{P}$.

$\square$

From now on we will let $\mathcal{P}$ be the prime model of T and we will assume that K is cofinal in P. Returning to the situation described in section 6.1 we will assume also that $\mathcal{R}$ defines (and hence 0-defines) a non-constant restricted exponential with right-hand derivative at 0 equal to 1. It immediately follows from this that E is differentiable on $(0, 1)$, and has left-hand derivative at 1, with the derivative at $x \in [0, 1]$ given by $\mathrm{E}(x)$.

The object of this chapter is to prove the following theorem.

Theorem 6.3.2. *If $\mathcal{R} \models T$ there is an elementary extension $\mathcal{S}$ of $\mathcal{R}$ which supports a (global) exponential function $\tilde{\mathrm{E}}$ which extends E. Furthermore the expansion of $\mathcal{S}$ by this exponential function is o-minimal.*

If we can find $\mathcal{P}_{\tilde{\mathrm{E}}}$ an o-minimal expansion of $\mathcal{P}$ with a global exponential $\tilde{\mathrm{E}}$ extending E then, given a model $\mathcal{R}$ of T we take a sufficiently saturated elementary extension $\mathcal{S}$ of $\mathcal{P}_{\tilde{\mathrm{E}}}$. The L-reduct of $\mathcal{S}$ will then elementarily embed $\mathcal{R}$. So the problem is to get an exponential o-minimal expansion of $\mathcal{P}$.

6.4 Defining exponentiation on the whole of $\mathcal{P}$

$\mathcal{P}$ is a power-bounded o-minimal expansion of a real closed field whose field of exponents K is cofinal in P and which defines a restricted exponential $\mathrm{E} : [0,1] \to P$. We extend E to a map $\tilde{\mathrm{E}} : P \to P$ as follows:

$$\tilde{\mathrm{E}}(x) = \begin{cases} \mathrm{E}(\frac{x}{k})^k & x \geq 0, k \in K \text{ and } \frac{x}{k} \in [0,1], \\ \tilde{\mathrm{E}}(-x)^{-1} & x < 0. \end{cases}$$

Lemma 6.4.1. $\tilde{\mathrm{E}}$ *is well-defined.*

Proof. Let $k \in K$ and assume that $k \geq 1$. Consider the definable maps $f, g : [0, \frac{1}{k}] \to P$ given by $f(x) = \mathrm{E}(kx)$ and $g(x) = \mathrm{E}(x)^k$. Now $f(0) = g(0) = 1$, furthermore

$$\left(\frac{f(x)}{g(x)}\right)' = 0.$$

Using the mean value property of differentiable functions definable in o-minimal structures we see that $f = g$.

Now let $x \geq 0$ and let $k, l \in K$ such that $0 \leq \frac{x}{k}, \frac{x}{l} \leq 1$. Without loss of generality we may assume $k \leq l$. Let $y = \frac{x}{l}$. Now $\frac{l}{k}y \leq 1$. By above $\mathrm{E}(\frac{l}{k}y) = \mathrm{E}(y)^{\frac{l}{k}}$. Thus $\mathrm{E}(\frac{x}{k})^k = \mathrm{E}(\frac{x}{l})^l$. □

Remark 6.4.2. Note that it follows from the definition of $\tilde{\mathrm{E}}$ that its restriction to any closed bounded interval of P is definable in $\mathcal{P}$.

For the proof of the next lemma we will make frequent use of the fact that the mean value theorem and the intermediate value theorem hold for functions definable in o-minimal expansions of fields (see section 2.2.3).

Lemma 6.4.3. $\tilde{\mathrm{E}} : P \to P$ *satisfies the following properties.*

1. *If* $x \in [0,1]$ *and* $k \in K$ *then* $\tilde{\mathrm{E}}(kx) = E(x)^k$.
2. $\tilde{\mathrm{E}}(x)^k = \tilde{\mathrm{E}}(kx)$*; for all* $k \in K$.
3. $\tilde{\mathrm{E}}(x+y) = \tilde{\mathrm{E}}(x)\,\tilde{\mathrm{E}}(y)$.
4. $\tilde{\mathrm{E}}$ *is everywhere differentiable and satisfies* $\tilde{\mathrm{E}}' = \tilde{\mathrm{E}}$.
5. $x < y \;\to\; \tilde{\mathrm{E}}(x) < \tilde{\mathrm{E}}(y)$.
6. $x > k^2 \to \tilde{\mathrm{E}}(x) > x^k$*; for each* $k \in K$ *with* $k \geq 1$.
7. $x > 0 \to \exists y(\tilde{\mathrm{E}}(y) = x)$.

Proof.

1. This is immediate from the definition of $\tilde{\mathrm{E}}$.

2. Suppose $x \geq 0$. Let $k \in \mathrm{Pos}(K)$. We want to show that $\tilde{\mathrm{E}}(kx) = \tilde{\mathrm{E}}(x)^k$. Choose $l \in K$ such that $l > x$. Let $y = \frac{x}{l}$. Now, using (1),

$$\begin{aligned}\tilde{\mathrm{E}}(kx) &= \tilde{\mathrm{E}}(kly) \\ &= \mathrm{E}(y)^{kl} \\ &= \tilde{\mathrm{E}}(ly)^k \\ &= \tilde{\mathrm{E}}(x)^k.\end{aligned}$$

   The other cases follow immediately.

3. We first consider the case where $x, y \geq 0$. Choose $k \in K$ such that $\frac{x+y}{k} \in [0,1]$. Now

$$\mathrm{E}\left(\frac{x+y}{k}\right) = \mathrm{E}\left(\frac{x}{k}\right)\mathrm{E}\left(\frac{y}{k}\right).$$

   So by raising to the power k we see that

$$\tilde{\mathrm{E}}(x+y) = \tilde{\mathrm{E}}(x)\,\tilde{\mathrm{E}}(y).$$

   Now suppose that $x \geq 0$ and $y < 0$ and $x + y \geq 0$. Let $z = x + y$. Then

$$\begin{aligned}\tilde{\mathrm{E}}(x) &= \tilde{\mathrm{E}}(z - y) \\ &= \tilde{\mathrm{E}}(z)\,\tilde{\mathrm{E}}(-y) \\ &= \tilde{\mathrm{E}}(z)\,\tilde{\mathrm{E}}(y)^{-1}.\end{aligned}$$

   So $\tilde{\mathrm{E}}(x+y) = \tilde{\mathrm{E}}(x)\,\tilde{\mathrm{E}}(y)$. The other cases are similar.

4. Note that

$$\frac{\tilde{\mathrm{E}}(x+h) - \tilde{\mathrm{E}}(x)}{h} = \tilde{\mathrm{E}}(x)\frac{\tilde{\mathrm{E}}(h) - 1}{h}.$$

   Since

$$\lim_{h \downarrow 0} \frac{\tilde{\mathrm{E}}(h) - 1}{h} = 1$$

   we get

$$\lim_{h \downarrow 0} \frac{\tilde{\mathrm{E}}(x+h) - \tilde{\mathrm{E}}(x)}{h} = \tilde{\mathrm{E}}(x).$$

   One easily deduces that $\tilde{\mathrm{E}}$ has a (two-sided) derivative at each point using the fact that $\tilde{\mathrm{E}}(-x) = \tilde{\mathrm{E}}(x)^{-1}$.

5. It is sufficient to prove that if $x > 0$ then $\tilde{\mathrm{E}}(x) > 1$. Since $\tilde{\mathrm{E}}$ has positive derivative at 0 there exists $\epsilon > 0$ such that $\tilde{\mathrm{E}}(x) > 1$ on $(0, \epsilon)$. Suppose for a contradiction that $\tilde{\mathrm{E}}(x) < 1$ for some $x > 0$. Let $I = [0, x]$. Since $\tilde{\mathrm{E}}$ is definable on I we may choose $y \in [\epsilon, x]$ to be least such that $\tilde{\mathrm{E}}(y) = 1$. But then the derivative of $\tilde{\mathrm{E}}$ must have a zero on $(0, y)$, that is there exists $z \in (0, y)$ such that $\tilde{\mathrm{E}}(z) = 0$. But then $\tilde{\mathrm{E}}$ must take value 1 on the interval $(0, y)$. This is a contradiction.

6. Using the fact that both the intermediate value property and the mean value property hold for functions definable in o-minimal expansions of fields we show that for all $x > 0$, $\tilde{\mathrm{E}}(x) > 2x$. Therefore $\tilde{\mathrm{E}}(x) > 1 + x^2$ for all $x > 0$. Let $k \in K$ and suppose that $k \geq 1$. We have

$$\tilde{\mathrm{E}}(k^2) = \tilde{\mathrm{E}}(k)^k > (1 + k^2)^k = (1 + k^{-2})^k (k^2)^k > (k^2)^k.$$

Furthermore, by differentiating $\frac{\tilde{\mathrm{E}}(x)}{x^k}$, we see that this is increasing for $x > k$. So if $x > k^2$ then

$$\frac{\tilde{\mathrm{E}}(x)}{x^k} > \frac{\tilde{\mathrm{E}}(k^2)}{(k^2)^k} > 1.$$

7. First consider the case where $z \geq 1$. We must find $y \in P$ such that $\tilde{\mathrm{E}}(y) = z$. Choose $k \in K$ such that $k > z$. Then $k > 1$ so $\tilde{\mathrm{E}}(k) > k$. Consider the $\mathcal{P}$-definable function $f : [0, 1] \to P$ given by $f(x) = \mathrm{E}(x)^k$. Now $f(0) = 1 \leq z < \tilde{\mathrm{E}}(k) = f(1)$, so by the intermediate value property for functions definable in o-minimal structures there exists $y \in (0, 1)$ such that $f(y) = z$, i.e. $\tilde{\mathrm{E}}(ky) = z$. The case where $z < 1$ follows immediately from the fact that $\tilde{\mathrm{E}}(-x) = \tilde{\mathrm{E}}(x)^{-1}$.

$\square$

By the above $\tilde{\mathrm{E}}$ is a bijection from P to $\mathrm{Pos}(P)$. We let $\mathrm{Log} : P \to P$ be given by

$$\mathrm{Log}(x) = \begin{cases} \tilde{\mathrm{E}}^{-1}(x) & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Now since $\tilde{\mathrm{E}}$ is $\mathcal{P}$-definable on closed bounded intervals of P, Log is definable on closed bounded intervals of P not containing 0. Furthermore since $\tilde{\mathrm{E}}(x)^k = \tilde{\mathrm{E}}(kx)$ for all $k \in K$, $k \,\mathrm{Log}(x) = \mathrm{Log}(x^k)$ for all $k \in K$ and $x > 0$.

6.5 A quantifier elimination result

Let L be the language of $\mathcal{P}$. We wish to prove that the expansion of $\mathcal{P}$ by $\tilde{\mathrm{E}}$ is o-minimal. To this end we may harmlessly assume that each 0-definable map $f : P^n \to P$ in the structure $\mathcal{P}$ is given by a function symbol of L so that T admits quantifier elimination and has a universal axiomatization (see remark 2.2.23).

Let $\mathcal{P}_{\tilde{\mathrm{E}}}$ be the expansion of $\mathcal{P}$ by $\tilde{\mathrm{E}}$ and $L_{\tilde{\mathrm{E}}}$ the language of $\mathcal{P}_{\tilde{\mathrm{E}}}$.

Let $T_{\tilde{\mathrm{E}}}$ be the $L_{\tilde{\mathrm{E}}}$-theory obtained by adding the following axiom schemes to T.

(A1) $\forall x \forall y (\tilde{\mathrm{E}}(x+y) = \tilde{\mathrm{E}}(x)\, \tilde{\mathrm{E}}(y))$,

(A2) $\forall x (0 \leq x \leq 1 \to \tilde{\mathrm{E}}(x) = \mathrm{E}(x)$,

(A3) $\forall x (\tilde{\mathrm{E}}(x)^k = \tilde{\mathrm{E}}(kx))$; for all $k \in K$,

(A4) $\forall x \forall y (x < y \ \to \ \tilde{\mathrm{E}}(x) < \tilde{\mathrm{E}}(y))$,

(A5) $\forall x (x > k^2 \to \tilde{\mathrm{E}}(x) > x^k)$; for each $k \in K$ with $k \geq 1$,

(A6) $\forall x (x > 0 \to \exists y (\tilde{\mathrm{E}}(y) = x))$.

Remark 6.5.1. Of course it follows from lemma 6.4.3 that $\mathcal{P}_{\tilde{\mathrm{E}}} \models T_{\tilde{\mathrm{E}}}$.

Let $L_{\tilde{\mathrm{E}},\mathrm{Log}}$ be the expansion of $L_{\tilde{\mathrm{E}}}$ by the unary function symbol Log and let $T_{\tilde{\mathrm{E}},\mathrm{Log}}$ be $T_{\tilde{\mathrm{E}}}$ together with the following axiom.

(A7) $\forall x ((x > 0 \to \tilde{\mathrm{E}}(\mathrm{Log}(x)) = x) \wedge (x < 0 \to \mathrm{Log}(x) = 0))$.

We will prove the following theorem, which is a generalization of Theorem B of [28]. The proof will closely follow the methods used in section 4 of [26].

Theorem 6.5.2.

1. *$T_{\tilde{\mathrm{E}},\mathrm{Log}}$ admits quantifier elimination and a universal axiomatization.*

2. *$T_{\tilde{\mathrm{E}}}$ axiomatizes* $\mathrm{Th}(\mathcal{P}_{\tilde{\mathrm{E}}})$.

6.5.1 Technical lemmas

From now onwards a map v defined on a model $\mathcal{A}$ of T will be the valuation map induced by the convex hull of the prime model.

Lemma 6.5.3. *Let $\mathcal{M} \models T_{\tilde{\mathrm{E}}}$ and let $\mathcal{A}$ be an L-substructure of $\mathcal{M}$. Then $\tilde{\mathrm{E}}$ maps $\{x \in A : v(x) \geq 0\}$ bijectively to $\{x \in \mathrm{Pos}(A) : v(x) = 0\}$.*

Proof. Let $y \in \{x \in A : v(x) \geq 0\}$. We will first show that $\tilde{\mathrm{E}}(y) \in A$. Since $\tilde{\mathrm{E}}(-y) = \tilde{\mathrm{E}}(y)^{-1}$ it is sufficient to consider the case when $y \geq 0$. Now $v(y) \geq 0$ so $y = kz$ for some $k \in K$ and $z \in [0,1] \cap A$. But $\tilde{\mathrm{E}}(y) = \tilde{\mathrm{E}}(z)^k = \mathrm{E}(z)^k \in A$. We must now show that $v(\tilde{\mathrm{E}}(y)) = 0$. Now $1 \leq \tilde{\mathrm{E}}(y) \leq \tilde{\mathrm{E}}(k) = \tilde{\mathrm{E}}(1)^k < l$ for some $l \in K$ since $V_{\mathcal{A}}$ is T-convex. So $v(\tilde{\mathrm{E}}(y)) = 0$. The injectivity of this map is clear. It remains to prove that it is surjective. So let $y \in \{x \in \mathrm{Pos}(A) : v(x) = 0\}$. We first consider the case when $y \geq 1$. As in part 7 of lemma 6.4.3 we find $c \in [0,1]$ and $k \in K$ such that $y = \mathrm{E}(c)^k = \tilde{\mathrm{E}}(kc)$. Now $v(kc) = v(k) + v(c) = v(c) \geq 0$. Now suppose that $0 < y < 1$. Find $x \in A$ with $v(x) \geq 0$ such that $\tilde{\mathrm{E}}(x) = \frac{1}{y}$. Then $\tilde{\mathrm{E}}(-x) = y$.

□

Lemma 6.5.4. *Let $\mathcal{M}, \mathcal{N} \models T_{\tilde{\mathrm{E}}}$, and suppose that $\mathcal{A}$ is an L-substructure of $\mathcal{M}$. Let $\sigma : \mathcal{A} \to \mathcal{N}$ be an L-embedding. Let $a \in \mathrm{Pos}(A)$ and suppose that $v(a) = 0$. Then $\sigma(\mathrm{Log}(a)) = \mathrm{Log}(\sigma(a))$.*

Proof. We consider the case when $a \geq 1$, the case where $0 < a < 1$ follows from the fact that $\mathrm{Log}(x^{-1}) = -\mathrm{Log}(x)$ for $x > 0$. Now, just as in part 7 of lemma 6.4.3 we find $c \in [0,1]$ and $k \in K$ such that $a = \mathrm{E}(c)^k$. Now

$$\begin{aligned}
\sigma(\mathrm{Log}(a)) &= \sigma(\mathrm{Log}(\mathrm{E}(c)^k)) \\
&= \sigma(kc)
\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{Log}(\sigma(a)) &= \mathrm{Log}(\sigma(\mathrm{E}(c)^k)) \\
&= \mathrm{Log}(\sigma(\mathrm{E}(c))^k) \\
&= k\,\mathrm{Log}(\sigma(\mathrm{E}(c)) \\
&= k\,\mathrm{Log}(\mathrm{E}(\sigma(c))) \\
&= k\sigma(c) \\
&= \sigma(kc).
\end{aligned}$$

□

Lemma 6.5.5. *Let $\mathcal{M} \models T_{\tilde{\mathrm{E}}}$. Let $x \in \mathrm{Pos}(M)$ and suppose that $v(x) < 0$. Then*

1. $v(\tilde{\mathrm{E}}(x)) < v(x)$.

2. $v(x) < v(\mathrm{Log}(x)) < 0$

Proof.

1. If $v(x) < 0$ then $x > k$ for all $k \in K$, so $\tilde{\mathrm{E}}(x) > x^2 > kx$ for all $k \in K$.

2. Since $x \in \mathrm{Pos}(M)$, $x = \tilde{\mathrm{E}}(y)$ for some $y \in M$. By lemma 6.5.3 we must have that $v(y) < 0$. By (1) we have $v(y) > v(\tilde{\mathrm{E}}(y))$, i.e. $v(\mathrm{Log}(x)) > v(x)$. If $v(\mathrm{Log}(x)) \geq 0$ then by lemma 6.5.3, $v(x) = 0$ which is a contradiction, so $v(\mathrm{Log}(x)) < 0$.

$\square$

6.5.2 Proof of theorem 6.5.2

We will say that an L-substructure $\mathcal{A}$ of a model of $T_{\tilde{\mathrm{E}}}$ is Log-*closed* if for all $a \in A$ we have $\mathrm{Log}(a) \in A$. We prove the following embedding theorem.

Theorem 6.5.6. *Suppose that $\mathcal{M} \models T_{\tilde{\mathrm{E}}}$ and $\mathcal{A}$ is an L-substructure of $\mathcal{M}$ which is* Log-*closed. Let $\mathcal{N}$ be an $|\mathcal{M}|^+$-saturated model of $T_{\tilde{\mathrm{E}}}$ and $\sigma : \mathcal{A} \to \mathcal{N}$ an* Log-*preserving embedding of L-structures. Then σ extends to a* Log-*preserving embedding of $\mathcal{M}$ into $\mathcal{N}$.*

Let us see now how theorem 6.5.6 implies theorem 6.5.2

Proof of theorem 6.5.2 assuming theorem 6.5.6.

1. In order to show that $T_{\tilde{\mathrm{E}},\mathrm{Log}}$ has quantifier elimination we will use corollary 2.3.4. So we let $\mathcal{M} \models T_{\tilde{\mathrm{E}},\mathrm{Log}}$ and suppose that $\mathcal{A}$ is an $L_{\tilde{\mathrm{E}},\mathrm{Log}}$-substructure of $\mathcal{M}$. Furthermore, we take $\mathcal{N}$ to be an $|\mathcal{M}|^+$-saturated model of $T_{\tilde{\mathrm{E}},\mathrm{Log}}$ and suppose that $\sigma : \mathcal{A} \to \mathcal{N}$ is an $L_{\tilde{\mathrm{E}},\mathrm{Log}}$-embedding. We must prove that σ extends to an $L_{\tilde{\mathrm{E}},\mathrm{Log}}$-embedding of $\mathcal{M}$ into $\mathcal{N}$. Theorem 6.5.6 tells us that σ extends to σ', a Log-preserving embedding of L-structures. Since Log is surjective on $\mathcal{M}$ the map σ' must in fact be an $L_{\tilde{\mathrm{E}},\mathrm{Log}}$-embedding. In order to see that $T_{\tilde{\mathrm{E}},\mathrm{Log}}$ has a universal axiomatization we note that axiom (A6) follows from axiom (A7).

2. We must prove that $T_{\tilde{\mathrm{E}}}$ is complete. Let $\mathcal{M} \models T_{\tilde{\mathrm{E}}}$ and let $\mathcal{M}'$ be its natural expansion to an $L_{\tilde{\mathrm{E}},\mathrm{Log}}$-structure. Since $L_{\tilde{\mathrm{E}},\mathrm{Log}}$ (and indeed L) has constants for all 0-definable elements of P and since $T_{\tilde{\mathrm{E}},\mathrm{Log}}$ has quantifier elimination we must have that $\mathcal{P}_{\tilde{\mathrm{E}},\mathrm{Log}}$ occurs as an elementary substructure of $\mathcal{M}'$. Cleary then $\mathcal{M} \equiv \mathcal{P}_{\tilde{\mathrm{E}}}$.

□

So it remains to prove theorem 6.5.6. We will do this via a number of lemmas. For $\mathcal{B}$ an L-substructure of $\mathcal{C} \models T_{\tilde{\mathrm{E}}}$ and $x \in C$, we will denote by $\mathcal{B}(x)$ the subfield of $\mathcal{C}$ generated by B and x and by $\mathcal{B}\langle x\rangle$ the L-substructure of $\mathcal{C}$ generated by B and x.

Lemma 6.5.7. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{A}$, σ be as in the statement of theorem 6.5.6. Suppose that $x \in M \setminus A$ and $v(\mathcal{A}(x)) = v(\mathcal{A})$. Let $\mathcal{A}' = \mathcal{A}\langle x\rangle$. Then $\mathcal{A}'$ is* Log*-closed and σ may be extended to an* Log*-preserving embedding of $\mathcal{A}'$ into $\mathcal{N}$.*

Proof. By theorem 2.2.61 we must have that $v(\mathcal{A}') = v(\mathcal{A})$. Let $x \in \mathrm{Pos}(A')$. We want to show that $\mathrm{Log}(x) \in A'$. Choose $y \in \mathrm{Pos}(A)$ with $v(y) = v(x)$. Now $\mathrm{Log}(x) = \mathrm{Log}(x/y) + \mathrm{Log}(y)$. Since $\mathcal{A}$ is Log-closed $\mathrm{Log}(y) \in A'$ and by lemma 6.5.3, since $v(x/y) = 0$, $\mathrm{Log}(x/y) \in \mathcal{A}'$ and therefore $\mathrm{Log}(x) \in A'$.

Now let $y \in N$ realize the image under σ of the cut made by x in A. By lemma 2.2.24, we can extend σ to an L-embedding $\sigma_0 : \mathcal{A}' \to \mathcal{N}$ by sending x to y.

It remains to prove that σ_0 is Log-preserving. Let $w \in \mathrm{Pos}(A')$. We may choose $z \in \mathrm{Pos}(A)$ and $a \in \mathrm{Pos}(A')$ such that $v(z) = v(w)$ and $v(a) = 0$ and $w = az$. As in the proof of lemma 6.5.3 we may choose $k \in K$ and $b \in [1, \mathrm{E}(1)] \cap A$ so that $a = b^k$. Now

$$\begin{aligned}
\sigma_0(\mathrm{Log}(w)) &= \sigma_0(\mathrm{Log}(z) + \mathrm{Log}(b^k)) \\
&= \sigma_0(\mathrm{Log}(z) + k\,\mathrm{Log}(b)) \\
&= \sigma_0(\mathrm{Log}(z)) + \sigma_0(k\,\mathrm{Log}(b)) \\
&= \mathrm{Log}(\sigma_0(z)) + k\,\mathrm{Log}(\sigma_0(b)) \\
&= \mathrm{Log}(\sigma_0(z)\sigma_0(b)^k) \\
&= \mathrm{Log}(\sigma_0(w)).
\end{aligned}$$

□

Lemma 6.5.8. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{A}$, σ be as in theorem 6.5.6. Suppose that $v(\mathcal{A}(x)) \neq v(\mathcal{A})$ for all $x \in M \setminus A$. Let $x \in A$ and suppose that $\tilde{\mathrm{E}}(x) \notin A$. Let $\mathcal{A}' = \mathcal{A}\langle\tilde{\mathrm{E}}(x)\rangle$. Then $\mathcal{A}'$ is* Log*-closed and σ may be extended to an* Log*-preserving embedding $\sigma_0 : \mathcal{A}' \to \mathcal{N}$ with $\sigma_0(\tilde{\mathrm{E}}(x)) = \tilde{\mathrm{E}}(\sigma(x))$.*

Proof. We first show that $v(\tilde{\mathrm{E}}(x)) \notin v(\mathcal{A})$. Suppose that $v(\tilde{\mathrm{E}}(x)) = v(a)$ where $a \in \mathrm{Pos}(A)$. Let $b = \mathrm{Log}(a)$, so $b \in A$. Now $\tilde{\mathrm{E}}(x) = a\,\tilde{\mathrm{E}}(x-b)$. So $v(\tilde{\mathrm{E}}(x-b)) = v(\tilde{\mathrm{E}}(x)) - v(a) = 0$. Since $x - b \in A$, by lemma 6.5.3, $\tilde{\mathrm{E}}(x-b) \in A$ and hence $\tilde{\mathrm{E}}(x) \in A$, which is a contradiction.

We must show that $\mathcal{A}'$ is Log-closed. Let $g = v(\tilde{\mathrm{E}}(x))$. By theorem 2.2.59 we have $v(\mathcal{A}') = v(\mathcal{A}) \oplus Kg$. Let $y \in \mathcal{A}'$, then $v(y) = v(a) + kg = v(a\,\tilde{\mathrm{E}}(kx))$ for some $a \in A$ and $b \in K$. So there exists $b \in A'$ with $v(b) = 0$ such that $y = ba\,\tilde{\mathrm{E}}(kx)$. Now since $v(b) = 0$, by lemma 6.5.3 applied to $\mathcal{A}'$ we have $\mathrm{Log}(b) \in A'$. Therefore $\mathrm{Log}(y) = \mathrm{Log}(b) + \mathrm{Log}(a) + kx \in \mathcal{A}'$.

We wish to extend our embedding so that $\sigma_0(\tilde{\mathrm{E}}(x)) = \tilde{\mathrm{E}}(\sigma(x))$. In order to show that this can be done it is sufficient to prove that $\tilde{\mathrm{E}}(\sigma(x))$ realises the image under σ of the cut of $\tilde{\mathrm{E}}(x)$ over A. Let $w \in \mathrm{Pos}(A)$ then

$$\begin{aligned} w < \tilde{\mathrm{E}}(x) &\Leftrightarrow \mathrm{Log}(w) < x \\ &\Leftrightarrow \sigma(\mathrm{Log}(w)) < \sigma(x) \\ &\Leftrightarrow \mathrm{Log}(\sigma(w)) < \sigma(x) \\ &\Leftrightarrow \sigma(w) < \tilde{\mathrm{E}}(\sigma(x)) \end{aligned}$$

It remains to prove that σ_0 is Log-preserving. Let $y \in A'$. As above we can find $a \in A$, $b \in A'$ with $v(b) = 0$ and $k \in K$ such that $y = ba\,\tilde{\mathrm{E}}(kx)$. Furthermore, as in part 7 of lemma 6.4.3, we may choose $c \in A'$ with $1 \leq c \leq \tilde{\mathrm{E}}(1)$ and $k \in K$ such that $b = c^k$. Now

$$\begin{aligned} \sigma_0(\mathrm{Log}(y)) &= \sigma_0(\mathrm{Log}(b) + \mathrm{Log}(a) + kx) \\ &= \sigma_0(k\,\mathrm{Log}(c)) + \sigma(\mathrm{Log}(a)) + \sigma(kx) \\ &= k\sigma_0(\mathrm{Log}(c)) + \mathrm{Log}(\sigma(a)) + \sigma(kx) \\ &= k\,\mathrm{Log}(\sigma_0(c)) + \mathrm{Log}(\sigma(a)) + \sigma(kx) \\ &= \mathrm{Log}(\sigma_0(c^k)\sigma(a)\sigma(\tilde{\mathrm{E}}(kx))) \\ &= \mathrm{Log}(\sigma_0(y)). \end{aligned}$$

□

Lemma 6.5.9. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{A}$ and σ be as in theorem* 6.5.6. *Suppose that $\mathcal{A}$ is closed under $\tilde{\mathrm{E}}$ and $v(\mathcal{A}(x)) \neq v(\mathcal{A})$ for all $x \in M \setminus A$. Let $x \in M \setminus A$. Then there is a* Log*-closed $\mathcal{B} \models T$ such that $\mathcal{A}(x) \subseteq \mathcal{B} \subseteq \mathcal{M}$ and a* Log*-preserving L-embedding $\sigma_0 : \mathcal{B} \to \mathcal{N}$ extending σ.*

Proof. Since $v(\mathcal{A}) \neq v(\mathcal{A}(x))$, by lemma 2.2.62 there exists $a \in A$ such that $v(x-a) \notin v(A)$. Replacing x by $x-a$ we may assume that $v(x) \notin v(A)$. Replacing x by x^{-1} if necessary we may also assume that $v(x) < 0$. We will choose elements $\beta_0, \beta_1, \ldots \in A$ and $x_0, x_1, \ldots \in \mathrm{Pos}(M)$. For all n we will have $v(x_n) < 0$ and $v(x_n) \notin v(A)$ (*). We begin by letting $x_0 = x$. Suppose that we have chosen $x_0, x_1, \ldots, x_n$ satisfying (*). We see that $v(A(\mathrm{Log}(x_n)) \nsubseteq v(A)$; otherwise, by the maximality of $\mathcal{A}$ with value group $v(A)$, we must have $\mathrm{Log}(x_n) \in A$ and thus $x_n \in A$ which is a contradiction. By 2.2.62 there exists $\beta_n \in A$ such that $v(\mathrm{Log}(x_n) - \beta_n) \notin v(A)$. Let $x_{n+1} = |\operatorname{Log}(x_n) - \beta_n|$, so that $\mathrm{Log}(x_n) = \beta_n + \epsilon_n x_{n+1}$, where $\epsilon_n = \pm 1$. We claim that

$$v(x_n) < v(\mathrm{Log}(x_n)) \leq v(x_{n+1}) < 0 \tag{6.1}$$

1. $v(x_n) < v(\mathrm{Log}(x_n))$: this follows immediately from lemma 6.5.5.

2. $v(\mathrm{Log}(x_n)) \leq v(x_{n+1})$: now $x_{n+1} = |\operatorname{Log}(x_n) - \beta_n|$; it follows that $v(x_{n+1}) \geq \min\{v(\mathrm{Log}(x_n)), v(\beta_n)\}$. Hence it is sufficient to prove that $v(\mathrm{Log}(x_n)) \leq v(\beta_n)$. Suppose $v(\beta_n) < v(\mathrm{Log}(x_n))$. Then $v(x_{n+1}) = v(\beta_n) \in v(A)$, which is a contradiction.

3. $v(x_{n+1}) < 0$: suppose that $v(x_{n+1}) \geq 0$. By lemma 6.5.3, $v(\tilde{\mathrm{E}}(x_{n+1})) = 0$. Now $v(x_n) = v(\tilde{\mathrm{E}}(\beta_n)) + v(\tilde{\mathrm{E}}(x_{n+1})) = v(\tilde{\mathrm{E}}(\beta_n)) \in v(A)$, which is a contradiction.

We now show that $v(x_0), \ldots, v(x_n), \ldots$ are K-linearly independent over $v(A)$. Suppose that

$$v(x_m) = \left(\sum_{i=m+1}^{n} k_i v(x_i) \right) + v(w) = v\left(w \prod_{m+1}^{n} x_i^k \right),$$

for some $k_i \in K$ and $w \in A$. Then there is $c \in M$ with $v(c) = 0$, such that

$$x_m = cw \prod_{i=m+1}^{n} x_i^{k_i},$$

So

$$\mathrm{Log}(x_m) = \mathrm{Log}(c) + \mathrm{Log}(w) + \sum_{i=m+1}^{n} k_i \operatorname{Log}(x_i),$$

therefore

$$\epsilon_m x_{m+1} = \mathrm{Log}(c) + \mathrm{Log}(w) - \beta_m + \sum_{i=m+1}^{n} k_i \operatorname{Log}(x_i).$$

So

$$v(x_{m+1}) \geq \min\{v(\mathrm{Log}(c)), v(\mathrm{Log}(w) - \beta_n), v(\mathrm{Log}(x_{m+1})), \ldots, v(\mathrm{Log}(x_n)\},$$

with equality if all terms on the right-hand side are distinct. By (6.1)

$$v(x_{m+1}) < v(\mathrm{Log}(x_{m+1})) < \ldots < v(\mathrm{Log}(x_n)) < 0.$$

Furthermore, since $v(c) = 0$, $v(\mathrm{Log}(c)) \geq 0$. Therefore $v(x_m) = v(\mathrm{Log}(w) - \beta_n) \in v(A)$, contradiction.

Now let $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_{n+1} = \mathcal{A}_n\langle x_n\rangle$ for $n \geq 0$. Let $\mathcal{B} = \bigcup_{n\geq 0} \mathcal{A}_n$. Since T has a universal axiomatization, $\mathcal{B} \models T$. Let $y \in N$ realize the image under σ_0 of the cut of x over A. Now define a sequence (y_n) in $\mathcal{N}$ by setting $y_0 = y$ and $y_{n+1} = \frac{\mathrm{Log}(y_n) - \sigma(\beta_n)}{\epsilon_n}$. We claim that y_n realises the image under σ of the cut of x_n over A. Suppose this is true for y_n. We will assume here that $\epsilon_n = 1$ (the case where $\epsilon_n = -1$ is similar). Let $w \in A$.

$$\begin{aligned}
w < x_{n+1} &\Leftrightarrow w + \beta_n < x_{n+1} + \beta_n \\
&\Leftrightarrow \tilde{\mathrm{E}}(w)\,\tilde{\mathrm{E}}(\beta_n) < x_n \\
&\Leftrightarrow \tilde{\mathrm{E}}(\sigma(w))\,\tilde{\mathrm{E}}(\sigma(\beta_n)) < y_n \quad \text{(since } \sigma \text{ is Log-preserving and } \tilde{\mathrm{E}}(w), \tilde{\mathrm{E}}(\beta_n) \in A) \\
&\Leftrightarrow \tilde{\mathrm{E}}(\sigma(w)) < y_n\,\tilde{\mathrm{E}}(-\sigma(\beta_n)) \\
&\Leftrightarrow \sigma(w) < \mathrm{Log}(y_n) - \sigma(\beta_n) = y_{n+1}
\end{aligned}$$

We now show that we can extend σ to an L-embedding $\sigma_n : \mathcal{A}_n \to \mathcal{N}$ by setting $\sigma_n(x_i) = y_i$. For $n = 1$ this follows from above. So we assume that we have such an L-embedding from $\mathcal{A}_n$ to $\mathcal{N}$. We must show that y_{n+1} realises the image under σ_n of the cut of x_{n+1} over $\mathcal{A}_n$.

Now, by theorem 2.2.59 and the fact that $v(x_1), \ldots, v(x_{n-1})$ are K-linearly independent over $v(A)$ we have $v(A_n) = v(A) \oplus K v(x_1) \oplus \ldots \oplus K v(x_{n-1})$. Let $w \in \mathrm{Pos}(A_n)$. We may choose $a \in \mathrm{Pos}(A_n)$ with $v(a) = 0$, $b \in \mathrm{Pos}(A)$ and $k_1, \ldots, k_n \in K$ such that

$$w = ab \prod_{i=0}^{n-1} x_i^{k_i}.$$

Let m be least such that $k_m \neq 0$. Let $c = a^{-\frac{1}{k_m}}$, $d = b^{-\frac{1}{k_m}}$, $r_i = -\frac{k_i}{k_m}$ for $i \leq n-1$ and $r_n = \frac{1}{k_m}$. We assume that $k_m < 0$. The other case is similar. Now

$$\begin{aligned}
w < x_{n+1} &\Leftrightarrow x_m > cd \prod_{i=m+1}^{n} x_i^{r_i} \\
&\Leftrightarrow \epsilon_m x_{m+1} > \mathrm{Log}(c) + \mathrm{Log}(d) - \beta_m + \sum_{i=m+1}^{n} r_i \,\mathrm{Log}(x_i).
\end{aligned}$$

Now, $v(x_{m+1}) < v(\mathrm{Log}(x_i)) < v(\mathrm{Log}(c))$ for all $i \geq m+1$. Furthermore, since $\mathrm{Log}(d) - \beta_m \in A$, $v(x_{m+1}) \neq v(\mathrm{Log}(d) - \beta_m)$. Therefore

$$\begin{aligned} w < x_{n+1} &\Leftrightarrow \epsilon_m x_{m+1} > \mathrm{Log}(d) - \beta_m \\ &\Leftrightarrow \epsilon_m y_{m+1} > \mathrm{Log}(\sigma_0(d)) - \sigma_0(\beta_m). \end{aligned}$$

Since σ_n is an L-embedding

$$\sigma_n(w) = \sigma_n(a)\sigma_n(b)\prod_{i=1}^{n} y_i^{k_i}.$$

As above we see that $\sigma_n(w) < y_{n+1} \Leftrightarrow \epsilon_m y_{m+1} > \mathrm{Log}(\sigma_0(d)) - \sigma_0(\beta_m)$ and so indeed y_{n+1} realises the image under σ_n of the cut of x_{n+1} over $\mathcal{A}_n$.

Now let $\sigma' = \bigcup \sigma_n : \mathcal{B} \to \mathcal{N}$. It remains to prove that $\mathcal{B}$ is Log-closed and σ' is Log-preserving. Let $w \in A_{n+1}$. As above we may choose $a \in A_{n+1}$ with $v(a) = 0$, $b \in A$ and $k_1, \ldots, k_n \in K$ such that

$$w = ab\prod_{i=0}^{n} x_i^{k_i}.$$

Then $\mathrm{Log}(w) = \mathrm{Log}(a) + \mathrm{Log}(b) + \sum_{i=0}^{n} k_i x_i$. Now $\mathrm{Log}(a) \in A_{n+1}$ since $v(a) = 0$, $\mathrm{Log}(b) \in A_0$ and $\mathrm{Log}(x_i) \in A_i$. So $\mathrm{Log}(w) \in A_{n+1}$. Furthermore

$$\begin{aligned} \sigma_{n+1}(\mathrm{Log}(w)) &= \sigma_{n+1}(\mathrm{Log}(a)) + \sigma_0(\mathrm{Log}(b)) + \sum_{i=0}^{n} k_i \sigma_{n+1}\mathrm{Log}(x_i) \\ &= \mathrm{Log}(\sigma_{n+1}(a)) + \mathrm{Log}(\sigma_0(b)) + \sum_{i=0}^{n} k_i \sigma_{n+1}(\beta_i + \epsilon_i x_{i+1}) \\ &= \mathrm{Log}(\sigma_{n+1}(a)) + \mathrm{Log}(\sigma_0(b)) + \sum_{i=0}^{n} k_i (\sigma_{n+1}(\beta_i) + \epsilon_i y_{i+1}) \\ &= \mathrm{Log}(\sigma_{n+1}(a)) + \mathrm{Log}(\sigma_0(b)) + \sum_{i=0}^{n} k_i \mathrm{Log}(y_i) \\ &= \mathrm{Log}(\sigma_{n+1}(w)). \end{aligned}$$

□

Proof of Theorem 6.5.6. Choose $\mathcal{B}$ maximal such that

1. $\mathcal{B}$ is a Log-closed L-substructure of $\mathcal{M}$ containing $\mathcal{A}$,

2. σ extends to a Log-preserving L-embedding $\sigma' : \mathcal{B} \to \mathcal{N}$.

We want to show that $\mathcal{B} = \mathcal{M}$. Suppose not. If there exists $x \in M \setminus B$ such that $v(x) \in v(B)$ then lemma 6.5.7 provides a contradiction to the maximality of $\mathcal{B}$. So for all $x \in M \setminus B$ we have $v(x) \notin v(B)$. If there exists $x \in B$ such that $\tilde{\mathrm{E}}(x) \notin B$ then lemma 6.5.8 gives us a contradiction to the maximality of $\mathcal{B}$. So $\mathcal{B}$ is closed under $\tilde{\mathrm{E}}$. But then applying lemma 6.5.9 we again contradict the maximality of $\mathcal{B}$. So we must have $\mathcal{B} = \mathcal{M}$. □

6.6 The o-minimality of $\mathcal{P}_{\tilde{\mathrm{E}}}$

In this section we prove that $\mathcal{P}_{\tilde{\mathrm{E}}}$ is o-minimal. The methods used will be very similar to those used in [26] apart from at one important stage where the authors use the local compactness of the real line. We give some additional arguments to complete the proof in our more general setting.

6.6.1 Fields of germs at infinity

We will let $\mathcal{G}$ be the ring of germs at $+\infty$ of (not necessarily definable) functions $f : P \to P$. We will not distinguish notationally between a function f and its germ at $+\infty$. This clearly does not cause any problems when talking about properties of f which hold for all sufficiently large x. We will say that a subfield N of $\mathcal{G}$ is a *$\mathcal{G}$-field* if for all $f \in N$, ultimately (i.e. for all sufficiently large x) $f(x)$ has constant sign. So a $\mathcal{G}$-field forms an ordered field. For each primitive[2] n-ary function F of $\mathcal{P}$ we define $F : \mathcal{G}^n \to \mathcal{G}$ by setting $F(f_1, \ldots, f_n)$ equal to the germ at $+\infty$ of the map $x \mapsto F(f_1(x), \ldots, f_n(x))$. We say that a $\mathcal{G}$-field N is a *$\mathcal{P}$-field* if it is closed under $F_{\mathcal{G}}$ for all primitive functions F of $\mathcal{P}$. If N is a $\mathcal{P}$-field and $f \in \mathcal{G}$ then we define

$$N\langle f\rangle = \{t(f_1, \ldots, f_n, f) : f_1, \ldots, f_n \in N \text{ and } t \text{ is an } L\text{-term}\}.$$

Of course if N is a $\mathcal{P}$-field we can give it an L-structure by interpreting the primitive n-ary function F on N as the restriction of F on $\mathcal{G}^n$.

Lemma 6.6.1. *Let N be a $\mathcal{P}$-field. Then, with its natural L-structure, $N \models T$. Furthermore, if $\phi(x_1, \ldots, x_n)$ is an L-formula and $f_1, \ldots, f_n \in N$ then $N \models \phi(f_1, \ldots, f_n)$ if and only if $\mathcal{P} \models \phi(f_1(x), \ldots, f_n(x))$ for all sufficiently large $x \in P$.*

Proof. Let $\mathcal{P}^+ = \langle \mathcal{P}, (f)_{f \in P^P}\rangle$, i.e. $\mathcal{P}^+$ is the structure expanding $\mathcal{P}$ by a function symbol for every (not necessarily definable in $\mathcal{P}$) function $P \to P$. Let $\mathcal{M}$ be an

[2]By a primitive function of $\mathcal{P}$ we mean a function that is the interpretation of a function symbol.

elementary extension of $\mathcal{P}^+$ containing a element a which is positive and infinite with respect to $\mathcal{P}$. We define a map $i_a : N \to M$ by $i_a(f) = f(a)$. To show that this map is well-defined we must show that if $f, g : P \to P$ have the same germ at $+\infty$ in P then $f(a) = g(a)$. Since f, g have the same germ at $+\infty$ there exists $m \in P$ such that

$$\mathcal{M} \models \forall x > m(f(x) = g(x)).$$

Therefore $f(a) = g(a)$. We now must show that i_a is an L-embedding. Let F be an n-ary function symbol of L and let $f_1, \ldots, f_n \in N$, then

$$\begin{aligned} i_a(F(f_1, \ldots, f_n)) &= F(f_1(a), \ldots, f_n(a)) \\ &= F(i_a(f_1), \ldots, i_a(f_n)). \end{aligned}$$

Furthermore, if $f, g \in N$ and $f < g$, then there exists $m \in P$ such that

$$\mathcal{M} \models \forall x > m(f(x) < g(x)).$$

So $f(a) < g(a)$. So $i_a : N \to i_a(N)$ is an L-isomorphism and hence $i_a(N)$ is an L-substructure of $\mathcal{M}$. Since T has a universal axiomatization in the language L, $i_a(N) \models T$ and hence $N \models T$. Since T also admits quantifier elimination $i_a(N)$ is an L-elementary substructure of $\mathcal{M}$. Thus the map i_a is an L-elementary embedding of N into $\mathcal{M}$. Furthermore if b is another positive infinite element of M then the evaluation map $i_b : N \to i_b(N)$ is also an L-isomorphism. So, using the notation in the statement of the lemma, if $N \models \phi(f_1, \ldots, f_n)$ then $\mathcal{M} \models \phi(f_1(c), \ldots, f_n(c))$ for all positive infinite elements $c \in M$. This implies that $\mathcal{M} \models \exists y \forall x(x > y \to \phi(f_1(x), \ldots, f_n(x))$. Since $(\mathcal{P}, (f)_{f \in P^P}) \preccurlyeq \mathcal{M}$ we have $(\mathcal{P}, (f)_{f \in P^P}) \models \exists y \forall x(x > y \to \phi(f_1(x), \ldots, f_n(x)),$ so for all sufficiently large $x \in P$ we have $\mathcal{P} \models \phi(f_1(x), \ldots, f_n(x))$. To establish the converse, consider $\neg\phi$. $\square$

Let N be a $\mathcal{P}$-field and let $f \in \mathcal{G}$. We say that f is comparable to N if for all $g \in N$, $f(x) - g(x)$ has ultimately constant sign.

Lemma 6.6.2. *Let N be a $\mathcal{P}$-field and suppose that $g \in \mathcal{G}$ is comparable to N. Then $N\langle g \rangle$ is a $\mathcal{P}$-field.*

Proof. Let $t(x_1, \ldots, x_{n+1})$ be an L-term and $f_1, \ldots, f_n \in N$. We must show that $t(f_1(x), \ldots, f_n(x), g(x))$ has ultimately constant sign. We may assume that $g \notin N$. We let $\mathcal{M}$, a and i_a be as in lemma 6.6.1. Since g is comparable to N, g determines a cut in N. It is straightforward to see that $g(a)$ realizes the image of this cut under i_a. Since T is o-minimal there are $h_0, h_1 \in N \cup \{\pm\infty\}$ such that $h_0 < g < h_1$

and $t(f_1, \ldots, f_n, y)$ has constant sign for $y \in (h_0, h_1) \cap N$. We will assume that $t(f_1, \ldots, f_n, y) > 0$ for all $y \in (h_0, h_1) \cap N$ and prove that $t(f_1(x), \ldots, f_n(x), g(x))$ is ultimately positive. The other cases are similar. Now

$$i_a(N) \models \forall y(i_a(h_0) < y < i_a(h_1) \rightarrow t(i_a(f_1), \ldots, i_a(f_n), y) > 0).$$

So

$$\mathcal{M} \models \forall y(i_a(h_0) < y < i_a(h_1) \rightarrow t(i_a(f_1), \ldots, i_a(f_n), y) > 0).$$

Therefore

$$\mathcal{M} \models t(i_a(f_1), \ldots, i_a(f_n), g(a)) > 0.$$

This is true for all positive infinite elements a. Hence $t(f_1(x), \ldots, f_n(x), g(x))$ is ultimately positive. □

6.6.2 Rings of functions

Let $\mathcal{O}$ be the ring of all functions $f : P \rightarrow P$. We will call a subring $\mathcal{Q}$ of $\mathcal{O}$ *locally* $\mathcal{P}$ if:

1. $\mathcal{Q}$ is closed under the primitive functions of $\mathcal{P}$, i.e., if $f_1, \ldots, f_n \in \mathcal{Q}$ and $F : P^n \rightarrow P$ is a primitive function of $\mathcal{P}$, then the function $x \mapsto F(f_1(x), \ldots, f_n(x))$ is in $\mathcal{Q}$.

2. the germs at $+\infty$ of functions in $\mathcal{Q}$ form a $\mathcal{P}$-field.

3. if $f \in \mathcal{Q}$ then there exists $a_1, \ldots, a_n \in P$ such that $a_1 < \ldots < a_n$ and if I is a closed bounded interval of P not containing a_i for any $i = 1, \ldots, n$, then f is $\mathcal{P}$-definable and continuously differentiable on I.

We will say that $\mathcal{Q}$, a locally $\mathcal{P}$ subring of $\mathcal{O}$, is *specially locally* $\mathcal{P}$ if it also satisfies:

4. if $f \in \mathcal{Q}$ and $q : P \rightarrow P$ is a rational function then $f \circ q \in \mathcal{Q}$ (we make our rational functions totally defined by setting them to 0 at points of discontinuity).

Note that the ring of $\mathcal{P}$-definable functions $f : P \rightarrow P$ is specially locally $\mathcal{P}$.

If $\mathcal{Q}$ is a subring of $\mathcal{O}$ and $f \in \mathcal{O}$ we will let

$$\mathcal{Q}\langle f \rangle = \{t(f_1, \ldots, f_n, f) : f_1, \ldots, f_n \in \mathcal{Q} \text{ and } t \text{ is an } L\text{-term}\}.$$

Lemma 6.6.3. *Let $f_1, \ldots, f_n : P \rightarrow P$ and let S be a definable subset of P^n. Let $a \in P$ and suppose that the functions $x \mapsto f_1(a - \frac{1}{x}), \ldots, x \mapsto f_n(a - \frac{1}{x})$ lie in a $\mathcal{P}$-field. Then there exists $\epsilon \in P$ such that $\epsilon > 0$ and $(f_1(x), \ldots, f_n(x)) \in S$ for all $x \in (a - \epsilon, a)$ or $(f_1(x), \ldots, f_n(x)) \notin S$ for all $x \in (a - \epsilon, a)$.*

Proof. Apply lemma 6.6.1 to the functions $x \mapsto f_1(a - \frac{1}{x}), \ldots, x \mapsto f_n(a - \frac{1}{x})$. $\square$

Lemma 6.6.4. *Suppose $f : P \to P$ lies in a subring $\mathcal{Q}$ of $\mathcal{O}$ that is specially locally $\mathcal{P}$. Then the sets $\{x \in P : f(x) > 0\}$ and $\{x \in P : f(x) = 0\}$ are the unions of finitely many intervals and points.*

Proof. Since f lies in a locally $\mathcal{P}$ subring of $\mathcal{O}$ we may choose $-\infty = a_0 < \ldots < a_n = +\infty$ such that f is $\mathcal{P}$-definable on closed bounded subintervals of P not containing any a_i. Let $X = \{x \in P : f(x) > 0\}$. It is sufficient to prove that $X \cap (a_i, a_{i+1})$ is a union of finitely many intervals and points. By lemma 6.6.3 for all x sufficiently close to a_i and for all x sufficiently close to a_{i+1} we have that $f(x)$ has constant sign (here we use that $\mathcal{Q}$ is closed under right composition with rational functions). So choose $I = [c, d]$ a closed bounded subinterval of (a_i, a_{i+1}) such that $f(x)$ has fixed sign on (a_i, c) and (d, a_{i+1}). Now $f(x)$ is $\mathcal{P}$-definable on I so the result follows. $\square$

Lemma 6.6.5. *Let $\mathcal{Q}$ be locally $\mathcal{P}$ and let $f \in \mathcal{Q}$. Then $\mathcal{Q}\langle \tilde{\mathrm{E}}(f)\rangle$ and $\mathcal{Q}\langle \mathrm{Log}(f)\rangle$ satisfy properties 1. and 2. in the definition of being locally $\mathcal{P}$.*

Proof. That property 1 is satisfied is immediate from the definitions of $\mathcal{Q}\langle \tilde{\mathrm{E}}(f)\rangle$ and $\mathcal{Q}\langle \mathrm{Log}(f)\rangle$.

By Lemma 6.6.2 it is sufficient to prove that germs at $+\infty$ of $\tilde{\mathrm{E}}(f), \mathrm{Log}(f)$ are comparable to the germs at $+\infty$ of functions in $\mathcal{Q}$. We first consider $\mathcal{Q}\langle \tilde{\mathrm{E}}(f)\rangle$. We show that for each $g \in \mathcal{Q}$ the sign of $\tilde{\mathrm{E}}(f(x)) - g(x)$ is ultimately constant. We may assume that $\tilde{\mathrm{E}}(f) \notin \mathcal{Q}$. Let

$$\Omega = \{h \in \mathcal{Q}\langle \tilde{\mathrm{E}}(f)\rangle : h \text{ has arbitrarily large zeros}\}.$$

Now since f and g lie in $\mathcal{Q}$ which is locally $\mathcal{P}$, for all sufficiently large x, y with $x < y$ f and g are $\mathcal{P}$-definable on $[x, y]$. Furthermore, in o-minimal expansions of fields the image of a closed, bounded set under a continuous definable function is closed and bounded. Since $\tilde{\mathrm{E}}$ is definable on any closed bounded subinterval of $\mathcal{P}$ we see that $\tilde{\mathrm{E}}(f) - g$ is $\mathcal{P}$-definable on $[x, y]$. Consequently for a sufficiently large x, $\tilde{\mathrm{E}}(f) - g$ satisfies the mean value property and the intermediate value property on (x, ∞). Therefore, in order to show that the sign of $\tilde{\mathrm{E}}(f) - g$ is ultimately constant it is sufficient to prove that $\tilde{\mathrm{E}}(f) - g \notin \Omega$. Suppose then for a contradiction that $\tilde{\mathrm{E}}(f) - g$ has arbitrarily large zeros. Then $h = 1 - g\,\tilde{\mathrm{E}}(-f) \in \Omega$. Therefore $h' \in \Omega$. Now $h' = \tilde{\mathrm{E}}(-f)(f'g - g')$ so $f'g - g' \in \Omega \cap \mathcal{Q} = \{0\}$. So $f' = \frac{g'}{g} = \mathrm{Log}(g)'$. By o-minimality of $\mathcal{P}$, on each closed bounded interval with sufficiently large left end point there exists $r \in P$ such that $\tilde{\mathrm{E}}(f) = rg$ for some $r \in P$. But clearly we must have the same r on each such interval. So for all sufficiently large x we have $\tilde{\mathrm{E}}(f(x)) = rg(x)$. Since $\tilde{\mathrm{E}}(f) - g \in \Omega$, $r = 1$ and so $\tilde{\mathrm{E}}(f) \in \mathcal{Q}$, which is a contradiction.

Now we consider $\mathcal{Q}\langle \mathrm{Log}(f)\rangle$. Let $g \in \mathcal{Q}$. We assume that $\mathrm{Log}(f) \notin \mathcal{Q}$. Then f must be ultimately positive. As above let

$$\Omega = \{h \in \mathcal{Q}\langle \mathrm{Log}(f)\rangle : h \text{ has arbitrarily large zeros}\}.$$

Since Log is $\mathcal{P}$-definable on any closed bounded subinterval of $\mathrm{Pos}(P)$, in order to prove that the sign of $\mathrm{Log}(f) - g$ is ultimately constant it is sufficient to prove that $\mathrm{Log}(f) - g \notin \Omega$. If $h = \mathrm{Log}(f) - g \in \Omega$ then $h' = \frac{f'}{f} - g' \in \Omega \cap \mathcal{Q} = \{0\}$. As above we see that there exists $r \in P$ such that for all sufficiently large x we have $g = \mathrm{Log}(f) + r$ and that in fact $r = 0$. So $g = \mathrm{Log}(f) \in \mathcal{Q}$ which is a contradiction. $\square$

Lemma 6.6.6. *Let $\mathcal{Q}$ be specially locally $\mathcal{P}$ and let $\langle f_\alpha : \alpha < \kappa\rangle$ be a κ-sequence of functions from $\mathcal{Q}$. We recursively define a sequence $\langle \mathcal{Q}_\alpha : \alpha < \kappa\rangle$ by*

1. *$\mathcal{Q}_0 = \mathcal{Q}$,*
2. *$\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha\langle \tilde{\mathrm{E}}(f_\alpha)\rangle$,*
3. *$\mathcal{Q}_\lambda = \bigcup_{\beta<\lambda} \mathcal{Q}_\beta$; for λ a limit ordinal.*

Then for each α, $\mathcal{Q}_\alpha$ is locally $\mathcal{P}$ and $\bigcup_{\alpha<\kappa} \mathcal{Q}_\alpha$ is locally $\mathcal{P}$.

Proof. We prove that each $\mathcal{Q}_\alpha$ is locally $\mathcal{P}$ by transfinite induction. That $\bigcup_{\alpha<\kappa} \mathcal{Q}_\alpha$ is locally $\mathcal{P}$ is an immediate consequence.

$\boldsymbol{\alpha = 0}$ By assumption.

$\boldsymbol{\alpha}$ **a successor ordinal** Let $\beta + 1 = \alpha$. Properties 1 and 2 follow from lemma 6.6.5. We must establish property 3 for $\mathcal{Q}_\beta\langle \tilde{\mathrm{E}}(f_\beta)\rangle$. Take $g \in \mathcal{Q}_\beta\langle \tilde{\mathrm{E}}(f_\beta)\rangle$. Then $g = t(f_1, \ldots, f_n, \tilde{\mathrm{E}}(f_\beta))$ for some $f_1, \ldots, f_n \in \mathcal{Q}_\beta$ and some L-term t. We must find $a_1, \ldots, a_n \in P$ such that g is $\mathcal{P}$-definable and continuously differentiable on closed bounded intervals of P not containing any a_i. We consider first the case where $g = \tilde{\mathrm{E}}(f_\beta)$. Since f_β lies in $\mathcal{Q}$ there exists $a_1 < \ldots < a_n$ such that f_β is $\mathcal{P}$-definable and continuously differentiable on closed bounded intervals of P not containing any a_i. We show that the same $a_1, \ldots, a_n$ will do for g. Let I be a closed bounded subinterval of P not containing any a_i. Now f_β is $\mathcal{P}$-definable and continuous on I and so $f_\beta(I)$ is closed and bounded. Therefore $\tilde{\mathrm{E}}$ is $\mathcal{P}$-definable on $f_\beta(I)$. Thus g is $\mathcal{P}$-definable on I. Since $\tilde{\mathrm{E}}$ is $\mathcal{C}^\infty$, g is continuously differentiable at all $x \notin \{a_1, \ldots, a_n\}$.

Now consider the case where g is a general element of $\mathcal{Q}_{\beta+1}$ so that $g = t(h_1, \ldots, h_n, \tilde{\mathrm{E}}(f_\beta))$ for some $h_1, \ldots, h_n \in \mathcal{Q}_\beta$ and t an L-term. We may certainly choose $-\infty = a_1 < \ldots < a_n = +\infty$ such that each h_i and $\tilde{\mathrm{E}}(f_\beta)$ are $\mathcal{P}$-definable and continuously differentiable on closed bounded intervals not containing any a_i. Then, since t is an L-term, g is $\mathcal{P}$-definable on closed bounded intervals not containing any a_i. Let

$$C = \{x \in P^{n+1} : t \text{ is not continuously differentiable at } x\},$$
$$D = \{x \in P : g \text{ is not continuously differentiable at } x\}.$$

It remains to prove that D is finite. Suppose, for a contradiction, that D is infinite. Then $X = D \cap (a_i, a_{i+1})$ is infinite for some i. Since g is $\mathcal{P}$-definable on any closed bounded subinterval of (a_i, a_{i+1}), X cannot contain an interval. For the same reason we must have that one of the endpoints of (a_i, a_{i+1}) is a limit point of X. Without loss of generality we assume that a_{i+1} is. Now consider the sequence $\langle k_\gamma : \gamma < \kappa \rangle$ of functions from P to P given by $k_\gamma(x) = f_\gamma(a_{i+1} - \frac{1}{x})$. Since $\mathcal{Q}$ is specially locally $\mathcal{P}$ we have $k_\gamma \in \mathcal{Q}$ for each $\gamma < \kappa$. Let $\mathcal{Q}'_\gamma$ be defined as $\mathcal{Q}_\gamma$ but with f_γ's being replaced by k_γ's. By our inductive hypothesis and lemma 6.6.5 the germs at $+\infty$ of functions in $\mathcal{Q}'_{\beta+1}$ form a $\mathcal{P}$-field, furthermore $h_1(a_{i+1} - \frac{1}{x}), \ldots, h_n(a_{i+1} - \frac{1}{x}), \tilde{\mathrm{E}}(f_\beta(a_{i+1} - \frac{1}{x})) \in \mathcal{Q}'_{\beta+1}$. Now, by lemma 6.6.3, for some $\epsilon > 0$ we must have that $(h_1(x), \ldots, h_n(x), \tilde{\mathrm{E}}(f_\beta(x))) \in C$ for all $x \in (a_{i+1} - \epsilon, a_{i+1})$. By the o-minimality of $\mathcal{P}$ and theorem 2.2.28 we have $\dim(C) < n + 1$. We now consider

$$C_1 = \{x \in C : t : C \to P \text{ is not continuously differentiable at } x\}.$$

Again, by o-minimality, $\dim(C_1) < \dim(C)$. By lemma 6.6.3 for some $\epsilon_1 > 0$ we must have that $(h_1(x), \ldots, h_n(x), \tilde{\mathrm{E}}(f_\beta)) \in C_1$ for all $x \in (a_{i+1} - \epsilon_1, a_{i+1})$. Continuing in this way we obtain a finite set $Y \subseteq P^n$ with the property that $(h_1(x), \ldots, h_n(x), \tilde{\mathrm{E}}(f_\beta)) \in Y$ for all x sufficiently close to a_{i+1}. We conclude that $(h_1(x), \ldots, h_n(x), \tilde{\mathrm{E}}(f_\beta))$ becomes constant as x approaches a_{i+1}. This contradicts the supposition that a_{i+1} is a limit point for X.

α a limit ordinal This is immediate.

□

Lemma 6.6.7. *Let $\mathcal{Q}$ be specially locally $\mathcal{P}$ and let $\langle f_\alpha : \alpha < \kappa \rangle$ be a sequence of functions from $\mathcal{Q}$. We recursively define a sequence $\mathcal{Q}_\alpha$ by*

1. $\mathcal{Q}_0 = \mathcal{Q}$,

2. $\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha\langle \mathrm{Log}(f_\alpha)\rangle$,

3. $\mathcal{Q}_\lambda = \bigcup_{\beta<\lambda} \mathcal{Q}_\beta$, *for* λ *a limit ordinal.*

Then for each α, $\mathcal{Q}_\alpha$ *is locally* $\mathcal{P}$ *and* $\bigcup_{\alpha<\kappa} \mathcal{Q}_\alpha$ *is locally* $\mathcal{P}$.

Proof. The proof is almost identical to the proof of lemma 6.6.6 except at the point where we assume that $\mathcal{Q}_\beta$ is locally $\mathcal{P}$ and must prove that for $\mathrm{Log}(f_\beta)$ there exists $c_1, \ldots, c_n$ such that $\mathrm{Log}(f_\beta)$ is $\mathcal{P}$-definable and continuously differentiable on closed bounded intervals not containing any c_i. By our inductive hypothesis we can find such $a_1, \ldots, a_m$ for the function f_β. Now Log is continuously differentiable except at 0 and definable on any closed bounded intervals of P not containing 0. We must show that f_β has only finitely many isolated zeros. This follows from lemma 6.6.4. Let $b_1, \ldots, b_k$ be the isolated zeros of f_β. Then $\mathrm{Log}(f_\beta)$ has property 3 of being locally $\mathcal{P}$ with respect to the points $a_1, \ldots, a_m, b_1, \ldots, b_k$. $\square$

Lemma 6.6.8. *Let* $\mathcal{Q}$ *be specially locally* $\mathcal{P}$ *and let* $f \in \mathcal{Q}$. *Then there exists* $\mathcal{Q}'$ *which is specially locally* $\mathcal{P}$ *and contains* $\tilde{\mathrm{E}}(f)$, *similarly there exists* $\mathcal{Q}''$ *which is specially locally* $\mathcal{P}$ *and contains* $\mathrm{Log}(f)$.

Proof. Let $\langle p_\alpha : \alpha < \kappa\rangle$ be an enumeration of the rational functions of $\mathcal{P}$. We now apply lemma 6.6.6 to the sequence $\langle f \circ p_\alpha : \alpha < \kappa\rangle$. So $\bigcup_{\alpha<\kappa} \mathcal{Q}_\alpha$ is specially locally $\mathcal{P}$ and contains $\tilde{\mathrm{E}}(f)$. We find $\mathcal{Q}''$ by applying lemma 6.6.7 in the same way. $\square$

Theorem 6.6.9. $\mathcal{P}_{\tilde{\mathrm{E}},\mathrm{Log}}$ *is o-minimal.*

Proof. By lemma 6.6.8 and the fact that the ring of $\mathcal{P}$-definable functions is specially locally $\mathcal{P}$ any term definable function in $\mathcal{P}_{\tilde{\mathrm{E}},\mathrm{Log}}$ lies in a locally $\mathcal{P}$ subring of $\mathcal{O}$. The o-minimality of $\mathcal{P}_{\tilde{\mathrm{E}},\mathrm{Log}}$ now follows from quantifier elimination and lemma 6.6.4. $\square$

Chapter 7

Future Work

There is one particular line of work arising from this thesis that I feel is especially worthy of pursuit. In chapter 5, we prove that the theory T_{exp} is decidable if the existential part of T_∞ is recursively enumerable. So a natural question to ask is: can one prove that the existential part of T_∞ is recursively enumerable?

As indicated in chapter 5, Wilkie and Macintyre prove that T_{exp} is decidable if the existential part of T_{exp} is recursively enumerable, and then prove that this is the case under the assumption of Schanuel's conjecture. In recent work [1] Bays, Kirby and Wilkie prove the following Schanuel-type statement.

Theorem 7.0.10. *Let F be an exponential field and let $\lambda \in F$ be exponentially transcendental. Let $\bar{x} \in F^n$ be such that* $\exp(\bar{x})$ *is multiplicatively independent. Then*

$$\operatorname{trd}\left(\exp(\bar{x}), \exp(\lambda\bar{x})/\lambda\right) \geq n,$$

i.e. the field extension $\mathbb{Q}\left(\lambda, \bar{x}, \exp(\lambda\bar{x})\right)/\mathbb{Q}\left(\lambda\right)$ has transcendence degree at least n.

We won't give the general definition of exponential algebraicity,[1] but in the special case of $\mathcal{R} \models T_{\mathrm{exp}}$, an element $a \in R$ is exponentially algebraic over $A \subseteq R$ if and only if $a \in \operatorname{dcl}(A)$, and so a is exponentially transcendental if and only if a is not 0-definable. Applying theorem 7.0.10 in $\mathcal{R}$ one gets the following statement.

Corollary 7.0.11. *Let $\lambda \in R \setminus \operatorname{dcl}\{\emptyset\}$ and let $\bar{x} \in \operatorname{Pos}(R)^n$ be multiplicatively independent. Then*

$$\operatorname{trd}\left(\bar{x}, \bar{x}^\lambda/\lambda\right) \geq n.$$

Notice that $\operatorname{dcl}\{\emptyset\} \subseteq \operatorname{Fin}(R)$, so if $\mathcal{R}$ is non-Archimedean and λ is positive infinite then λ is exponentially transcendental. Consequently the Schanuel condition holds in models of T_∞.

Given this, one might hope to emulate the result of Macintyre and Wilkie (that the existential part of T_{exp} is recursively enumerable if Schanuel's conjecture holds) to prove outright that the existential part of T_∞ is recursively enumerable (and hence T_{exp} is decidable). Indeed, in a forthcoming paper, Jones and Servi use corollary 7.0.11 and the methods of Wilkie and Macintyre to prove that the structure $\langle \overline{\mathbb{R}}, x^r \rangle$ is decidable if r is both exponentially transcendental and the cut of r in $\mathbb{Q}$ is recursive (note that for models of T_∞ the cut of the exponent of the power function is trivially recursive).

Unfortunately complications arise. The work of Macintyre and Wilkie (and subsequently Jones and Servi) makes essential use of the fact that

[1] Exponentially transcendental means not exponentially algebraic.

1. $\mathbb{Q}$ is dense in $\mathbb{R}$, the domain of the standard model,
2. if $\bar{q} \in \mathbb{Q}^n$ and $P(\bar{x}, \exp(\bar{x})) \in \mathbb{Q}[\bar{x}, \exp(\bar{x})]$, then one can effectively determine the sign of $P(\bar{q}, \exp(\bar{q}))$.

Let $\mathcal{P}$ be the prime model of T_∞. In order to apply the method of Macintyre and Wilkie to T_∞ it seems that one must find a subset D of P which is recursively enumerable, dense in P and has the property that given $\bar{p} \in D$ one can effectively determine the sign of a power-polynomial (defined in analogy to exponential polynomial) evaluated at $\bar{p}$. As a first step towards finding such a D, one would like to prove that if $\mathcal{P} = \langle P, x^\lambda, \lambda \rangle$, then the sequence $\lambda, \lambda^\lambda, \lambda^{\lambda^2}, \ldots$ is cofinal in P.

Bibliography

[1] M. Bays, J. Kirby, and A. Wilkie. A Schanuel property for exponentially transcendental powers. *arXiv*, 2008.

[2] J. Denef and L. van den Dries. p-adic and real subanalytic sets. *Ann. of Math. (2)*, 128(1):79–138, 1988.

[3] T. Foster. Uniform model completeness for the real field expanded by power functions. *J. Symbolic Logic*, 75(4), 2010.

[4] A. G. Hovanskiĭ. A class of systems of transcendental equations. *Dokl. Akad. Nauk SSSR*, 255(4):804–807, 1980.

[5] G. O. Jones and A. J. Wilkie. Locally polynomially bounded structures. *Bull. Lond. Math. Soc.*, 40(2):239–248, 2008.

[6] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[7] O. Le Gal and J.-P. Rolin. An o-minimal structure which does not admit C^∞ cellular decomposition. *Ann. Inst. Fourier (Grenoble)*, 59(2):543–562, 2009.

[8] A. Macintyre and A. J. Wilkie. On the decidability of the real exponential field. In *Kreiseliana*, pages 441–467. A K Peters, Wellesley, MA, 1996.

[9] D. Macpherson and L. L. Jt. Notes on o-minimality and variations, 1999.

[10] D. Marker. Introduction to model theory. In *Model theory, algebra, and geometry*, volume 39 of *Math. Sci. Res. Inst. Publ.*, pages 15–35. Cambridge Univ. Press, Cambridge, 2000.

[11] C. Miller. Expansions of the real field with power functions. *Ann. Pure Appl. Logic*, 68(1):79–94, 1994.

[12] C. Miller. Infinite differentiability in polynomially bounded o-minimal structures. *Proc. Amer. Math. Soc.*, 123(8):2551–2555, 1995.

[13] C. Miller. A growth dichotomy for o-minimal expansions of ordered fields. In *Logic: from foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 385–399. Oxford Univ. Press, New York, 1996.

[14] C. Miller. An upgrade for "Expansions of the real field by power functions". available at http://www.math.ohio-state.edu/∼miller/, 2003.

[15] J.-P. Ressayre. Integer parts of real closed exponential fields (extended abstract). In *Arithmetic, proof theory, and computational complexity (Prague, 1991)*, volume 23 of *Oxford Logic Guides*, pages 278–288. Oxford Univ. Press, New York, 1993.

[16] G. E. Sacks. *Saturated Model Theory.* W.A. Benjamin, Inc., 1972.

[17] P. Speissegger. The Pfaffian closure of an o-minimal structure. *J. Reine Angew. Math.*, 508:189–211, 1999.

[18] A. Tarski. *A Decision Method for Elementary Algebra and Geometry.* RAND Corporation, Santa Monica, Calif., 1948.

[19] J. Tyne. *T-levels and T-convexity.* PhD thesis, University of Illinois at Urbana-Champaign, 2003.

[20] L. van den Dries. Remarks on Tarski's problem concerning $(\mathbf{R}, +, \cdot, \exp)$. In *Logic colloquium '82 (Florence, 1982)*, volume 112 of *Stud. Logic Found. Math.*, pages 97–121. North-Holland, Amsterdam, 1984.

[21] L. van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bull. Amer. Math. Soc. (N.S.)*, 15(2):189–193, 1986.

[22] L. van den Dries. o-minimal structures. In *Logic: from foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 137–185. Oxford Univ. Press, New York, 1996.

[23] L. van den Dries. T-convexity and tame extensions. II. *J. Symbolic Logic*, 62(1):14–34, 1997.

[24] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1998.

[25] L. van den Dries and A. H. Lewenberg. T-convexity and tame extensions. *J. Symbolic Logic*, 60(1):74–102, 1995.

[26] L. van den Dries, A. Macintyre, and D. Marker. The elementary theory of restricted analytic fields with exponentiation. *Ann. of Math. (2)*, 140(1):183–205, 1994.

[27] L. van den Dries and C. Miller. On the real exponential field with restricted analytic functions. *Israel J. Math.*, 85(1-3):19–56, 1994.

[28] L. van den Dries and P. Speissegger. The field of reals with multisummable series and the exponential function. *Proc. London Math. Soc. (3)*, 81(3):513–565, 2000.

[29] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.