# PSEUDOPLANES AND NON-LOCAL MODULARITY – Talk 3

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(1)

Let $T$ be a complete, stable, strongly minimal theory and let $D \models T$ be a monster model.

**Aim:** characterisation of (non-)local modularity, there:

> **Theorem:** Tfae:
> (i) $D$ is 1-based
> (ii) $D$ is linear
> (iii) $D$ is locally modular
> (iv) $D$ has no generic pseudoplane

References:
- Martin Bays: Geometric Stability Theory
- Pillay: Geometric Stability Theory
- Tent, Ziegler: A Course in Model Theory
- Palacín: An Intro. to Stability Theory
- Martin Bays (thanks :))
- Talk 1, Talk 2

## 1. MODULARITY

**Definition 1** (recall): A pregeometry $(X, cl)$ is
- **modular** if f.a. finite dimensional closed $A, B \subseteq X$:
$$\dim(A/B) = \dim(A/A\cap B), \qquad (*)$$
- **locally modular** if the localisation at any $p \in X \setminus cl(\emptyset)$ is modular, i.e. if $(X, cl_p)$ is modular, where $cl_p(A) := cl(Ap)$.

**Lemma 2:** A pregeometry $(X, cl)$ is modular if $(*)$ holds for all finite dimensional closed sets $A, B \subseteq X$ with $\dim(A) = 2$.

**Proof:** ▸ Step 1: Show: $(*)$ for $\dim(A)=2 \implies (*)$ for $\dim(A/A\cap B)=2$.
Let $a_1, a_2$ be a basis for $A/A\cap B$, let $A' := cl(a_1 a_2)$.
Then $A'\cap B = cl(\emptyset)$, hence
$$\dim(A/B) = \dim(A'/B) \overset{\text{ass.}}{=} \dim(A'/A'\cap B) = \dim(A'/A'\cap B) = \dim(A')$$
$$= 2 \underset{\text{ass.}}{=} \dim(A/A\cap B).$$

▸ Step 2: Show: $(*)$ for $\dim(A/A\cap B) = 2 \implies (*)$.
Let $n := \dim(A/B) = \dim(AB) - \dim(B)$.
TS: $n = \dim(A/A\cap B)$.
Consider an immediate chain $B = B_0 \subsetneq B_1 \subseteq \ldots \subseteq cl(AB)$ of closed sets and let $A_i := A\cap B_i$.
STS: $\dim(A_{i+1}/A_i) \le 1$.
Else we can assume $\dim(A_{i+1}/A_i) = 2$ (the case $>2$ is similar).
Then $2 = \dim(A_{i+1}/A_i) = \dim(A_{i+1}/A_{i+1}\cap B_i) \underset{\text{ass.}}{=} \dim(A_{i+1}/B_i)$
$\le \dim(B_{i+1}/B_i) = 1$. ↯

□

## 2. 1-BASEDNESS

<u>Definition 3.</u> $D$ is <u>1-based</u> if f.a. $a \in D^{eq}$ and all $B \subseteq D^{eq}$:
$$Cb(a/B) \subseteq acl^{eq}(a).$$

<u>Lemma 4:</u> $D$ is 1-based iff f.a. $acl^{eq}$-closed $A, B \subseteq D^{eq}$:
$$A \underset{A \cap B}{\perp} B. \qquad (*2)$$

<u>Proof:</u> • "$\Rightarrow$": By finite character of $\perp$ can assume that $A$ is a finite tuple $a \subseteq D^{eq}$. With 1-basedness we get $Cb(a/B) \subseteq acl^{eq}(a) \cap acl^{eq}(B) =: S$, so $a \underset{Cb(a/B)}{\perp} B$ (which holds more generally) implies $a \underset{S}{\perp} B$.

<u>Recall:</u> $RM(a/Cb(a/B)) = RM(a/B)$ and $Cb(a/B) \subseteq acl^{eq}(B)$.

• "$\Leftarrow$": Let $a \in D^{eq}$, $B \subseteq D^{eq}$. Then by $(*2)$ $a \underset{S}{\perp} acl^{eq}(B)$, hence $Cb(a/B) \subseteq S \subseteq acl^{eq}(a)$.

<u>Recall:</u> $a \underset{C}{\perp} B \iff Cb(a/BC) \subseteq acl^{eq}(C)$

<u>Lemma 5:</u> 1-basedness is preserved under adding and removing parameters.

<u>Proof:</u> • <u>adding:</u> Let $D$ be 1-based, $a \in D^{eq}_A$, $B \subseteq D^{eq}_A$. Then
$$Cb_A(a/B) \subseteq dcl^{eq}_A(Cb_A(a/B)) = dcl^{eq}_A(Cb(a/AB)) \overset{\text{1-basedness}}{\subseteq} dcl^{eq}_A(acl^{eq}(a)) \subseteq acl^{eq}_A(a).$$

• <u>removing:</u> Let $D_A$ be 1-based. Let $a \in D^{eq}$, $B \subseteq D^{eq}$. We can assume $aB \perp A$. Then
$$Cb(a/B) \subseteq dcl^{eq}_A(Cb(a/B)) = dcl^{eq}_A(Cb(a/AB)) = dcl^{eq}_A(Cb_A(a/B)) \underset{D_A \text{ 1-based}}{\subseteq} acl^{eq}_A(a) = acl^{eq}(Aa).$$
So $Cb(a/B) \subseteq acl^{eq}(B) \cap acl^{eq}(Aa) \underset{\boxed{A \underset{a}{\perp} B}}{\subseteq} acl^{eq}(a)$. □

<u>Lemma 6:</u> If $D$ has gEI, then $D$ is modular iff $D$ is 1-based.

<u>Proof:</u> Note that
- $A = acl(A) \Rightarrow A = acl^{eq}(A) \cap D$
- $A = acl^{eq}(A) \overset{gEI}{\Longrightarrow} A = acl^{eq}(A \cap D)$.

Now for $A = acl^{eq}(A)$, $B = acl^{eq}(B)$:
$$A \underset{A \cap B}{\perp} B \iff acl^{eq}(A \cap D) \underset{acl^{eq}(A \cap B \cap D)}{\perp} acl^{eq}(B \cap D) \iff A \cap D \underset{A \cap B \cap D}{\perp} B \cap D,$$
which implies the claim. □

# 3. LINEARITY

**Definition 7:** $D$ is *linear* if f.a. $a, b \in D$ and all $C \subseteq D^{eq}$:

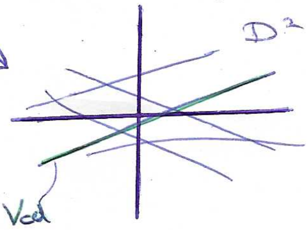
$$RM(ab/C) = 1 \implies RM(Cb(ab/C)) \leq 1.$$

↳ Intuition: "$D$ has no 2-dimensional family of plane curves."

**Example 8:** ACF's are not linear: take a Morley sequence $(a, c, d) \models p^{(3)}$ and let $b := c \cdot a + d$. Consider the algebraic variety $V_{cd} = \{(x,y) \mid y = cx + d\}$. Then $Cb(ab/cd) = dcl^{eq}(cd)$ and $RM(ab/cd) = 1$, but $RM(c,d) = 2$.

global generic type of $D$

varying c and d gives different algebraic varieties; this 'induces' a 2-dimensional family of plane curves




# 4. GENERIC PSEUDOPLANES

**Definition 9:** A *generic pseudoplane* (over $A \subseteq D^{eq}$) is the set $r(D^{eq}, D^{eq})$ ("points", "lines") of realisations of a complete type $r(x,y) = tp(b,c/A)$ with $b, c \in D^{eq}$, s.t.

(1) $c \notin acl^{eq}(b)$ (3) $r(x, c_1) \cup r(x, c_2)$ is algebraic for $c_1 \neq c_2$,

(2) $b \notin acl^{eq}(c)$ (4) $r(b_1, y) \cup r(b_2, y)$ is algebraic for $b_1 \neq b_2$.

**Lemma 10:** Let $b, c \in D^{eq}$ st.

(A) $RM(b) = 2$ (B) $RM(b/c) = 1$ (C) $Cb(b/c) = c$

(A') $RM(c) = 2$ (B') $RM(c/b) = 1$ (C') $Cb(c/b) = b$.

Then $tp(b/c)$ defines a generic pseudoplane.

*Proof:* By symmetry sts (1) and (3), since (1) follows from (B'). If (3) fails, then ex. $c_1 \neq c_2 \models tp(c/b)$ st. $r(x,c_1) \cup r(x,c_2)$ is non-alg, where we can assume $c = c_1$. Pick a realisation $b'$ of a common, non-algebraic extension of $r(x,c)$ and $r(x,c_2)$. Can assume that $b' = b$, so $b \notin acl^{eq}(c, c_2)$. So $RM(b/cc_2) = 1 = RM(b/c) = RM(b/c_2)$, so $p = tp(b/cc_2)$ is a non-forking extension of both $r(x,c)$ and $r(x,c_2)$. By (C), $r(x,c)$ and $r(x,c_2)$ are stationary, so a global non-forking extension $\mathfrak{p} \supseteq p$ is the unique global non-forking extension of both $r(x,c)$ and $r(x,c_2)$ and $Cb(\mathfrak{p}) = c$. So if $\sigma \in Aut(D^{eq})$ maps $c$ to $c_2$, then $\sigma(r(x,c)) = r(x,c_2)$ and hence $\sigma(\mathfrak{p}) = \mathfrak{p}$, which implies by def. of Cb $\sigma(c) = c = c_2$. ↯ $c \neq c_2$. □

# 5. CHARACTERISATION OF NON-LOCAL MODULARITY

**Theorem 11:** Tfae:

(i) $D$ is 1-based
(ii) $D$ is linear
(iii) $D$ is locally modular
(iv) $D$ has no generic pseudoplane

**Proof:** Plan: (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

- (ii) $\Rightarrow$ (iii): Plan: localise at $e \in D \setminus acl(\emptyset)$ and show modularity with Lemma 2. Let $a, b \in D$ st. $a \perp b$ and $ab \perp e$, hence $RM(ab/e) = 2$. Let $e \in C = acl(C) \subseteq D$.
  I.S. $ab \perp_I C$, where $I := acl(abe) \cap C$.

  If
  - $RM(ab/C) = 0$, then $ab \in C$, hence $ab \in I$, so $RM(ab/I) = 0$. ✓
  - $RM(ab/C) = 2$, then $RM(ab/I) = 2$. ✓

  So consider the remaining case $RM(ab/C) = 1$.

  Sts: $acl(e) \subsetneq I$.

  If so, then ex. $z \in I \setminus acl(e)$ st. $RM(ez) = 2$. Then since $RM(abe) = 3$ and $ez \in acl(abe)$, we have $RM(ab/ez) = 1$. So $RM(ab/I) = 1$ since $ez \in I$ and $RM(ab/C) = 1$. ✓

  Sts: $acl(e) \subsetneq acl^{eq}(ed) \cap D \subseteq I$, where $d := Cb(ab/C) \subseteq acl^{eq}(C)$.

  - For $acl^{eq}(ed) \cap D \subseteq I$ consider
  $$RM(d/ab) = \underbrace{RM(ab/d)}_{\boxed{\substack{=RM(ab/C) \\ =1}}} + \underbrace{RM(d)}_{1} - \underbrace{RM(ab)}_{2} = 0.$$
  $\boxed{\substack{\text{since} \\ RM(ab)=2 \\ RM(ab/d)=1 \\ RM(ab/C)=1}}$

    Then $d \in acl^{eq}(ab)$, so $acl^{eq}(ed) \cap D \subseteq acl^{eq}(abe) \cap D \subseteq I$.
  - For $acl(e) \neq acl^{eq}(ed) \cap D$:
    - if $a \not\perp d$: ✓
    - if $a \perp d$: then $tp(a/d) = p|_d \overset{\boxed{\substack{\text{since } e \perp d \\ \text{since } e \perp ab \\ \& \ d \in acl^{eq}(ab)}}}{=} tp(e/d)$, so we can find an $f$ st. $ab \equiv_d ef$, hence st.
      - $f \notin acl(e)$ since $b \notin acl(a)$
      - $f \in acl(ed)$ since $b \in acl(ad)$ since $RM(ab/d) = 1$
      - $f \in D$.

$\square$

- (iii) $\Rightarrow$ (i): Let $c \notin acl(\emptyset)$ and add an infinite parameter set $C \ni c$ to the language. Then $D_C$ is modular and has wEI (by Martin's talk), hence has gEI. So $D_C$ is 1-based by Lemma 6, so $D$ was already 1-based by Lemma 5. □

- (i) $\Rightarrow$ (iv): Let $D$ be 1-based. Assume that $D$ has a generic pseudoplane defined by $r(x,y) = tp(b,c)$. (Can assume that this is over $\emptyset$ by putting parameters into language.)

  <u>Sts:</u> $c \in Cb(b/c)$

  $\longrightarrow$ Then $c \in acl^{eq}(b)$ by 1-basedness ↯ (1) of Def. 9.

  Let $r'(x) \supseteq stp(b/c) \supseteq r(x,c)$ be a global non-forking extension and let $\sigma \in Aut(D^{eq})$ fix $r'$.

  <u>Want:</u> $\sigma(c) = c$ $\left(\overset{\text{Def}}{\Longleftrightarrow} c \in Cb(b/c)\right)$.

  Now $r'(x) \supseteq r(x,c)$ gl. nf. ext. $\Longleftrightarrow$ $r'(x) \supseteq r(x,\sigma(c))$ gl. nf. ext.

  If $\sigma(c) \neq c$, then $r'(D^{eq}) \subseteq r(D^{eq},c) \cap r(D^{eq},\sigma(c))$ is finite by (3) of Def. 9, i.e. $RM(r') = 0 \underset{(2)}{<} RM(r(x,c))$, so $r'$ cannot be a non-forking extension of $r(x,c)$. ↯ □

- (iv) $\Rightarrow$ (ii): Let $D$ be non-linear, i.e. ex. $a, a' \in D$, $A \subseteq D^{eq}$ s.t. $RM(aa'/A) = 1$, $RM(Cb(aa'/A)) \geq 2$.

  Let $b := aa'$, $c := Cb(aa'/A)$.

  <u>TS:</u> $b$ and $c$ satisfy (A) – (C') from Lemma 10.

  Then $tp(b,c)$ defines a generic pseudoplane.

  - (A): $RM(b) = RM(aa') = 2$. ✓
  - (B): $RM(b/c) = RM(b/A) = 1$. ✓
  - (C): $Cb(b/c) = Cb(b/A) = c$. ✓

  - (A'): $RM(c) \geq 2$ <u>TODO</u>
  - (B'): $RM(c/b) = \underbrace{RM(b/c)}_{1} + RM(c) - \underbrace{RM(b)}_{2} = RM(c) - 1 \overset{\boxed{\text{once (A') is fixed}}}{=} 1$. ✓

  - (C'): $Cb(c/b) = b$ ? <u>TODO</u>

• Fix (A'): add suitable parameters to language in order to drop RM(b), tbme: pick Morley sequence $b b_1 \dots b_n$, $n := RM(c)-2$, in $tp(b/c)$ and add all $b_i$ to language. Successively adding $b_1 - b_n$ inductively gives us $RM(b) = 2$:

- $RM_{b_1}(c) = RM(c/b_1) = \underbrace{RM(b_1/c)}_{1} + RM(c) - \underbrace{RM(b_1)}_{2}$
  $= RM(c) - 1$,
- etc.

Preservation of (A), (B), (C) is guaranteed by Morley's $\perp$.

• Fix (C'): Replace $b$ by $b' := Cb(c/b)$.
Then $Cb(c/b') = Cb(c/Cb(c/b)) = Cb(c/b) = b'$. ✓

Preservation of (A) – (B') is guaranteed by interalgebraicity of $b$ (the original $b$) and $b'$.

Lemma-in-proof: For $RM(a) = RM(a') = 2$, $R(a'/a) = 1$, $e := Cb(a'/a)$, $a, a' \in D^{eq}$, tfae:
(i) $acl^{eq}(a) = acl^{eq}(e)$
(ii) $acl^{eq}(a) \cap acl^{eq}(a') = acl^{eq}(\emptyset)$.

Then $acl^{eq}(c) \underset{\text{Exercise}}{=} acl^{eq}(c) \xRightarrow{\boxed{\text{Lemma with } (a,a',e)=(c,b,c)}} acl^{eq}(c) \cap acl^{eq}(b) = acl^{eq}(\emptyset)$
$\xRightarrow{\boxed{\text{Lemma with } (a,a',e)=(b,c,b')}} acl^{eq}(b) = acl^{eq}(b')$.

Proof of lemma-in-proof: "⟹": Else pick $d \in acl^{eq}(a) \cap acl^{eq}(a') \setminus acl^{eq}(\emptyset)$, i.e. $RM(d) > 0$ and we can assume $RM(d) = 1$. Then
$RM(a'/d) = 1 = RM(a'/ad)$, i.e. $a' \underset{d}{\perp} a$,
so $e = Cb(a'/ad) \subseteq acl^{eq}(d)$, hence $RM(e) \leq RM(d) = 1$.
But then $acl^{eq}(a) \neq acl^{eq}(e)$ since $RM(a) = 2$.

"⟸": Else $acl^{eq}(e) \subsetneq acl^{eq}(a)$, hence $RM(e) < RM(a) = 2$. Since $RM(a'/e) = RM(a'/Cb(a'/a)) = RM(a'/a) = 1$ and $RM(a') = 2$, have $a' \not\perp e$, i.e. $RM(e/a') < RM(e) \leq 2$. So must have $RM(e/a') = 0$ and $RM(e) = 1$, i.e. $e \in acl^{eq}(a) \cap acl^{eq}(a') = acl^{eq}(\emptyset)$, but also $e \notin acl^{eq}(\emptyset)$ since $RM(e) = 1$. ↯

□
□
Th. 11

Remark 12: By a compactness argument, we can even get a "proper" pseudoplane from a generic one (in our setting).

i.e. definable by a single formula