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A NOTE ON CM-TRIVIALITY AND THE GEOMETRY OF FORKING

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§1. Introduction. CM-triviality of a stable theory is a notion introduced by Hrushovski [1]. The importance of this property is first that it holds of Hrushovski's new non 1-based strongly minimal sets, and second that it is still quite a restrictive property, and forbids the existence of definable fields or simple groups (see [2]). In [5], Frank Wagner posed some questions about CM-triviality, asking in particular whether a structure of finite rank, which is "coordinatized" by CM-trivial types of rank 1, is itself CM-trivial. (Actually Wagner worked in a slightly more general context, adapting the definitions to a certain "local" framework, in which algebraic closure is replaced by P-closure, for P some family of types. We will, however, remain in the standard context, and will just remark here that it is routine to translate our results into Wagner's framework, as well as to generalise to the superstable theory/regular type context.) In any case we answer Wagner's question positively. Also in an attempt to put forward some concrete conjectures about the possible geometries of strongly minimal sets (or stable theories) we tentatively suggest a hierarchy of geometric properties of forking, the first two levels of which correspond to 1-basedness and CM-triviality respectively. We do not know whether this is a strict hierarchy (or even whether these are the "right" notions), but we conjecture that it is, and moreover that a counterexample to Cherlin's conjecture can be found at level three in the hierarchy.

In the rest of the paper T will denote a stable theory. We work, as usual in a structure M^{eq}, where M is a big saturated model of T. We assume familiarity with the basics of stability theory (forking, canonical bases etc.), as well as the theory of generic types in stable groups, which which can be found in Chapter 1 of [3] and Chapter 5 of [4]. a, b, etc. denote possibly infinite (but of small length) tuples of elements of M^{eq} and A, B small subsets of M^{eq}, unless we say otherwise.

§2. Coordinatization and CM-triviality. In [1] several equivalent definitions were given of CM-triviality. The most suggestive for us was:

DEFINITION 2.1. T is CM-trivial if whenever $A \subseteq B$ and c satisfy $\mathrm{acl}(c, A) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$, then $\mathrm{Cb}(\mathrm{stp}(c/A)) \subseteq \mathrm{Cb}(\mathrm{stp}(c/B))$.

We pointed out in [2] that CM-triviality is invariant under naming parameters.

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We will find an equivalent definition which is quite easily seen to pass from all rank 1 types to the whole theory in the finite rank case. In fact it is convenient (bearing in mind later definitions) to define the opposite property.

DEFINITION 2.2. T is 2-ample if, possibly after naming some parameters, there are a, b, c such that

(i) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$,
(ii) $\mathrm{acl}(ab) \cap \mathrm{acl}(ac) = \mathrm{acl}(a)$,
(iii) $\mathrm{tp}(c/ab)$ does not fork over b,
(iv) c forks with a over $\emptyset$.

LEMMA 2.3. *T is 2-ample if and only if T is not CM–trivial.*

PROOF. Suppose T to be 2-ample, witnessed by a, b, c over some parameter set D which we name for now. Set $A = a$, and $B = ab$. So $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$. By (iii) $\mathrm{Cb}(\mathrm{stp}(c/B)) \subseteq \mathrm{acl}(b)$. By (iv), $\mathrm{Cb}(\mathrm{stp}(c/A))$ is not contained in $\mathrm{acl}(\emptyset)$ and so by (i) is not contained in $\mathrm{acl}(b)$. Thus $\mathrm{Cb}(\mathrm{stp}(c/A))$ is not contained in $\mathrm{acl}(\mathrm{Cb}(\mathrm{stp}(c/B)))$. Thus T is not CM trivial over D. So T is not CM-trivial.

Conversely, suppose T is not CM-trivial. So there are $A \subseteq B$, and c such that $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$ but $\mathrm{Cb}(\mathrm{stp}(c/A))$ is *not* contained in $\mathrm{Cb}(\mathrm{stp}(c/B))$. Let $b = \mathrm{Cb}(\mathrm{stp}(c/B))$. Let $a = A$ and let $D = \mathrm{acl}(a) \cap \mathrm{acl}(b)$. So, working over D, we clearly have (i)–(iv) of Definition 2.2 satisfied. ⊣

LEMMA 2.4. *Suppose a, b, c satisfy Definition 2.2 over some set of parameters which we name. Let $b' = \mathrm{Cb}(\mathrm{stp}(c/b))$ and $a' = \mathrm{Cb}(\mathrm{stp}(b'/a))$. Then a', b', c satisfy Definition 2.2 over the same set of parameters.*

PROOF. It is clear firstly that a, b', c satisfy (i)–(iv) too. In order to show the same thing for a', b', c the only nonobvious things to check are (a) $\mathrm{acl}(a'b') \cap \mathrm{acl}(a'c) = \mathrm{acl}(a')$ and (b) c forks with a' over $\emptyset$. We first prove (a): $\mathrm{acl}(a'b') \cap \mathrm{acl}(a'c')$ is contained in $\mathrm{acl}(a)$ (by hypothesis), so contained in $\mathrm{acl}(a'b') \cap \mathrm{acl}(a)$. But b' is independent from $\mathrm{acl}(a)$ over a', so $\mathrm{acl}(a'b') \cap \mathrm{acl}(a) = \mathrm{acl}(a')$.
We now prove (b): We have that b' is independent from a over a'. On the other hand c is independent from a over b', so also over $b'a'$. So $\mathrm{stp}(a/cb'a')$ does not fork over $b'a'$. But $\mathrm{stp}(a/b'a')$ does not fork over a'. Thus $\mathrm{stp}(a/ca')$ does not fork over a'. By symmetry, $\mathrm{stp}(c/a)$ does not fork over a'. But c forks with a over $\emptyset$. Thus c forks with a' over $\emptyset$. ⊣

DEFINITION 2.5. Let p be a partial type over a set D_0. We say that p is 2-ample, if there are a, b, c satisfying Definition 2.2 over some set of parameters including D_0, where moreover c is a tuple of realisations of p. We say that p is CM-trivial if it is not 2-ample.

REMARK 2.6. By the proof of 2.5 of [2], it would be equivalent to require in Definition 2.5 that c is a tuple from p^{eq}.

LEMMA 2.7. *Suppose p is a partial type over D_0 which is 2-ample. Then we can find a, b, c contained in p^{eq} and D containing D_0 contained in p^{eq} such that a, b, c satisfies Definition 2.2 over D.*

PROOF. By Lemma 2.4. we can find a, b, c in p^{eq} satisfying Definition 2.2 over some set C of parameters. Now replace C by $\mathrm{Cb}(\mathrm{stp}(a, b, c/C))$. ⊣

REMARK 2.8. Lemma 2.7 shows that, for example, the 2-ampleness (and so also CM-triviality) of a strongly minimal set p is a function of the geometry of p.

LEMMA 2.9. *Suppose that a, b, c and A_0 satisfy* $\mathrm{acl}(abA_0) \cap \mathrm{acl}(acA_0) = \mathrm{acl}(aA_0)$. *Let $A \supseteq A_0$ be such that (a, b, c) is independent from A over A_0. Then* $\mathrm{acl}(abA) \cap \mathrm{acl}(acA) = \mathrm{acl}(aA)$.

PROOF. Let $D = aA$. Then D contains aA_0 and (b, c) is independent from D over aA_0. So Fact 2.4 of [2] applies to yield the lemma. ⊣

PROPOSITION 2.10. *Suppose T has finite U-rank (namely every type has finite U-rank). Suppose that every stationary type of U-rank 1 is CM-trivial. Then T is CM-trivial.*

PROOF. Suppose by way of contradiction that T is not CM-trivial, so T is 2-ample, and let a, b, c satisfy Definition 2.2, without loss of generality over $\emptyset$. Also we may assume c to be a an element of M^{eq} (rather than an infinite tuple of such elements). Let M be a saturated model which is independent from (a, b, c) over $\emptyset$. Then, using Lemma 2.9, (i)–(iv) of Definition 2.2 hold of a, b, c over M. As $\mathrm{tp}(c/M)$ has finite U-rank (and M is saturated) there is a finite M-independent tuple d each of whose elements realises a U-rank 1 type over M, and such that c is domination equivalent to d over M. Thus $d \in \mathrm{acl}(cM)$. So (i), (ii) and (iii) of Definition 2.2 hold of a, b, d over M. As d dominates c over M, we conclude that d forks with a over M, so actually also (iv) holds of a, b, d over M. We may assume that $d = (d_1, \dots, d_n)$ where each d_i is a tuple of realisations of the U-rank 1 type $p_i \in S(M)$, and where the p_i are pairwise orthogonal. It follows that some d_i must fork with a over M. But then clearly a, b, d_i satisfy (i)–(iv) of 2.2, over M. This contradicts the assumption that p_i is CM-trivial. ⊣

§3. Higher-dimensional generalizations. Our definition of 2-ampleness suggests a hierarchy of strengthenings. We give the following tentative definition. Quite possibly the "correct" notion is somewhat stronger.

DEFINITION 3.1. Let n be a natural number greater than or equal to 1. We will say that T is n-ample if after possibly naming some set of parameters, there exist $a_0, \dots, a_n$ such that

(i) for each $i = 0, \dots, n-1$,

$$\mathrm{acl}(a_0, \dots, a_i) \cap \mathrm{acl}(a_0, \dots, a_{i-1}, a_{i+1}) = \mathrm{acl}(a_0, \dots, a_{i-1}).$$

(ii) for each $i = 0, \dots, n-1$, a_n is independent from $(a_0, \dots, a_i)$ over a_i, and

(iii) a_n forks with a_0 over $\emptyset$.

As a matter of notation, in the case $i = 0$, (i) in the Definition means that $\mathrm{acl}(a_0) \cap \mathrm{acl}(a_1) = \mathrm{acl}(\emptyset)$. Note also that (by definition) n-ampleness of T is invariant under naming parameters, and also that this definition agrees with Definition 2.2 in the case $n = 2$.

LEMMA 3.2. *Assume that* $a_0, \dots, a_n$ *satisfy Definition* 3.1 *over some set of parameters which we now name. Then for each* $0 < i < n$, a_n *forks with* a_i *over* $(a_0, \dots, a_{i-1})$.

PROOF. By induction on i. $i = 1$: If not, a_n is independent from a_1 over a_0. But by (ii) a_n is independent from a_0 over a_1. Thus $\mathrm{Cb}(\mathrm{stp}(a_n/a_0a_1)) \subseteq \mathrm{acl}(a_0) \cap \mathrm{acl}(a_1)$ and the latter equals $\mathrm{acl}(\emptyset)$ by (i). So a_n is independent from a_0 over $\emptyset$, contradicting (iii). The inductive step has the same proof with $(a_0, \dots, a_{i-1})$ named. ⊣

COROLLARY 3.3. *Suppose that* $a_0, \dots, a_n$ *witness n-ampleness of T over some set D of parameters. Then for any* $0 < m < n$, $a_m, \dots, a_n$ *witness* $(n - m)$*-ampleness of T over* $Da_0 \dots a_{m-1}$.

REMARK 3.4. T is 1-ample iff T is not 1-based.

PROOF. Remember that 1-basedness of T means that for any a and b, a is independent from b over $\mathrm{acl}(a) \cap \mathrm{acl}(b)$ and that this property is invariant under naming parameters. It is clear that 1-ampleness contradicts 1-basedness. On the other hand suppose that T is not 1-based, witnessed by a_0, a_1 such that a_1 forks with a_0 over $D = \mathrm{acl}(a_0) \cap \mathrm{acl}(a_1)$. Then a_0, a_1 witness 1-ampleness over D. ⊣

LEMMA 3.5. *Suppose that* $a_0, \dots, a_n$ *witness n-ampleness of T over D. Define by downward induction on* $i < n$, a_i', *as follows*: $a_{n-1}' = \mathrm{Cb}(\mathrm{stp}(a_n/a_{n-1}D))$, *and* $a_{i-1}' = \mathrm{Cb}(\mathrm{stp}(a_i'/a_{i-1}D))$. *Then* $a_0', \dots, a_{n-1}', a_n$ *witness n-ampleness of T over D*.

PROOF. Like the proof of Lemma 2.4. ⊣

DEFINITION 3.6. Let p be a partial type over D_0. We say that p is n-ample if there are $a_0, \dots, a_n$ satisfying Definition 3.1 over some set of parameters including D_0 such that a_n is a tuple of realisations of p.

REMARK 3.7. By Lemma 3.5, it follows, as in 2.7, that if p is n-ample then this can be witnessed by elements and base set from p^{eq}.

PROPOSITION 3.8. *Suppose T has finite U-rank,and is n-ample. Then some stationary type of U-rank* 1 *is n-ample.*

PROOF. Like the proof of Proposition 2.10, using repeatedly Lemma 2.9. ⊣

CONJECTURE 3.9. *For each* $n > 0$ *there is a strongly minimal set which is n-ample but not* $(n + 1)$*-ample.*

Hrushovski's construction gives the conjecture for the case $n = 1$. We would imagine that some higher-dimensional versions of his constructions would yield the full conjecture.

On the other hand the main result of [2] says that an infinite simple (noncommutative) group of finite Morley rank is 2-ample.

CONJECTURE 3.10. *There is an infinite simple noncommutative group of finite Morley rank which is not* 3*-ample.*

Finally we will show that if an infinite field F is type-definable in M then T is n-ample for all $n > 0$. As this is similar in spirit and details to the proof of Proposition 3.2 of [2] (where it is shown that such T is not CM-trivial), we will be brief with the proofs. Let us begin by making a couple of remarks. Firstly, as we are working in a stable structure which may impose more structure on F

than just the field structure, what we are claiming is more than simply a fact of algebraic geometry. Secondly, if we assumed that T has finite U-rank, or is even just superstable, then U-rank arguments would make the proofs easier. In our general stable context, the proofs depend on the theory of generic types in stable groups.

Let us now fix an infinite type-definable field F, defined without loss over $\emptyset$. We will work for now in affine $n+1$-space $V = F^{n+1}$. By an m-dimensional affine subspace A of V we mean an additive translate of an m-dimensional vector subspace of V. We call A generic (as an m-dimensional affine subspace of V) if A is defined by a system of equations:

$$x_{m+1} = a_{m+1,0} + a_{m+1,1}.x_1 + \dots + a_{m+1,m}.x_m$$
$$x_{m+2} = a_{m+2,0} + a_{m+2,1}.x_1 + \dots + a_{m+2,m}.x_m$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_{n+1} = a_{n+1,0} + a_{n+1,1}.x_1 + \dots + a_{n+1,m}x_m$$

where $\{a_{i,j} : i = m+1, \dots, n+1, j = 0, \dots, m\}$ is an independent (over $\emptyset$) set of generic (over $\emptyset$) elements of F.

Note that the tuple of $a_{i,j}$'s forms a canonical tuple of definition for A in the sense of M. In any case let A be as above. Then by a generic hyperplane of A we mean an $m-1$-dimensional affine subspace B of V which is contained in A and is determined (modulo being contained in A) by an equation:

$$x_m = b_0 + b_1.x_1 + \dots b_{m_1}.x_{m-1},$$

where $b_0, \dots, b_{m-1}$ are generic independent elements of F over the set of $a_{i,j}$'s.

Now let $A_0, \dots, A_n$ be defined as follows: A_0 is a generic hyperplane (i.e. n-dimensional affine subspace) of V, and A_{i+1} is a generic hyperplane of A_i. Note that A_n is simply a point of V. Identifying each A_i with its canonical parameter, we will show that the sequence $A_0, \dots, A_n$ satisfies Definition 3.1, proving that T is n-ample. The main point is:

LEMMA 3.11. *For each* $r = 0, \dots, n-1$ *we have*:

(i) A_{r+1} *is a generic* $n-r$*-dimensional affine subspace of* V, *and*
(ii) $\mathrm{acl}(A_0, \dots, A_{r-1}, A_r) \cap \mathrm{acl}(A_0, \dots, A_{r-1}, A_{r+1}) = \mathrm{acl}(A_0, \dots, A_{r-1})$.

PROOF. We give the proof only in the case $r = 0$. So we have to prove:

(i) A_1 is a generic $n-1$-dimensional affine subspace of V, and
(ii) $\mathrm{acl}(A_0) \cap \mathrm{acl}(A_1) = \mathrm{acl}(\emptyset)$.

Let A_0 be defined by:

$$x_{n+1} = a_0 + a_1.x_1 + \dots + a_n.x_n,$$

where the $a_0, \dots, a_n$ are generic independent in F over $\emptyset$. Let A_1 be the unique hyperplane of A_0 defined by:

$$x_n = b_0 + b_1.x_1 + \dots + b_{n-1}.x_{n-1},$$

where $b_0, \dots, b_{n-1}$ are generic independent over $\{a_0, \dots, a_n\}$.

We first prove (ii). Let A_0' be a conjugate of A_0 which is independent from A_0 over $\emptyset$, namely A_0' is defined by $x_{n+1} = a_0' + a_1'.x_1 + \dots + a_n'.x_n$ where $a_0', \dots, a_n'$ are generic independent over $\{a_0, \dots, a_n\}$. Then A_0 and A_0' clearly intersect in a unique $n-1$-dimensional affine subspace B of V, satisfying the equation

$$x_n = (a_0' - a_0)/(a_n - a_n') + \cdots ((a_{n-1}' - a_{n-1})/a_n - a_n')).x_{n-1}.$$

CLAIM. $(a_0' - a_0)/(a_n - a_n'), \dots, (a_{n-1}' - a_{n-1})/(a_n - a_n')$ *are generic independent over* $\{a_0, \dots, a_n\}$, *as well as over* $\{a_0', \dots, a_n'\}$.

PROOF OF CLAIM. Note that a_0' is generic over $\{a_0, \dots, a_n, a_1, \dots, a_n'\}$. So by properties of generic types we see that first $a_0' - a_0$ and then $(a_0' - a_0)/(a_n - a_n')$ is generic over $\{a_0, \dots, a_n, a_1', \dots, a_n'\}$. So $(a_0' - a_0)/(a_n - a_n'), a_1', \dots, a_{n-1}'$ are generic independent over $\{a_0, \dots, a_n, a_n'\}$. Continuing this way (replacing successively a_i' by $(a_i' - a_i)/(a_n - a_n')$) we prove the first part of the claim. The second part follows in the same way. ⊣

By uniqueness and stationarity of the generic type of F, we see from the claim that

$$\begin{aligned} &\mathrm{tp}((a_i)_{i=0,\dots,n}, ((a_j' - a_j)/(a_n - a_n'))_{j=0,\dots,n-1}) \\ &= \mathrm{tp}((a_i')_{i=0,\dots,n}, ((a_j' - a_j)/a_n - a_n'))_{j=0,\dots,n-1}) \\ &= \mathrm{tp}((a_i)_{i=0,\dots,n}, (b_j)_{j=0,\dots,n-1}). \end{aligned}$$

So clearly $\mathrm{tp}(A_0, B) = \mathrm{tp}(A_0', B) = \mathrm{tp}(A_0, A_1)$. So we may assume that $B = A_1$, and we see that $\mathrm{acl}(A_0) \cap \mathrm{acl}(A_1) \subseteq \mathrm{acl}(A_0) \cap \mathrm{acl}(A_0')$. But A_0 is independent from A_0' over $\emptyset$, so $\mathrm{acl}(A_0) \cap A_0' = \mathrm{acl}(\emptyset)$. So $\mathrm{acl}(A_0) \cap \mathrm{acl}(A_1) = \mathrm{acl}(\emptyset)$, proving (ii).

To prove (i), note that A_1 is defined in V by two equations, the first being the given $x_n = b_0 + b_1.x_1 + \dots + b_{n-1}.x_{n-1}$ and the second an equation $x_{n+1} = c_0 + c_1x_1 + \dots + c_{n-1}.x_{n-1}$, which is obtained by substituting the first into the equation defining A_0. Computing the values of c_i and doing another argument involving generic points of F, we see that $b_0, \dots, b_{n-1}, c_0, \dots, c_{n-1}$ are generic independent over $\emptyset$. This proves the $r = 0$ case of the lemma. ⊣

We finally leave the reader to prove, using similar arguments:

LEMMA 3.12. (i) A_n *forks with* A_0 *over* $\emptyset$, *and*

(ii) *for each* $r = 0, \dots, n-1$, A_n *is independent from* $A_0, \dots, A_r$ *over* A_r.

From Lemmas 3.11 and 3.12, we see that $A_0, \dots, A_n$ witness n-ampleness of T. So:

PROPOSITION 3.13. *Suppose that an infinite field is type-definable in* $\mathbb{M}$. *Then* T *is* n*-ample for all* n.

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