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A new strongly minimal set

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Abstract

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We construct a new class of $\aleph_1$ categorical structures, disproving Zilber's conjecture, and study some of their properties.

1. Introduction

A structure M is called $\aleph_1$-categorical if its first-order theory has exactly one model of power $\aleph_1$. Every $\aleph_1$-categorical structure admits a dimension theory, and in particular contains irreducible one-dimensional sets. These are called strongly minimal and can be characterized as follows:

Definition. A definable subset D of a model M is called *strongly minimal* if every M-definable subset of D is finite or cofinite, uniformly in the parameters. More precisely: If R is a definable subset of M^{n+1}, then there exists an integer d such that for every $\bar{a} \in M^n$, one of the sets

$$\{x \in D : (x, \bar{a}) \in R\} \quad \text{or} \quad \{x \in D : (x, \bar{a}) \notin R\}$$

has at most d elements.

The reader is referred to [1] for this and the ensuing discussion.

A structure is called strongly minimal if it is strongly minimal as a definable subset of itself. If D is strongly minimal, one can define a closure relation (algebraic closure) on D:

$$a \in \mathrm{acl}(B) \text{ iff } a \text{ is in some finite } B\text{-definable set.}$$

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This closure relation is a *pregeometry*, i.e., it satisfies the *Steinitz exchange property*:

if $a \in \mathrm{acl}(B, c) - \mathrm{acl}(B)$ then $c \in \mathrm{acl}(B, a)$.

It also enjoys a strong homogeneity property: if a_1, a_2 are not in $\mathrm{acl}(B)$, then there exists an automorphism of D fixing B pointwise and taking a_1 to a_2.

If (D, cl) is any pregeometry and X is a subset of D, one obtains *the localization at* X, (D_X, cl_X) by letting $D_X = D$, $\mathrm{cl}_X(A) = \mathrm{cl}(X \cup A)$. A pregeometry is called a *geometry* if $\mathrm{cl}(\{x\}) = \{x\}$ for every element x. Every pregeometry (J, cl) has an associated geometry $J^{\sim} = (J^{\sim}, \mathrm{cl}^{\sim})$: $J^{\sim} = (J - \mathrm{cl}(\emptyset))/\sim$, where $x \sim y$ iff $\mathrm{cl}(x) = \mathrm{cl}(y)$. If (J, cl) is a geometry, the *localization* of J at X is $(J_X)^{\sim}$. Two geometries J_1, J_2 are called (finitely) locally isomorphic if they have finite subsets X_1, X_2 such that $(J_{1X_1})^{\sim} \approx (J_{2X_2})^{\sim}$.

While an $\aleph_1$ categorical structure can have many strongly minimal definable sets, the local isomorphism class of their associated geometries is an invariant of the structure. Call two $\aleph_1$ categorical structures geometrically equivalent if their associated geometries are locally isomorphic. Until now there were three known types of $\aleph_1$ categorical structures, up to geometric equivalence:

(I) (Combinatorial type) A set with no structure.

$(\mathrm{II})_F$ (Linear type) A vector space $(V, +, \cdot, \alpha)_{\alpha \in F}$ over a fixed division ring F.

$(\mathrm{III})_{p,\lambda}$ An algebraically closed field $(F, +, \cdot, c)_{c \in F_0}$ of characteristic p, with elements of a subfield F_0 of transcendence degree λ distinguished.

Classes I and II admit many abstract descriptions; for example, in lattice-theoretic language, a structure is in class I iff its lattice of algebraically closed sets (in C^{eq}) is distributive, in class II iff this lattice is modular but not distributive. In the other direction, more concrete descriptions exist in each class. If M is in classes II or III, one can fully identify a 0-definable structure M_0 interpreted in M (see [4, 6]). There is then a theory of 'co-ordinization' of M over M_0, involving groups definable over M_0. See [8] for a survey. The theory is especially extensive in the degenerate and linear cases.

Zilber drew attention to this trichotomy, and conjectured (in a different language) that every $\aleph_1$-categorical structure is in fact in one of three classes ('the structural conjecture', [16]). The ideology behind the conjecture was highly influential and turned out to be powerful in surprisingly different contexts. In particular this is true in the more special totally categorical context: there the dichotomy I or II was proved in [17], and yet the shadow of III is central in the proof. The importance of the trichotomy is also becoming increasingly clear in the more general superstable context; see for instance [10]. However, we show in this paper that the conjecture itself is false; the class of possibilities of nonlinear $\aleph_1$ categorical structures is much richer than the conjecture suggests.

The construction is carried our in Sections 2 and 3. It makes it clear that further dividing lines are needed among nonlinear $\aleph_1$ categorical structures. With this in mind we isolate some of the properties of the new strongly minimal sets in

Section 4. (Further study is needed to determine which of these, if any, are decisive.) In Section 5 we mention some modifications that solve some other problems of interest in the subject: an $\aleph_1$ categorical structure without an acl($\emptyset$)-definable strongly minimal set; and examples of symmetric almost-orthogonality, an issue raised in [12, 9].

2. Finite substructures and dimension theory

Let L be a relational language containing at least one ternary relation symbol. We will build a strongly minimal structure $D(L, \mu)$ in this language; μ is a specification of data concerning multiplicities that will be explained below. For definiteness, we will assume L consists of a single ternary relation R. ($D(L, \mu)$ will be defined even if L is a binary language, but the strongly minimal set obtained will be of the degenerate type (I).) A, B, B_1, etc. will denote *finite* L-structures; M, N will denote possibly infinite ones. $A \subseteq M$ means that A is a substructure of M.

Any strongly minimal structure D gives rise to theories of dimension and multiplicity on finite L-structures. Given a finite L-structure $A = \{a_1, \ldots, a_n\}$, consider the definable set $E^*(A, D) = \{(fa_1, \ldots, fa_n)$: f an embedding of L-structures from A to $D\}$. Then the dimension (Morley rank) of $E^*(A, D)$ is an integer depending on the isomorphism type of A, but not on the enumeration. It is thus reasonable to inquire, before constructing a strongly minimal structure, what the corresponding dimension function would be. We answer this by specifying directly a very simple such function.

Similar considerations will be required for multiplicity. For example the dimension function may assign the value zero to certain finite structures. This should mean that the strongly minimal set contains only a finite number of copies of each such structure. Again we may ask what the number of copies should be. The answer will be provided in advance by a numerical function μ (but the details are slightly more complicated).

Definition. Let M be an L-structure, $A \subseteq M$ finite.

(i) $n(A)$ is the size of A. $r(A)$ is the number of triples $\bar{a}$ from A such that $M \models R(\bar{a})$. $d_0(A) = n(A) - r(A)$. $d_0(A/B) = d_0(A \cup B) - d_0(B)$.

(ii) $d(A, M) = \min(d_0(B) : B$ finite, $A \subseteq B \subseteq M\}$. $d(A/B, M) = d(A \cup B, M) - d(B, M)$.

Definition. A substructure A of M is *self-sufficient* in M if $d(A, A) = d(A, M)$. Write $A \leqslant M$ for this.

Let $\mathscr{C}_0 = \{A : \emptyset \leqslant A\}$. If $M \in \mathscr{C}_0$, $X \subseteq M$, X finite, say that c *depends on* X in M if $d(X \cup \{c\}, M) = d(X, M)$.

Definition. A is *simply algebraic over* B (in M) if A, B are finite subsets of M, $B \leqslant A \cup B$, $A \cap B = \emptyset$, $d_0(A \cup B) = d_0(B)$, and there is no proper nonempty subset A' of A such that $d_0(A' \cup B) = d_0(B)$. A is minimally simply algebraic over B if in addition, there is no proper subset B' of B such that A is simply algebraic over B'.

Now fix an integer-valued function $\mu(A, B)$, defined when $A \neq \emptyset$ and A is minimally simply algebraic over B. μ is assumed to be a function of the atomic type of (A, B). it must also satisfy the lower bound: $\mu(A, B) \geqslant d_0(B)$. Note that for any A, B with A simply algebraic over B, there is a unique smallest $B' \subseteq B$ with A simply algebraic over B'. Namely, $B' = \{b \in B$: for some $a \in A - B$, both a, b are members of some tuple in $R\}$. It is convenient to write $\mu(A, B)$ for $\mu(A, B')$ in this situation.

Remark. The requirement $\mu(A, B) \geqslant d_0(B)$ (when A is minimally simply algebraic over B) can be relaxed somewhat. However, it is shown in [7] that one cannot have, for example, $\mu(A, B) = 1$ (or even, for fixed A, $\mu(A, B)$ bounded with B.) It would be interesting to find the precise lower bound.

We now define a class of finite L-structures; it will turn out to be the class of finite substructures of $D(L, \mu)$.

Definition. Let $\mathscr{C}$ be the class of finite structures M of L, such that
(i) $\emptyset \leqslant M$.
(ii) Let B, A_i $(i = 1, \ldots, n)$ be pairwise disjoint subsets of M $(A_j \neq \emptyset)$. Suppose the atomic type of (A_i, B) is constant with i, and that A_i is minimally simply algebraic over B. Then $n \leqslant \mu(A_i, B)$.

Remark. It is possible to write (i) and (ii) together, by defining $\mu(A, B)$ even when A has 'negative dimension' over B, and letting it equal 0 in that case. However, this will hide the fact that there are two quite different mechanisms for algebraicity in the final model, one inherent in the dimension theory (so it persists even if we build a regular type of higher rank), the other obtained by force to get strong minimality.

Lemma 1. *Let $A \subseteq N$ be L-structures. Suppose $A \leqslant N$.*
(i) *$d_0(X \cap A) \leqslant d_0(X)$ whenever $X \subseteq N$.*
(ii) *$d(A', A) = d(A', N)$ whenever $A' \subseteq A$.*
(iii) *In particular, if $A' \leqslant A \leqslant N$ then $A' \leqslant N$.*

Proof. (ii) is immediate from (i) and the definition of d, and (iii) from (ii). To prove (i), let $Y = X - A$. Let r' be the size of $R^N \cap (X^3 - A^3)$. Then $d(A, N) \leqslant d_0(A \cup Y) = d_0(A) + (n(Y) - r')$, so from $d(A, N) = d_0(A)$ we get $n(Y) - r' \geqslant 0$.

Thus

$$d_0(X) = d_0(X \cap A) + (n(Y) - r') \geq d_0(X \cap A). \qquad \square$$

Definition. Let B_i be substructures of an L-structure M, such that any two distinct B_i intersect in a given set A. Let B be the substructure whose universe is the union of the sets B_i. B is called a *free join* of the B_i over A if whenever $R(c)$ holds and $c \in B^3$, then $c \in B_i^3$ for some i.

Remark. (i) The isomorphism type of a free join of B_1, B_2 over A is uniquely determined.

(ii) In general, $d_0(B_1 \cup B_2) < d_0(B_1) + d_0(B_2) - d_0(B_1 \cap B_2)$. Equality holds iff $B_1 \cup B_2$ is a free join of B_1, B_2 over $B_1 \cap B_2$.

Lemma 2. *Let $M \in \mathscr{C}_0$. Let $A \subseteq M$, and suppose B_i is simply algebraic over A, and $A \leqslant (A \cup \bigcup_i B_i)$ $(i \in I)$. Then:*

(i) *The distinct B_i's are disjoint.*

(ii) *$A \cup \bigcup_i B_i$ is a free join of the B_i over A.*

(iii) *Suppose $A \subseteq \bar{A} \subseteq M$, $\bar{A} \leqslant \bar{A} \cup B_i$, and $B_i \nsubseteq \bar{A}$ $(i = 1, 2)$. Then any isomorphism of B_1 with B_2 over A extends to an isomorphism over $\bar{A}$. In fact $\bar{A} \cup B_i$ is a free join of $\bar{A}$ and B_i over A.*

Prior to proving the lemma we make the following observation: if $A \leqslant B$ then $d_0(X/A) \geqslant d_0(X/B)$. Indeed let $A' = A \cup (X \cap B)$. Then

$$d_0(X/A) = d_0(X/A') + d_0(A'/A) \geqslant d_0(X/A') \geqslant d_0(X/B).$$

The first inequality follows from the assumption $A < B$, the second from Remark (ii) directly above.

Proof of Lemma 2. (i) We show that $B_1 \cap B_2 = \emptyset$.

$$\begin{aligned} d_0(A) &\leqslant d_0(A \cup B_1 \cup B_2) \\ &\leqslant d_0(A \cup B_1) + d_0(A \cup B_2) - d_0(A \cup (B_1 \cup B_2)) \\ &= 2d_0(A) - d_0(A \cup (B_1 \cap B_2)). \end{aligned}$$

So $d_0(A) \geqslant d_0(A \cup (B_1 \cap B_2))$. As $A \leqslant (A \cup B_1 \cup B_2)$, equality holds. But B_1 and B_2 are simply algebraic over A, so by the minimality condition in the definition of simply algebraic, $B_1 \cap B_2$ is empty, or else equal to both B_1 and B_2.

(ii) Similar argument.

(iii) We have $0 \leqslant d_0(B_i/\bar{A}) < d_0(B_i/A \cup (B_i \cap \bar{A})$. Thus as B_i is minimally algebraic over A, $B_i \cap \bar{A} = \emptyset$ or $B_i \cap \bar{A} = B_i$. The latter possibility is assumed not to hold, so $B_i \cap \bar{A} = \emptyset$. No relations can hold between B_i and $\bar{A}$ other than those holding between B_i and A, or else again we would have $d_0(B_i/\bar{A}) < 0$. $\square$

Lemma 3 (Algebraic amalgamation). *Suppose A, B_1, $B_2 \in \mathscr{C}$, A is a substructure of B_1 and B_2, and $B_1 - A$ is simply algebraic over A (in B_1). Assume for convenience that $B_1 \cap B_2 = A$, and let E be a free join of B_1 and B_2 over A. Then $E \in \mathscr{C}$, unless either*

(1) *$B_1 - A$ is minimally simply algebraic over some $F \subseteq A$, and B_2 contains $\mu(B_1 - A, F)$ disjoint sets, each realizing the atomic type of $B_1 - A$ over F; or*

(2) *there exists a set $X \subseteq B_2$ such that $X \cap A \nleqslant X$, and B_1 contains an isomorphic copy of X.*

Observe that (2) is a strong negation of: $A \leqslant B$.

Proof. (i) If $X \subseteq E$, then

$$r(X) = r(X \cap B_1) + r(X \cap B_2) - r(X \cap A),$$

$$n(X) = n(X \cap B_1) + n(X \cap B_2) - n(X \cap A),$$

so $d_0(X) = d_0(X \cap B_1) + d_0(X \cap B_2) - d_0(X \cap A)$. As $A \leqslant B_1$, $d_0(X \cap B_1) \geqslant d_0(X \cap A)$ by Lemma 1(i). Thus $d_0(X) \geqslant 0$.

(ii) Let $F, C^1, \ldots, C^r$ be pairwise disjoint subsets of E, with C^i minimally simply algebraic over F. We assume the sets C^i are implicitly enumerated, so that they all realize the same atomic type over F. We will show that if (2) fails then $r \leqslant \mu(C^1, F)$, unless $F = A$ and one of the C^i's is $B_1 - A$. Let $C^i_0 = C^i \cap A$, $C^i_\nu = C^i \cap B_\nu$ $(\nu = 1, 2)$, and define F_0, F_1, F_2 similarly. Let $k^i = \operatorname{card}(C^i)$, $k^i_\nu = \operatorname{card}(C^i_\nu)$. Write $r(X/Y)$ for $r(X \cup Y) - r(Y)$. Let $\beta^i_\nu = r(C^i_\nu/F)$ $(\nu = 0, 1, 2)$. Renumber so that $\beta^i_1 - \beta^i_0 > k^i_1 - k^i_0$ iff $i \leqslant r_0$, and for $i > r_0$, $C^i = C^i_2$ iff $i \leqslant r_1$ $(r_1 \geqslant r_0)$.

Claim 1. $r_0 \leqslant d_0(F_1/A)$.

Proof. Let $i \leqslant r_0$. SInce B_1, B_2 are freely joined over A, there are no relations between $C^i_1 - C^i_0$ and $A \cup F$ except those holding between $C^i_1 - C^i_0$ and $A \cup F_1$. Also $C^i_1 - C^i_0$ and $A \cup F$ are disjoint. Thus

$$d_0(C^i_1 - C^i_0/A \cup F_1) = d_0(C^i_1 - C^i_0/A \cup F).$$

Again by the disjointness, and the definition of d_0,

$$d_0(C^i_1 - C^i_0/A \cup F) \leqslant d_0(C^i_1 - C^i_0/C^i_0 \cup F).$$

Now

$$d_0(C^i_1 - C^i_0/C^i_0 \cup F) = d_0(C^i_1/F) - d_0(C^i_0/F)$$
$$= (k^i_1 - \beta^i_1) - (k^i_0 - \beta^i_0) < 0.$$

So $d_0(C^i_1 - C^i_0/A \cup F_1) < 0$.

Let $C^* = \bigcup_{i \leqslant r_0} C^i_1$; then $d_0(C^*/A \cup F_1) \leqslant r_0 \cdot (-1) = -r_0$. But $A \leqslant B_1$, so $A \leqslant (A \cup C^* \cup F_1)$, so $d_0(C^* \cup F_1/A) > 0$. Thus $d_0(F_1/A) \geqslant r_0$. □ Claim 1

Claim 2. *If $i > r_1$ then $C^i_2 = \emptyset$.*

Proof. Let $i > r_0$. So $\beta_1^i - \beta_0^i \leq k_1^i - k_0^i$. From the structure of E, and the fact that C^i is simply algebraic over F, it is clear that $\beta_1^i + \beta_2^i - \beta_0^i = k_1^i + k_2^i - k_0^i$. Also $F \leq (C^i \cup F)$ implies $F \leq (C_2^i \cup F)$, so $\beta_2^i \leq k_2^i$. The three relations together imply $\beta_2^i = k_2^i$. So C_2^i is simply algebraic over F; by minimality, $C_2^i = \emptyset$ or $C_2^i = C^i$. □ Claim 2

Case (1): $F \subseteq B_2$.

Then $F_1 \subseteq A$, so by Claim 1, $r_0 = 0$. By Claim 2, $C_2^i = \emptyset$ or $C_2^i = C^i$ for every i. If $C_2^i = C^i$ for every i, then all the C^i's as well as F are in B_2, so the required bound on r is a consequence of the fact that $B_2 \in \mathscr{C}$. So suppose $C_2^i = \emptyset$ for some i, i.e., $C^i \subseteq B_1 - A$. Since B_1 and B_2 are freely amalgamated over A in E, and C^i is simply algebraic over $F = F_2$, it follows that C^i is imply algebraic over $F \cap A$; but C^i is minimally simply algebraic over F, so $F \subseteq A$. But now $C^i \subseteq B_1 - A$ is simply algebraic over $F \subseteq A$ while $B_1 - A$ is simply algebraic over A, so $C^i = B_1 - A$. Hence every C^j realizes the atomic type of $B_1 - A$ over F; so if $r > \mu(C_1, F)$ then (1) holds. □ Case (1)

Assume from now on that $F \neq F_2$.

Claim 3. $r_1 - r_0 \leq d_0(F_1/F_0) - d_0(F_1/A)$.

Proof. Let $r_0 < i \leq r_1$. Then $C^i = C_2^i$; since C^i is minimally simply algebraic over F, and $F \neq F_2$, $d_0(C^i/F_2) > d_0(C^i/F) = 0$. It follows that a relation (an instance of R) holds between C^i and F on top of the relations between C^i and F_2. All instances of R in E take place between elements of B_1 or of B_2; the instance in question must thus involve elements of B_1. Thus the relation holds between some elements of C^i, and of F_1; and at least one element of $F_1 - F_0$ must be involved. We may thus say that $F_1 - F_0$ and $F_0 \cup C_0^i$ are related. Since the sets C_0^i are pairwise disjoint, and contained in A, there are $r_1 - r_0$ relations between $F_1 - F_0$ and A. Thus $d_0(F_1/F_0) - d_0(F_1/A) \geq (r_1 - r_0)$. (We could also deduce this from the general principles governing d_0, as in Claim 1.) □ Claim 3

Case (2): $r > r_1$.

So $C_2^r = \emptyset$, i.e., $C^r \subseteq B_1$; and as was shown above with $_2$ in place of $_1$, it follows that $F \subseteq B_1$ as well. If $C^i \subseteq B_1$ for all i, then everything takes place in B_1, and $B_1 \in \mathscr{C}$. Assume that $C^i \nsubseteq B_1$ for some i. If $C^i \subseteq B_2 - A$ then, as C^i is minimally simply algebraic over F and $F \subseteq B_1$, it must be that $F \subseteq A$, so $F \subseteq B_2$, contradicting our assumption. Thus $C_0^i \neq \emptyset$. By minimality again, $d_0(C_0^i/F) > 0$. So

$$d_0(C_2^i - C_0^i / C_0^i \cup F) = d_0(C_2^i/F) - d_0(C_0^i/F) < d_0(C_2^i/F).$$

But $d_0(C_2^i/F) = d_0(C_2^i/F \cup C_1^i)$ because $(F \cup C_1^i) \subseteq B_1$ while $C_2^i \subseteq B_2$. By simple algebraicity, $d_0(C_2^i/F \cup C_1^i) \leq 0$. Thus $d_0(C_2^i - C_0^i/C_0^i \cup F) < 0$.

Let $X = F_0 \cup C_2^i$. Then

$$\begin{aligned} d(X/X \cap A) &= d_0(C_2^i/C_0^i \cup F_0) = d_0(C_2^i \cup F_0) - d_0(C_0^i \cup F_0) \\ &= d_0(C_2^i \cup F) - d_0(C_0^i \cup F) < 0. \end{aligned}$$

The atomic type of X is realized in B_1, inside $F \cup C^r$. Thus (2) holds. □ Case (2)

If neither of the special cases hold, then

$$r = r_1 = (r_1 - r_0) + r_0 \leqslant (d_0(F_1/F_0) - d_0(F_1/A)) + d_0(F_1/A)$$
$$= d_0(F_1/F_0) \leqslant d_0(F).$$

By the choise of μ, $d_0(F) \leqslant \mu(C^1, F)$. This proves the lemma. □

Lemma 4 (Self-sufficient amalgamation). *Suppose $A, A_1, A_2 \in \mathscr{C}$, $A \leqslant A_1$, $A \leqslant A_2$. Then there exists $E \in \mathscr{C}$ and embeddings f_1, f_2 of A_1, A_2 into E, so that $f_1 \mid A = f_2 \mid A$, and $f_i A_i \leqslant E$.*

Proof. We use induction on $|A_1 - A| + |A_2 - A|$. Let

$$d(A, A_1) = d(A, A_2) = d(A, A) = d.$$

Case 1: *there exists X with $A \subset X \subset A_1$ such that $d_0(X) = d_0(A) = d$.*

In this case $A \leqslant X \leqslant A_1$. By induction, X can be amalgamated with A_2 over A, and then A_1 can be amalgamated with XA_2 over X. That gives E with $A_1 \leqslant E$ and $A_2 \leqslant XA_2 \leqslant E$.

Case 2: *$d(A_1, A_1) > d$, and case* (1) *fails.*

Choose any $b \in A_1$. Then there is no X with $A \cup \{b\} \subseteq X \subseteq A_1$ and $d_0(X) = d$. Thus $d(A \cup \{b\}, A_1) = d + 1$. It is extremely easy to check in this case that the free amalgamation of $A \cup \{b\}$ with A_2 is in $\mathscr{C}$, and satisfies the requirements. Since $d(A \cup \{b\}, A \cup \{b\}) \leqslant d(A, A) + 1 = d + 1$, it follows that $A \cup \{b\} \leqslant A_1$. By induction, we can continue and amalgamate A_1 with A_2 over $A \cup \{b\}$.

Case 3: *Case* 1 *fails, and $d(A_1, A_1) = d$.*

In this case $A_1 - A$ is simply algebraic over A; say it is minimally simply algebraic over $F \subseteq A$. Case (2) of Lemma 2 cannot apply, as $A \leqslant A_2$. If case (1) applies, the atomic type of $A_1 - A/F$ is realized $\mu(A_1 - A, F)$ times in A_2. As $A \leqslant A_2$, each such realization is either contained in A, or else freely joined with A over F (Lemma 2(iii)). If the latter case occurs at all, then the realization in question realizes the atomic type of $A_1 - A$ over F; so the amalgamation may be achieved by identifying $A_1 - A$ with this part of A_2. In the other case A contains $\mu(A_1 - A, F)$ disjoint realizations of the atomic type of $A_1 - A$ over F, so A_1 contains $\mu(A_1 - A, F) + 1$ such, contradicting the fact that $A_1 \in \mathscr{C}$.

Finally, if neither (1) nor (2) apply, then Lemma 3 says that A_1, A_2 can be freely joined in $\mathscr{C}$ over A. It remains only to check that the amalgam E satisfies $A_1 \leqslant E$, $A_2 \leqslant E$. This is easy, and uses only $A \leqslant A_1$, $A \leqslant A_2$. □

3. The strongly minimal set

Consider the following description of a model M:

(1) M is a countable L-structure.

(2) Every finite substructure of M is in $\mathscr{C}$.

(3) Let $B \leqslant M$ and $B \leqslant C$, $C \in \mathscr{C}$. Then there exists an embedding $f: C \to M$ such that $fC \leqslant M$, and $F \mid B = \mathrm{Id}_B$.

Note that in any L-structure satisfying (2), for every finite $A \subseteq M$, there exists a finite B, $A \subseteq B \subseteq M$, $d(B, M) = d(A, M)$, such that B is self-sufficient. (Choose any B with $d(A, M) = d_0(B)$ and $A \subseteq B$). By a standard back-and-forth argument (cf. [3]), there exists a structure M satisfying (1)–(3); and it is unique up to isomorphism. Fix such an M for a moment.

Let $d(A) = d(A, M)$. Note that there exist $A \subseteq M$ with $d(A)$ arbitrarily large.

Lemma 5. (i) $d(C) \leqslant d(aC) \leqslant d(C) + 1$.

(ii) *If* $d(aCb) = d(Cb) = d(C)$ *then* $d(aC) = d(C)$.

(iii) *If* $d(aCb) = d(Cb)$, *and* $d(aC) > d(C)$, *then* $d(aCb) = d(Ca)$.

(iv) *If* $d(Ca) = d(C) = d(Cb)$, *then* $d(C) = d(aCb)$.

(v) *If* $d(aC) = d(C)$, *then* $d(aCb) = d(Cb)$.

Proof. (i), (ii) are evident.

(iii) $d(aCb) = d(Cb) \leqslant d(C) + 1 \leqslant d(aC)$.

(iv) Let E_1, E_2 be such that $Ca \subseteq E_1$, $Cb \subseteq E_2$, $d_0(E_1) = d(C) = d_0(E_2)$, and each is minimal. If $a \in E_2$ or $b \in E_1$ there is no problem. Let $E = E_1 \cap E_2$. E_1 adds at least as many relations as elements to E, otherwise E would show $d(C) < d_0(E_1)$. So $d_0(E_1 \cup E_2) \leqslant d_0(E_2) = d(C)$.

$$d(C) < d(Cb) < d(aCb) \leqslant d(aC) + 1 = d(C) + 1.$$

So either $d(C) = d(Cb)$ or $d(Cb) = d(aCb)$; in the former case (iv) applies. □

This shows that the relation '$d(aC) = d(C)$' is a dependence relation. Further note that if $d(aC) > d(C)$, $E \supset C$, $d_0(E) = d(C)$, then $d(E) = d(C)$, so E is self-sufficient, and also aE is self-sufficient; and no relations hold between a and E; and so $\mathrm{tp}(aC)$ is fully determined by this, i.e., by $\mathrm{tp}(C)$ and the fact that a does not depend on C.

However, (3) is not a 1st-order property, so we replace it by:

(3′) M contains an infinite set I such that $d(A) = \mathrm{card}(A)$ for finite $A \subseteq I$.

(3″) Suppose $B \subseteq M$, $B \leqslant C$, $C \in \mathscr{C}$, and $C - B$ is simply algebraic over B. Supoose also that whenever $X \subseteq M$ realizes an atomic type realized in C, $(X \cap B) \leqslant X$. Then there are $\mu(C - B, B)$ distinct solutions C' in M of the atomic type of C over B.

Claim. (1, 2, 3) *and* (1, 2, 3′, 3″) *are equivalent.*

Proof. Assume $(1,2,3)$. Then $(3'')$ is clear from Lemma 3, and $(3')$ is trivial. Conversely, assume $(1,2,3',3'')$; then (3) follows just as in the proof of Lemma 4. (We may assume there is no C' with $B < C' \leqslant C$ except $C' = B$ or $C' = C$. This implies that C is simply algebraic over B, or else $C = B \cup \{c\}$ and c does not depend on B. In the second case, let $I' \subseteq I$ have more elements that card(B); then there is some $c' \in I$ such that c' does not depend on B; $c \mapsto c'$ is the required embedding. In the algebraic case, say $C - B$ is minimally simply algebraic over $B' \subseteq B$. By $(3'')$, there are $r = \mu(C - B, B')$ solutions $C_1, \ldots, C_r$ of $\text{tp}_{\text{atomic}}(C - B/B')$. They are disjoint. Also since $B \leqslant M$, $B \cap (B' \cup C_i) \leqslant (B' \cup C_i)$, so if $C_i \cap B \neq \emptyset$ then $C_i \subseteq B$. If every C_i is contained in B, then in C there are $r + 1$ solutions of the atomic type (including $C - B$), contradicting the fact that $C \in \mathscr{C}$. If one C_i is missing from B, then it is disjoint from B, and moreover it has no relations with B except the ones it has with B' (as $B \leqslant M$.) Thus $C \mapsto C_i$ gives the sought-for embedding.) □

Lemma 6. *Let M_1, M_2 satisfy $(1,2,3',3'')$. Let $f: A_1 \to A_2$ be an L-embedding, where $A_\nu \leqslant M_\nu$, A_i finite. Then f extends to an isomorphism of M_1 with M_2.*

Proof. $(1,2,3)$ allow back-and-forth between self-sufficient substructures. □

Corollary 7. *M is saturated.*

Proof. M is isomorphic to every countable elementary extension of itself. It follows that there are only countably many types realized in elementary extensions of M (in each sort). Hence there exists a saturated elementary extension of M, to which M must be isomorphic. □

Corollary 8. *M is strongly minimal.*

Proof. Let $A \subseteq M$ be a finite set, and let x be any element. We already know that there is a unique orbit of elements that do not depend on A. Since M is saturated, it suffices to show that every other orbit is finite.

1. There exists a self-sufficient $A' \supseteq A$, algebraic over A. Choose A' minimal so that $A' \supseteq A$ and $d_0(A') = d(A)$. Let A'' by any conjugage of A' over A. I claim that $A'' = A'$. If not, call the intersection A'''. Then $d_0(A''') > d(A)$, so $d_0(A'/A''') < 0$; hence $d_0(A'/A'') < 0$; so $d_0(A' \cup A'') < d_0(A'') = d(A)$. This contradicts the definition of $d(A) = d(A, M)$.

2. If x depends on A then x is algebraic over A. $d(\{x\} \cup A) = d(A)$. Find $B \supseteq A \cup \{x\}$ with $d_0(B) = d(\{x\} \cup A) = d(A)$. Clearly A' (can be chosen to be) $\subseteq B$. Let $B_0 = A'$, and choose $B_1, \ldots, B_n$ so that $B_{i+1} - B_i$ is nonempty and simply algebraic over B_i, and $B_n = B$. Clearly B_{i+1}/B_i is algebraic for each i; since A'/A is algebraic and $x \in B_n$ we are done. □

Proposition 9. *M has quantifier elimination to the level of Boolean combinations of formulas of the form* $(\exists \bar{x})(R(\bar{x}, \bar{y}))$, *where R is a conjunction of atomic formulas and inequalities.*

The proof is left to the reader. We note that such formulas are not necessarily equations in the sense of Srour. We will show in Section 4.1 that the definable subsets of M are also Boolean combinations of equational sets, but we have not determined these explicitly.

4. Geometrical properties

4.1. Strong equationality and CM-triviality

We define a property of stable theories stating in some sense that the fundamental order is algebraically trivial. We show that the new structures have this property. We also define 'strong equationality' and show that it follows. In [11] it is shown that strong equationality implies equationality. We refer the reader to [14], [15] for the notions of 1-basedness, equationality, pseudo-plane; however these will only serve as a background, and aside from one or two isolated remarks, the section is self-contained.

To further motivate the definition of CM-triviality, recall that a theory is 1-based if it does not interpret a pseudo-plane; this is a structure modeled on the incidence between points and lines in a plane, over an infinite field. Intuitively, CM-triviality forbids the existence of a richer structure, modeled on the incidence relations between points, lines, and planes in 3-space. The critical properties: the intersection of two distinct lines is finite; the intersection of two distinct planes is contained in the union of finitely many lines; given a point and a plane, there are infinitely many lines passing through the point, and contained in the plane.

Proposition 10. *For a stable structure M, the following conditions are equivalent:*

(CMT1) *Suppose* B_1, B_2 *are independent over* $E = \mathrm{acl}(E)$; *and* $\mathrm{acl}(B_1, B_2) \cap \mathrm{acl}(E, B_i) = B_i$, *and* $B_i \cap E = A$. *Then* B_1, B_2 *are independent over A.*

(CMT2) *If E is algebraically closed,* $C_1 \perp\!\!\!\perp C_2 \mid E$, *then* $C_1 \perp\!\!\!\perp C_2 \mid (\mathrm{acl}(C_1, C_2) \cap E)$.

(CMT3) *Let C, A, B be algebraically closed. Assume* $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$. *Then* $\mathrm{Cb}(C/A \cup B) \supseteq \mathrm{Cb}(C/A)$.

Proof. (1)$\to$(2) Let C_1, C_2, E be as in (CMT2). Let $B_i = \mathrm{acl}(C_1, C_2) \cap \mathrm{acl}(E, C_i)$. Then $C_i \subseteq B_i \subseteq \mathrm{acl}(C_1, C_2)$, so $\mathrm{acl}(B_1, B_2) = \mathrm{acl}(C_1, C_2)$. Similarly $C_i \subseteq B_i \subseteq \mathrm{acl}(E, C_i)$ implies that $\mathrm{acl}(E, B_i) = \mathrm{acl}(E, C_i)$. Thus $\mathrm{acl}(B_1, B_2) \cap \mathrm{acl}(E, B_i) = B_i$. Also $B_1 \cap E = \mathrm{acl}(C_1, C_2) \cap E = B_2 \cap E \overset{\mathrm{def}}{=} A$. So (CMT1) applies, and $B_1 \perp\!\!\!\perp B_2 \mid A$.

(2)→(3) Let C, A, B be as in (CMT3); we may assume $A \subseteq B$. Let $Y = \mathrm{Cb}(C/B)$. So $C \perp B \mid Y$, and in particular $C \perp A \mid Y$. By (CMT2), $C \perp A \mid (Y \cap \mathrm{acl}(C \cup A))$. But $B \cap \mathrm{acl}(C \cup A) = A$ by assumption, so $Y \cap \mathrm{acl}(C \cap A) \subseteq Y \cap A$. Thus $C \perp A \mid Y \cap A$, so $\mathrm{Cb}(C/A) \subseteq Y$.

(3)→(1) Let B_1, B_2, E, A be as in (CMT1). Let $A' = B_2$, $B' = \mathrm{acl}(B_2 \cup E)$, $C' = B_1$. Then $\mathrm{acl}(A' \cup C') \cap \mathrm{acl}(A' \cup B') = A'$. Hence by (CMT3), $\mathrm{Cb}(C'/A' \cup B') \supseteq \mathrm{Cb}(C'/A')$, i.e., $\mathrm{Cb}(B_1/B_2 \cup E) \supseteq \mathrm{Cb}(B_1/B_2)$. But $\mathrm{Cb}(B_1/B_2 \cup E) \subseteq E$. Thus $\mathrm{Cb}(B_1/B_2) \subseteq E$. Since also $\mathrm{Cb}(B_1/B_2) \subseteq B_2$ and $B_2 \cap E = A$, we get $\mathrm{Cb}(B_1/B_2) \subseteq A$, so $B_1 \perp B_2 \mid A$. □

Definition. If T is a stable theory all of whose models satisfy (CMT1), we say that T is CM-trivial. A structure is CM-trivial if its theory is.

Definition. Let M be a saturated structure of finite Morley rank. M is called *strongly equational* if whenever E is algebraically closed in M^{eq}, $C_1 \perp C_2 \mid E$, and $\alpha_i : C_i \to M$ is elementary over E, then $\mathrm{rk}(C_1C_2) \geq \mathrm{rk}(\alpha_1 C_1 \alpha_2 C_2)$; and moreover, if equality holds, then $\alpha_1 \cup \alpha_2$ is elementary.

Proposition 11. *A CM-trivial structure of the finite Morley rank is strongly equational.*

Proof. Let C_1, C_2, E, α_i be as in the definition of strong equationality; extend α_i to an elementary map (fixing E) defined on $\mathrm{acl}(E \cup C_i)$, and let $\alpha = \alpha_1 \cup \alpha_2$. Let $A = \mathrm{acl}(C_1, C_2) \cap E$. Then by (CMT2), C_1, C_2 are independent over A. Since α_i fixes A, we have:

$$\begin{aligned}\mathrm{rk}(C_1, C_2) &= \mathrm{rk}(C_1C_2/A) + \mathrm{rk}(A) = \mathrm{rk}(C_1/A) + \mathrm{rk}(C_2/A) + \mathrm{rk}(A)\\ &= \mathrm{rk}(\alpha_1C_1/A) + \mathrm{rk}(\alpha_2C_2/A) + \mathrm{rk}(A) \geq \mathrm{rk}(\alpha C_1 \alpha C_2/A) + \mathrm{rk}(A)\\ &= \mathrm{rk}(\alpha C_1 \alpha C_2).\end{aligned}$$

If equality holds, then αC_1, αC_2 must be independent over A; since A is algebraically closed in M^{eq}, α is elementary on AC_1C_2 (Shelah's finite equivalence relation theorem). □

Remark 12. A 1-based structure satisfies (CMT1). For 1-basedness implies that $B_i \perp E \mid B_j$ ($\{i, j\} = 1, 2$), so $\mathrm{Cb}(E/B_1B_2) \subseteq C_1 \cap B_2$ and thus $E \perp (B_1B_2) \mid (B_1 \cap B_2)$; but $E \perp B_2 \mid A$; so $E \perp (B_1B_2) \mid A$, and thus E, B_1, B_2 are independent over A.

Lemma 13. *$D(L, \mu)$ is CM-trivial.*

Proof. We will show (CMT1) holds. Let $\bar{B}_1 = \mathrm{acl}(B_1 \cup E)$. Then $\bar{B} = \bar{B}_1 \cup \bar{B}_2 \leq M$. By assumption, $\mathrm{acl}(B_1 \cup B_2) \cap \bar{B}_i = \bar{B}_i$, so $\mathrm{acl}(B_1 \cup B_2) \cap (\bar{B}_1 \cup \bar{B}_2) =$

$(\bar{B}_1 \cup \bar{B}_2)$, and hence $(B_1 \cup B_2) \leq \bar{B}_1 \cup \bar{B}_2$. So $B_1 \cup B_2 \leq M$. Clearly $B_1 \cap B_2 = A$, and (as $\bar{B}_1$, $\bar{B}_2$ are freely joined over E) there are no explicit relations between $B_1 - A = B_1 - E$ and $B_2 - A = B_2 - E$. This shows $B_1 \perp\!\!\!\perp B_2 \mid A$. □

Putting together Lemmas 10, 11 and 13, we see that each of the strongly minimal sets $D(L, \mu)$ is strongly equational. It would be worthwhile to show (or contradict) the existence of a structure of finite Morley rank, not CM-trivial, and not interpreting an infinite field.

4.2. Flatness

Since CM-triviality follows from 1-basedness, it is consistent with the existence of an infinite definable group. We consider a second geometrical property, that forbits any 'relations among the relations of the strongly minimal set' (such as an associative law.) When E_i $(i \in I)$ are sets and s is a nonempty subset of I, let E_s denote $\bigcap_{i \in s} E_i$; and let $E_\emptyset = \bigcup_i E_i$.

A *combinatorial geometry* is just a pregeometry (as defined in the introduction), in which no singleton point depends on any other.

Definition. A combinatorial geometry J is *flat* if whenever E_i $(i \in I)$ are a finite number of finite-dimensional closed subsets of J, s ranges over the subsets of I, then $\sum_s (-1)^{\mathrm{card}(s)} d(E_s) \leq 0$.

Lemma 14. *Let D be a saturated strongly minimal set whose geometry is flat. Then D does not interpret an infinite group.*

Proof. Suppose G is a group interpreted in D, of dimension g. Let a_1, a_2, a_3 be generic elements of G. For $i = 1, 2, 3$ let $E_i = \mathrm{cl}\{a_j : i \neq j\})$, and let $E_4 = \mathrm{cl}(a_1^{-1}a_2, a_1^{-1}a_3)$. Then $d(E_\emptyset) = 3g$, $d(E_i) = 2g$, $d(E_{ij}) = g$, and the intersection of any three of the E_i's equals $\mathrm{cl}(\emptyset)$. Thus the formula predicts: $3g - 4(2g) + 6(g) \leq 0$, so $g = 0$. □

Lemma 15. *$D(L, \mu)$ is flat.*

Proof. We may view E_i as a closed, finite-dimensional subset of $D = D(L, \mu)$. For $s \subseteq I$ let F_s be a finite subset of D such that $\mathrm{cl}(F_s) = E_s$. Let G_i be a finite self-sufficient subset of E_i containing $\bigcup\{F_s : i \in s \subseteq I\}$. Then $E_s = \mathrm{cl}(G_s)$, and $d(E_s) = d(G_s) = d_0(G_s)$. Thus

$$\sum_s (-1)^{|s|} d(E_s) = \sum_s (-1)^{|s|} d_0(G_s) = \sum_s (-1)^{|s|} n(G_s) - \sum_s (-1)^{|s|} r(G_s).$$

Now the first summand equals 0 (as can be seen for example by expanding the product $\prod_i (1 - 1_{G_i}) = 0$ in the ring Q^D). For the same reason, if r is the cardinality of $\bigcup\{R(G_i): i\}$, then $r = \sum_{s \neq \emptyset} (-1)^{|s|} r(G_s)$; so the second summand equals $r - r(\bigcup_i G_i)$, and the difference $r(\bigcup_i G_i)$-r is non-positive. □

Remark. Cm-triviality follows from flatness. Let B_1, B_2, E be as in (CMT1); then any two of these three sets intersect in A. Let $b_i = \text{rk}(B_i \cup E)$, $a = \text{rk}(A)$, $e = \text{rk}(E)$. Then $\text{rk}(B_1 \cup B_2 \cup E) = b_1 + b_2 - e$. Applying the definition of flatness to the sets $\text{cl}(B_1 \cup B_2)$, $\text{cl}(B_1 \cup E)$, $\text{cl}(B_2 \cup E)$, we get:

$$(b_1 + b_2 - e) - (\text{rk}(B_1 \cup B_2) + b_1 + b_2) + (\text{rk}(B_1) + \text{rk}(B_2) + e) - \text{rk}(A) < 0.$$

In other words, $\text{rk}(B_1 \cup B_2) \geqslant \text{rk}(B_1) + \text{rk}(B_2) - \text{rk}(A)$, as required.

Flatness can be defined, by the same formula, for structures of finite Morley rank. It is natural then to use the formula in M^{eq} rather than M. We note however, for later use, that if M itself is flat, then it essentially (for geometric purposes) has elimination of imaginaries.

Proposition 16. *Let M be of finite Morley rank, and flat. For any $e \in M^{\text{eq}}$ there exists a finite $A \subseteq M$ with* $\text{acl}(e) = \text{acl}(A)$.

Proof. Let $E = \text{acl}(e) \cap M$. Choose $A_i = \text{acl}(A_i) \cap M$ of finite rank, A_i containing E, with $e \in \text{dcl}(A_i)$. Let A_i $(i = 1, 2, \ldots, n)$ by a Morley sequence over $\text{acl}(e)$. Let $I^* = \{(i, j)\colon 1 \leqslant i < j \leqslant n\}$, and for $(i, j) \in I^*$ let $A_{ij} = \text{acl}(A_i A_j) \cap M$. Note that $A_{ij} \cap A_{ik} = A_i$, since $A_j \perp\!\!\!\perp A_k \mid A_i$, and for distinct i, j, k, l, $A_{ij} \cap A_{ik} = E$. For convenience we work over E. After some computation flatness gives:

$$\text{rk}(A_1 \cup \cdots \cup A_n) - \sum_{i<j} \text{rk}(A_{ij}) + \sum_i \text{rk}(A_i) \cdot n \leqslant 0.$$

(Indeed for each subset s of I^*, $|s| \geqslant 2$, $\bigcap_{\nu \in s} A_\nu$ is either some A_i or E; and each A_i occurs $\binom{n-1}{2} - \binom{n-1}{3} + \binom{n-1}{4} - \cdots = n$ times as such an intersection.) Letting $d = \text{rk}(A_i)$, $d^* = \text{rk}(A_i/e)$ we get:

$$\text{rk}(e) + nd^* - (n(n-1)/2)(2d^* - \text{rk}(e)) + n^2 d \leqslant 0.$$

Consider the last expression as a polynomial in n; the quadratic coefficient is $(d + \frac{1}{2}\text{rk}(e) - d^*)$. Since the polynomial is negative for large n, we have $d + \frac{1}{2}\text{rk}(e) \leqslant d^*$. But clearly $d^* \leqslant d$. This forces $d^* = d$ and $\text{rk}(e) = 0$. □

5. Variations

5.1. Strictly minimal sets

Recall that a strongly minimal set is called *strictly minimal* if any two elements are independent generics.

Proposition 17. *Let k be a fixed integer. There exists a strongly minimal set D with the property that any two k-tuples of distinct elements have the same type, i.e., that any set of size k has rank k, while not every $k+1$-set has rank $k+1$.*

Proof. We choose a language with one $(k+1)$-ary relation. We choose μ and define $\mathscr{C}$ as in Section 2. We then let $\mathscr{C}' = \mathscr{C} \cap c_0'$, where

$$c_0' = \{A \in \mathscr{C}_0 \colon \text{for any } B \subseteq A,\ d_0(B) \geq \min(\operatorname{card}(B), k)\}.$$

It is easy to verify that if E is the free amalgam of B_1, B_2 over A, where $A \leq B_1$ and $A \leq B_2$ and each B_i is in $\mathscr{C}_0'$, then E is in $\mathscr{C}_0'$. Moreover, the condition $A \leq B_i$ may be replaced by the first-order approximation:

$$(A \cap X) \leq X \quad \text{for every } X \leq B_i \text{ with } \operatorname{card}(X) \leq k.$$

Thus the amalgamation lemmas for $\mathscr{C}$ hold also for $\mathscr{C}'$, and there exists M' defined by conditions (1)–(3) of Section 3, with respect to the class $\mathscr{C}'$. Clearly M' is strongly minimal, with dimension function d, and we have:

$$d(A) = \operatorname{card}(A) \quad \text{whenever } |A| \leq k.$$

5.2. *The number of geometries*

Proposition 18. *There is a continuum of strongly minimal sets with pairwise non-isomorphic geometries.*

One constructs strongly minimal sets in a language with a single ternary relation R, and with the property that $R(x_1x_2x_3)$ holds if, and only if, $x_1x_2x_3$ are dependent. Thus the model-theoretic structure can be read off from the geometry. By modifying μ one easily gets a continuum of distinct theories, hence distinct geometries.

To achieve the requirement, we consider only finite structures A on which R is symmetric (under permutations of the three variables) and applies to triples of *distinct* points only. We can thus consider R as a set of unordered triples. Define the dimension function $d_0 = n - r$, where r is the number of unordered triples of which R holds. We let

$$c_0 = \{A \colon \text{for any } B \subseteq A \text{ with } |B| \leq 3,\ B \leq A\}$$

We then proceed as in Sections 2 and 3. This yields a strongly minimal set with the desired property.

It is less clear, however, whether the geometries we obtain are locally isomorphic, or even whether there is more than one local isomorphism type of nontrivial geometries of flat strongly minimal sets.

5.3. *Symmetric almost-orthogonality*

Let D_1, D_2 be two strongly minimal sets, defined without parameters in a single structure. Let $p_i^{(n)}$ denote the type of an independent generic n-tuple of elements of D_i. We write $p_1^{(n)} \perp^a p_2^{(i)}$ if for any $\bar{a} \models p_1^{(n)}$ and $\bar{b} \models p_2^{(i)}$, $\bar{a}$, $\bar{b}$ are independent.

Until now the only known examples of almost-orthogonality (even of regular types) were asymmetric: either $p_1 \perp^a p_2^{(n)}$ for all n, or $p_2 \perp^a p_1^{(n)}$ for all n. In the locally modular context, this is equivalent to the 'uniqueness of parallel lines', and is the geometric fact underlying the existence of Zilber envelopes [2]. See also [12]. It is now easy to construct an example of symmetric non-orthogonality:

$$p_1 \not\perp^a p_2^{(2)}, \qquad p_2 \not\perp^a p_1^{(2)}, \qquad p_1 \perp^a p_2.$$

We simply take a language with a single ternary predicate R, and two unary predicates D_1, D_2; we consider only finite L-structures in which D_1, D_2 partition the universe, and we use the dimension function d_0 of Section 2 (ignoring the unary predicates.) The construction will now yield a structure of Morley rank 1, degree 2, consisting of a pair of strongly minimal sets. These strongly minimal sets are non-orthogonal: there exist a_1, a_2, b such that $R(a_1, a_2, b)$ (and hence $b \in \mathrm{acl}(a_1, a_2)$) while a_1, a_2 are independent elements of D_1. Similarly $p_1 \not\perp^a p_2^{(2)}$. This remains true if one imposes the constraint of 5.1, so that any pair of elements is independent. This implies in particular that $p_1 \perp^a p_2$.

In the above example, D_1 and D_2 are strictly minimal. This is not accidental; see [9, Corollary 7].

We can also construct a flat example where $p^2 \not\perp^a q$, $p \perp^a q^2$. This cannot be improved to:

$$p^2 \not\perp^a q, \qquad p \perp^a q^3$$

since in [9] it was shown that this can only occur in the presence of a definable group.

Certain patterns of non-orthogonality were ruled out in [9]; for instance $p^+ \not\perp^a q$, $p^4 \perp^a q^3$ is impossible. At the time the set of possibilities ruled out seemed arbitrary; it now seems likely that it was in fact best possible, and that all other patterns can in fact be constructed using the present technique. It would be good to determine whether this is so.

5.4. An $\aleph_1$-categorical structure with no acl(∅)*-definable strongly minimal set*[1]

The fact that a saturated $\aleph_1$-categorical structure contains a definable strongly minimal set was contained in Morley's [13]. It was shown in [1] that the parameter for such a set can be found in the prime model, and the dependence on the parameter was analyzed. For structures satisfying Zilber's conjecture, one can find a strongly minimal set with algebraic parameters, removing the need for this analysis (the strongly minimal set and the parameters may be in imaginary sorts). We construct an $\aleph_1$-categorical structure with no such strongly minimal set.

Evidently, we need to build a structure of finite Morley rank higher than one. The method of Section 2 can easily be adapted for this purpose. Let L be a many-sorted relational language; let $d^*(S)$ be a nonnegative integer for each sort S of L, and let $d^*(R)$ be a positive integer for each relation R of L. ($d^*(S)$ will be

[1] Similar constructions were independently obtained by Baldwin, following an early version of this paper.

the dimension of the sort S, and $d^*(R)$ the codimension of R.) Given a finite L-structure A, define

$$d_0(A) = \sum_S d^*(S)\,|S^A| - \sum_R d^*(R)\,|R^A|$$

where S ranges over the sorts of L, R over the relations. The rest of the definitions remain unchanged; a function μ is defined satisfying the same constraints, 'simply algebraic', '$\leqslant$', and the class $\mathscr{C}(L, \mu)$ are defined as before; and the structure M is again defined by the properties (1)–(3) of Section 3. One needs to show that M exists and is saturated of finite Morley rank. Principally, one needs to prove the amalgamation lemmas. The algebraic amalgamation lemma can be deduced from the strongly minimal case, by thinking of M, roughly speaking, as a reduct of D^n for some strongly minimal D. For example, if L has a single sort of weight n, and a single r-ary relation of weight k, let L' be the language with k relations of arity $n \cdot r$. If A is an L-structure, let A' be the L'-structure with universe $A \times n$, and let

$$R_j^{A'} = \{((a_1, 0), \ldots, (a_1, n-1), \ldots, (a_r, 0), \ldots, (a_r, n-1)) : (a_1, \ldots, a_r) \in R\}.$$

Then, if μ' extends μ in the obvious sense, then the amalgamation lemma for $\mathscr{C}(L', \mu')$ pulls back to the same for $\mathscr{C}(L, \mu)$. The other details are left to the reader.

We note that the Morley rank of M will not be n but $n/\gcd(n, k)$.

Proposition 19. *There exists an almost strongly minimal structure of Morley rank* 2, *without an* $\mathrm{acl}(\emptyset)$*-definable strongly minimal set; equivalently, with no* 0*-definable sets of Morley rank* 1.

This includes, of course, imaginary sorts.

Proof. The structure will be a graph of rank 2. Let R be a binary relation, $L_2 = \{R\}$. We consider only L_2-structures on which R is irreflexive and symmetric. We let $d_0(A) = 2n(A) - r(A)$, where $n(A)$ is the size of A, and $r(A)$ is the number of edges in the graph determined by R. We further restrict the class of structures $\mathscr{C}_0$ by demanding, as in 5.3, that $\{a\} \leqslant B$ for any $B \in \mathscr{C}_0$ and any $a \in B$. We choose any appropriate function μ, and consider the resulting class $\mathscr{C}$ and the amalgam M. We need to show that for any (imaginary) e, $\mathrm{rk}(e/\emptyset) \neq 1$. This is evident for M itself. The question is therefore one of elmination of imaginaries; it is settled by Lemma 16. □

5.5. The LM–SR dichotomy requires NOTOP

In [10] it was shown that in a superstable structure with NOTOP, every regular type is non-orthogonal either to a strongly regular type, or to a locally modular one. We show here that NOTOP is needed in the hypotheses: we construct a superstable structure with a type p orthogonal to every strongly regular type and to every locally modular regular type.

Let L consist of a single ternary predicate R, and let $\mathscr{C}_0$ be the class defined in Section 2. Let $L^+ = L \cup \{E_n : n \in \omega\}$, E_n a binary predicate. Let $\mathscr{C}_0^+$ be the class of all finite L^+-structures whose L-reduct is in $\mathscr{C}_0$, and such that E_n is an equivalence relation, $E_{n+1} \subseteq E_n$, and E_{n+1} refines each E_n class into at most 2 classes. Define d_0, d, $\leq$ as in Section 2, ignoring the E_n's. Let $ZD = \{(A, B)$: A is simply algebraic over $B\}$. (The word 'algebraic' is no longer appropriate, however; zero-dimensional would be correct.) If A, A_1, A_2 are in $\mathscr{C}_0^+$, $A = A_1 \cap A_2$, and $G = A_1 \cup A_2$, call G a free amalgam of A_1, A_2 over A if $R^G = R^{A_1} \cup R_2^A$. Observe that here there are many free amalgams, since the equivalence classes of the E_n may be identified in different ways. We assume however that the E_n remain equivalence relations on G, and that E_{n+1} refines each E_n class into at most 2 classes.

Lemma 20. *Suppose $A, A_1, A_2 \in \mathscr{C}^+$, $A \leq A_1$. Let G be a free amalgam of A_1, A_2 over A. Then $G \in \mathscr{C}_0^+$, and $A_2 \leq G$.*

Proof. By Section 2, Lemma 1(i). □

Lemma 21. *There exists a first-order theory T whose models are precisely the L-structures M satisfying*:

(1) *Every finite substructure of M is in $\mathscr{C}_0^+$.*

(2) *If $A \subseteq M$ is finite, $B \in \mathscr{C}_+^0$, $A \leq B$, and some E_n separates all the points of B, then there exists an embedding of B into M over A.*

If M is an $\aleph_0$-saturated model of T, then

(3) *If $A \subseteq M$ is finite, $B \in \mathscr{C}^+$, and $A \leq B$, then there exists an embedding of B into M over A.*

(4) *In* (3), *if $A \leq M$ then the image of B can be chosen self-sufficient in M.*

Proof. For (1) it suffices to state (1_a) that every finite substructure of M is in $\mathscr{C}$, and (1_b) the E_i are equivalence relations, and each E_i-class is refined into at most two classes of E_{i+1}.

For (2) observe that if E_n separates points on B, then the reduct of B to the language $L_n = L \cup \{E_i: i \leq n\}$ determines the structure of B completely. Thus the instance of (2) for the pair (A, B) can be stated with a single $\forall\exists$ sentence (given (1_a)).

(3) By compactness, it suffices to find an L_m-embedding of B into M with the required properties, for each m. For this limited purpose we may modify the structure of B, not touching the L_n-structure, but ensuring that any two points of B are L_n-inequivalent for an appropriately large n (say $n = m + \log_2(|B|) + 1$). Then the existence of the embedding is guaranteed by (2).

(4) Using induction on $|B|$, we reduce the two cases: (a) $B = A \cup \{b\}$ (b) $(B - A, A)$ is in ZD. In case (b) the embedding of (3) is automatically good, using the arguments of Lemma 2. In case (a) we use compactness and saturation

again. We must find $b' \in M$ inside a certain E_n-class (say the class of b'') such that $d(b'/A) = 1$. In other words, b' must satisfy, for each integer m:

$$(\#_m) \text{ for all } m\text{-element subsets } C \text{ of } M \text{ containing } b', \quad d_0(C/A) > 1.$$

By compactness, it suffices to find b_M satisfying $(\#_m)$ for $m \leqslant M$, where M is a given integer. We do this by embedding into M a larger set B_M. B_M can be taken to be a free join of B'_M with A over $\emptyset$, where B'_M is a fixed element of $\mathscr{C}_0^+$ containing b and enjoying the following property:

$$d_0(B_M - A) = 0, \quad \text{but } d_0(Y) \geqslant 1 \quad \text{for every } Y \subseteq B'_M \text{ with } 1 \leqslant |Y| \leqslant M.$$

The construction of B'_M is left to the reader.

Now consider B_M as embedded in M, and let C be an m-element subset of M, $m \leqslant M$, $b \in C$. Then $d_0(C/B_m) \geqslant 0$. Further $d_0(C \cap B_M/A) \geqslant 1$. Thus $d_0(C/B_m) \geqslant 1$, as required. □

Lemma 22. *T is complete, consistent, and superstable.*

Proof. Existence of a model of T is easy from Lemma 20. For completenesss, suppose (using absoluteness) that M_1, M_2 are two $\aleph_1$-saturated models of T of power $\aleph_1$. Build an isomorphism $f: M_1 \to M_2$ as follows. Suppose f is given on a countable $A_1 \leqslant M_1$, $fA_1 \leqslant M_2$. Let $c \in M_1 - A_1$. Find a finite $C \subseteq M_1$, $c \in C$, such that $d_0(C/C \cap A_1)$ is least possible. Then $A_1 \leqslant A_1 \cup C \leqslant M_1$. By Lemma 21(4) and saturation, one can extend f_2 to an atomic map on $A_1 \cup C$, whose image is self-sufficient in M_2. Thus a back-and-forth construction can continue.

Let $N \subseteq \mathbb{C}$ be an $\aleph_0$-saturated model of T. If $c \in \mathbb{C}$, we can find a finite C as above such that $c \in C$ and $d_0(C) - d_0(C \cap N)$ is least possible. Further we may require that for each $c \in C$ there is $c' \in C \cap N$ such that $c E_n c'$ for each n. Then $C \cup N \leqslant \mathbb{C}$, and $C \cup N$ is a free amalgam of C and N over $C \cap N$. tp(C/N) is determined by tp$(C/C \cap N)$ and the above information. So there are at most $2^{\aleph_0} \cdot |N|$ types over N. Thus T is superstable. □

Lemma 23. *T has infinity-rank* ω; *all types of rank* ω *are non-orthogonal, non-locally-modular. The rank* 1 *types are trivial. T has no strongly regular types (in any sort.)*

Proof. Note first that if $(A, B) \in ZD$, $A \cup B \leqslant M$, then tp(A/B) has rank 1. Indeed for any model N containing B, either $A \subseteq N$, or else $A \cup N$ is a free amalgam of C and N over B. In the latter case the type of A over N is determined by the E_n-type of A over N for each n; there are at most $2^{\aleph_0}$ possibilities. Thus tp(A/B) has rank 1. Further, by Lemma 2, if A_j realizes tp(A/B) and A_i, A_j have distinct universes for $i \neq j$, then $\{A_i : i \in I\}$ is independent over B. This shows that rank 1 types are trivial. The rest of the proof is left to the reader. □

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