Non-expansion

$X \subseteq_{fin} \mathbb{C}$ Typically, $|X+X| \sim |X|^2$.

But $A = \{\alpha + k\beta : k \in \{-N, \dots, N\}\}$

$|A+A| < 2|A|$ // $(\mathbb{C};+)$ admits non-expanding sets

Th^m [Elekes–Király '01]: $\forall c \exists m. \forall X \subseteq_{fin} SL_2(\mathbb{R})$

$|X \cdot X| \leq c|X| \Rightarrow \exists A \leq SL_2(\mathbb{R})$ abelian. $\exists \alpha_1, \dots, \alpha_m$.

$X \subseteq \bigcup_{i=1}^{m} \alpha_i A$

- $C \subseteq \mathbb{R}^2$ irr^{ble} alg^c curve

For $x \neq y \in C$, $x \circ y := (C \cap \text{line through } x, y) \setminus \{x, y\}$

deg $C = 3$: $x \circ y \sim$ ab^n gr^p on smooth pts

$(x + y := u \circ (x \circ y))$

A arith^c progr^n $\Rightarrow |A \circ A| < 2|A|$

Orchard Problem: Find large $X \subseteq_{fin} \mathbb{R}^2$

s.t. many lines meet X in 3 pts ("3-pt lines")

Sol^n: $X := A \cup A \circ A$ has $\sim |X|^2$ 3-pt lines

Th^m [Elekes–Szabó '13]: $\forall d > 3. \exists \eta > 0.$ deg $C = d$, $X \subseteq_{fin} C$

$\Rightarrow |X \circ X| \geq |X|^{1+\eta}$

and $|\{\text{3-pt lines}\}| \leq |X|^{2-\eta}$

Q^n: What about $S \subseteq \mathbb{R}^3$ (cubic) alg^c surface?

Psf dim^n

$\mathcal{U}$ non-princ^l ufilter on $\mathbb{N}$

$K := \mathbb{C}^{\mathcal{U}}$

$X \subseteq K^n$ is *internal* if $X = \prod_{i \to \mathcal{U}} X_i$, some $X_i \subseteq \mathbb{C}$

Then: $|X| := \lim_{i \to \mathcal{U}} |X_i| \in \mathbb{N}^{\mathcal{U}} \cup \{\infty\}$

Fix $\xi \in \mathbb{N}^{\mathcal{U}} \setminus \mathbb{N}$

$\delta(X) := st(\log_\xi |X|) \in \mathbb{R}_{\geq 0} \cup \{-\infty, +\infty\}$

Given c^{bly} many internal $X_i \subseteq K^{n_i}$

exists c^{ble} language $\mathcal{L}$ on K s.t.

- X_i is $\emptyset$-def^{ble}
- K-def^{ble} $\Rightarrow$ internal
- For $a, b \in K^{<\omega}$ and $C \subseteq K$ c^{ble},

setting $\delta(a/C) := \delta(tp(a/C)) = \inf\{\delta(\varphi(K)) : \varphi \in tp(a/C)\}$

- $a \equiv_C b \Rightarrow \delta(a/C) = \delta(b/C)$
- $\delta(ab/C) = \delta(a/bC) + \delta(b/C)$
- Φ partial tp $/C \Rightarrow \exists a \in \Phi(K). \delta(a/C) = \delta(\Phi)$

- $a \mathop{\downarrow}^{\delta}_{C} b \overset{}{\underset{def}{\iff}} \delta(a/bC) = \delta(a/C)$

- a is *broad* if $\delta(a) \in \mathbb{R}_{>0}$

Incidence Bounds

Th^m [Toth '14]: n points and e lines in $\mathbb{C}^2$

give $O(n^{2/3} e^{2/3} + n + e)$ incidences

Assume $+, \cdot \in \mathcal{L}$

$\rightsquigarrow$ For $p \in l$ point on line in K^2

$\delta(pl) \leq \max(\frac{2}{3}(\delta(p) + \delta(l)), \delta(p), \delta(l))$

Th^m [Elekes–Szabó '13, ..., Chernikov–Galvin–Starchenko '20]:

I constructible binary rel^n (or: def^{ble} in a distal th^y)

Let $(p, l) \in I$ with $\delta(p) > \frac{1}{2}\delta(l) > 0$.

If $\delta(l/p) \geq \frac{1}{2}\delta(l)$

then $(p, l) \in A \times B \subseteq I$ with $|A| = 2$ and B internal, $\delta(B) > 0$

Def^n: (a_1, a_2, a_3) is a *δ-triangle* if a_i is broad

and $a_i \mathop{\downarrow}^{\delta} a_j$ $\{i,j,k\} = \{1,2,3\}$

- $a_i \in acl^{ACF}(a_j, a_k)$

e.g. non-expansion $\rightsquigarrow$ δ-triangle $(a, b, a \circ b)$

~~$(a, b, a \circ b)$ witnessing non-expansion~~

Th^m [Elekes–Szabó '13] B–Breuillard '19

(a_1, a_2, a_3) δ-triangle ~~trd$(a_i) = 1$~~ a_i cgp

Then $\exists$ comm^{ve} conn^d alg^c gr^p G

$\exists a_i' \in G$ gen^c

s.t. $a_i' \sim^{ACF} a_i$

$a_1' + a_2' = a_3'$

Def^n: Broad a is *cgp* if $\forall b. (a \not\mathop{\downarrow}^{ACF} b \Rightarrow \delta(a/b) = 0)$

" " " *wgp* " " " $a \not\mathop{\downarrow}^{\delta} b$

Th^m [Breuillard–Green–Tao $\Rightarrow$, B–Dobrowolski–Zou $\Leftarrow$]

$(G; \cdot)$ conn^d complex alg^c gr^p

Exists δ-tri^{gle} $(a_1, a_2, a_3 = a_1 \cdot a_2)$ with $a_i \in G$ gen^c, a_i wgp

iff G nilpotent.

Qu^n: What are the wgp δ-triangles?

Collinearity on a cubic surface

Let $S \subseteq \mathbb{P}^3(K)$ be a smooth irr^{ble} cubic surface

WLOG $/\emptyset$

Fact: S contains 27 lines $L_1, \dots, L_{27}$

$\underline{S} := S \setminus \bigcup_{i=1}^{27} L_i$

Thm [B–Dobrowolski–Zou]:

If $(a_1, a_2, a_3 = a_1 \circ a_2)$ is a collinear ~~triangle~~ wgp δ-triangle

then exists plane $P \subseteq \mathbb{P}^3(K)$ over $\emptyset$ s.t. $a_i \in P$.

Hence $\forall \varepsilon > 0\ \exists \eta > 0,\ N_0 \in \mathbb{N}$.

$\forall S \subseteq \mathbb{P}^3(\mathbb{C})$ smooth irr^{ble} cubic

$\forall X \subseteq_{fin} S,\ |X| > N_0$

$|\text{3-pt lines}| > |X|^{2-\eta}$

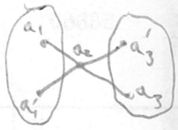
$\Rightarrow \exists P$ plane. $|X \cap P| > |X|^{1-\varepsilon}$

pf sketch:

Suppose $\mathrm{trd}(a_i) = 2$.

WLOG $\delta(a_i) = 1$

Let $a_1' a_3' \equiv_{a_2} a_1 a_3 \quad a_1' a_3' \mathop{\downarrow}^{\delta}_{a_2} a_1 a_3$



$e := ((a_1, a_1'), (a_3, a_3')) \in \{a_1 \circ a_3 = a_1' \circ a_3'\} =: \underset{\text{constr}^{ble}}{E} \subseteq S^4$

$\delta(e) = 3 = 2 + 1 = \delta(a_1 a_1') + \frac{1}{2}\delta(a_3 a_3')$

$\rightsquigarrow e \in A \times B \subseteq E \quad |A| = 2,\ \delta(B) > 0$

Key Lemma: Such $A \times B$ is coplanar (up to finite)

$\rightsquigarrow \delta(e/P) > 0$, $P :=$ plane spanned by e.

$\delta(e/P) < 3$ by wgp.

Tsth $\Rightarrow \delta(a_3 P / a_1) \leq \dots$

$\Rightarrow \times$

So $\mathrm{trd}(a_i) = 1$.

Let $C_i := \mathrm{Locus}(a_i)$

Want: $C_i = C$ planar cubic

Facts:

- S is blowup of $\mathbb{P}^2$ at 6 points
- $\mathrm{Pic}(S) \cong \mathbb{Z}^7$ generators $\ell, e_1, \dots, e_6$

  e_i = class of i^{th} exceptional div

  ℓ = class of strict transform of gen' line in $\mathbb{P}^2$
- $C \subseteq S$ planar iff $[C] = 3\ell - \sum e_i$

- $\deg(a\ell - \sum b_i e_i) = 3a - \sum b_i$
- For $x \in S$, $\gamma_x(y) := x \circ y$ "Geiser inv^n"
- $[C] = a\ell - \sum b_i e_i$, $C \notin T_x(S)$, $M_x(C) = m$, $\deg(C) = d$, then (by Cayley-Bacharach)

  $\cancel{[\gamma_x^*(C)] = (8a - 3\sum_i b_i - 3m)\ell - \sum_j (3a - \sum_i b_i - b_j - m) e_j}$

  $\cancel{M_x(\gamma_x^*(C)) = 3a - \sum b_i - 2m = \deg(C) - 2m}$

  $\cancel{\deg(\gamma_x^*(C)) = 2\deg(C) - 3m}$

  $[\gamma_x^*(C)] = (3d - a - 3m)\ell - \sum_i (d - b_i - m) e_i$

  $M_x(\gamma_x^*(C)) = d - 2m$

  $\deg(\gamma_x^*(C)) = 2d - 3m$

Now $C_i := \mathrm{locus}(a_i)$ $i = 1, 3$ $\quad (S/\gamma_a^*(C) / a \circ C / g\,?)$

$C_3 = \gamma_{a_2}^*(C_1)$

$M_{a_2}(C_i) \leq 1$ since $\mathrm{trd}(a_2) > 0$ $(\Leftarrow \delta(a_2) > 0)$

$\rightsquigarrow M_{a_2}(C_i) = 1$, $\deg(C_i) = 3$

so $a_2 \in C_1 \cap C_2$ so $C_1 = C_2 =: C$

$[C] = a\ell - \sum b_i e_i = \cancel{(8a - 3\sum b_i - 3)\ell - \sum (3a - \sum b_i - b_i - 1) e_i}$

$= (6 - a)\ell - \sum (2 - b_i) e_i$

$\Rightarrow [C] = 3\ell - \sum e_i \Rightarrow C$ planar

Key Lemma: $a, b, c, d \in S$ $\quad \mathrm{tr.d}(a,b) = 4 = \mathrm{tr.d}(c,d)$, $\mathrm{trd}(a,b,c,d) \geq 5$

$C = a \circ (b \circ (c \circ (d \circ C)))$ $\Rightarrow$ C planar