Density of compressibility

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1 Preliminaries

T a complete $\mathcal{L}$-theory.
$\mathcal{U} \vDash T$ sufficiently saturated.

2 Compressible types

Definition. Let $A \subseteq \mathcal{U}$ and $p(x) \in S^x(A)$.

- p is **isolated** if there exists $\psi \in p$ such that $\psi \vDash p$.
- p is **l-isolated** if for each $\mathcal{L}$-formula $\phi(x,y)$
  there exists $\psi \in p$ such that $\psi \vDash p_\phi$,
  where $p_\phi := \{\phi(x,a)^{\epsilon_a} \in p : a \in A^y\}$.
- p is **compressible** if for each $\mathcal{L}$-formula $\phi(x,y)$
  there exists $\psi \in p^*$ such that $\psi \vDash p_\phi$,
  where $(\mathcal{U}, A) \prec (\mathcal{U}^*, A^*)$, and $\mathcal{U} \ni b \vDash p$, and $p^* = \mathrm{tp}(b/A^*)$.

  Equivalently: for each $\phi(x,y)$ there exists $\zeta(x,z)$ s.t.
  for any $A_0 \subseteq_{\mathrm{fin}} A$ there exists $\zeta(x,a) \in p$ s.t.
  $\zeta(x,a) \vDash p_\phi|_{A_0}$.

Facts.

- *Isolated $\Rightarrow$ l-isolated $\Rightarrow$ compressible.*
- *If T is stable: compressible $\Leftrightarrow$ l-isolated.*
- *A theory is* **distal** *iff every type is compressible.*

Example. In $(\mathbb{R}; <)$, $\mathrm{tp}(\pi/\mathbb{Q})$ is compressible but not l-isolated.

Motivating question: how well does compressibility work as an isolation notion?
How much of Chapter IV of *Classification Theory* applies?
In particular, does compressibility provide an NIP version of the following classical fact?

Fact (Shelah)**.** *If T is a countable stable theory and A is a parameter set,
then there exists a model $\mathcal{M} \supseteq A$ which is* **l-atomic** *over A,
i.e. $\mathrm{tp}(b/A)$ is l-isolated for any tuple $b \in \mathcal{M}^{<\omega}$.*

3 Density

This leads to the first question: are compressible types dense in $S(A)$,
i.e. can any formula over A be completed to a compressible type in $S(A)$?

3.1 Finitary version

Question. *Given $\phi(x,y)$,
does there exist k such that for any finite A,
there is a type $p \in S_\phi(A)$ s.t.
$p_0 \vDash p$ for some $p_0 \subseteq p$ with $|p_0| \leq k$?
(i.e. $p(x) \vDash \bigwedge_{i<k} \phi(x,a_i)^{\epsilon_i} \vDash p(x)$.)
Call such a p* ***k*-isolated**.

Definition.

- $\mathrm{vc}^*(\phi)$ is the largest size of a finite A such that $|S_\phi(A)| = 2^{|A|}$,
  or ∞ if no such bound exists.
- ϕ is **NIP** if $\mathrm{vc}^*(\phi) < \infty$.
- T is **NIP** if every $\mathcal{L}$-formula $\phi(x,y)$ is.

Certainly we need $k \geq \mathrm{vc}^*(\phi)$.
In the following example, $\mathrm{vc}^*(\phi) = 2$, but we need $k = 3$.

```
1 0 1 0 1
0 1 0 1 1
1 0 1 1 0
0 1 1 0 1
1 1 0 1 0
1 0 0 0 1
0 0 0 1 1
0 0 1 1 0
0 1 1 0 0
1 1 0 0 0
```

(The (i,j)th entry is the truth value of $\phi(x,a_j)$ in $p_i \in S_\phi(A)$.)
It turns out that the above Question was asked by C. Kuhlmann in 1999, under the terminology of "recursive teaching dimension", and answered in 2016!

Theorem (Xi Chen, Yu Cheng, Bo Tang '16)**.** *$k = d2^{d+1}$ works, where $d = \mathrm{vc}^*(\phi)$.*

3.2 Local density

We adapt the proof of Chen-Cheng-Tang to the infinitary setting.

Definition.

- $p \in S_\phi(A)$ is ***k*-compressible** if:
  for any finite $p_0 \subseteq p$,
  there exists $p_1 \subseteq p$ with $|p_1| \leq k$,
  such that $p_1 \vDash p_0$.
  (i.e. $p(x) \vDash \bigwedge_{i<k} \phi(x,a_i)^{\epsilon_i} \vDash p_0(x)$.)
- p is **$*$-compressible** if it is k-compressible for some k.

Theorem ("Local density")**.** *If ϕ is NIP with $\mathrm{vc}^*(\phi) = d$,
then for any A, $S_\phi(A)$ contains a $d2^{d+1}$-compressible type.
Moreover, the same holds relative to any consistent partial type π: there exists $p \in S_\phi(A)$ which is consistent with π and is $d2^{d+1}$-compressible modulo π.
Hence, the $*$-compressible types are dense in $S_\phi(A)$.*

3.3 Global density

We conclude the global density we wanted:

Theorem ("Global density")**.** *Suppose T is countable and NIP.
Then the compressible types are dense in any $S^x(A)$.*

Proof idea. Enumerate the $\mathcal{L}$-formulas as $(\phi_i(x,y_i))_{i\in\omega}$.
Iteratively build a type by adding for each i a ϕ_i-type
which is k_i-compressible modulo the partial type we have built so far;
this exists by relative local density. □

We will say more about the global case later.

4 Strengthening local density

4.1 Averages of compressible types

Given $p_1, \ldots, p_n \in S_\phi(A)$ with n odd,
their **rounded average** is

$$\left\{\phi(x,a)^\epsilon : a \in A;\ |i : \phi(x,a)^\epsilon \in p_i| > \frac{n}{2}\right\}.$$

Theorem 1. *Let $\phi(x,y)$ be NIP.
There are n and k depending only on $\mathrm{vc}^*(\phi)$ s.t.
for any $A \subseteq \mathcal{U}$,
any $p \in S_\phi(A)$ is the rounded average of k-compressible types $p_1, \ldots, p_n \in S_\phi(A)$.*

4.2 Local uniform honest definitions

Corollary. *Let $\phi(x,y)$ be NIP.
Then ϕ has "uniform honest definitions":
If $A \subseteq \mathcal{U}^x$ and $b \in \mathcal{U}^y$ and $A_0 \subseteq_{<\omega} A$,
then there is $d \in A^z$ such that $\phi(b,A_0) \subseteq \theta(d,A) \subseteq \phi(b,A)$ where*

$$\theta(w,y) = \mathrm{Maj}_{i\in\{1,\ldots,n\}}\, \forall x. \left(\bigwedge_{j<k} \phi(x,w_{i,j})^{\epsilon_{i,j}} \to \phi(x,y)\right)$$

*for appropriate $\epsilon_{i,j}$ depending on b,
where k and n depend only on $\mathrm{vc}^*(\phi)$.
(We can then code the finitely many such θ into a single formula depending only on ϕ.)*

Sketch Proof. $\mathrm{tp}_\phi(b/A)$ is the rounded average of k-compressibles $p_1, \ldots, p_n$.
Given A_0, we have $p_i \vDash \bigwedge_{j<k} \phi(x,d_{i,j})^{\epsilon_{i,j}} \vDash p_i|_{A_0}$.
Set $d := (d_{i,j})_{i,j}$. □

4.3 Superdensity

Theorem 1 (every type is a bounded rounded average of compressibles) follows from a (p,q) argument and the following strengthening of local density.
Local density implies that any consistent $\bigwedge_{i<n} \phi(x,a_i)^{\epsilon_i}$ can be completed to a k-compressible type in $S_\phi(A)$, where $k = k(n, \mathrm{vc}^*(\phi))$.
We generalise this by replacing $\mathrm{tp}(a_i/\mathcal{U})$, which are types realised in A, with types that are merely finitely satisfiable in A.

Lemma ("Local superdensity of compressibility")**.**
*Let $\phi(x,y)$ be NIP.
Let $A \subseteq \mathcal{U}$, $b \in \mathcal{U}^x$, and $n \in \mathbb{N}$.
Let $q(y_1, \ldots, y_n) \in S_{\phi^{\mathrm{opp}},A\text{-}fs}(\mathcal{U})$.
Then there exists a k-compressible type $p \in S_\phi(A)$,
where $k = k(n, \mathrm{vc}^*(\phi))$,
such that "p agrees with b on q":*

$$q(y_1, \ldots, y_n) \otimes p(x) \vDash \bigwedge_i (\phi(x,y_i) \leftrightarrow \phi(b,y_i)).$$

Proof idea for countable A.
One can reduce to the case $n = 1$.
WLOG $q(y) \vDash \phi(b,y)$.
Using that A is countable and ϕ is NIP and q is fs, we have

Fact (Simon)**.** $q = \lim_{i\to\omega}(\mathrm{tp}_\phi(a_i/\mathcal{U}))$ *for some sequence $a_i \in A$.*

By Ramsey, assume $(a_i)_i$ is sufficiently indiscernable.
Take a ϕ-type p_0 over $(a_i)_i$ such that the truth value of $\phi(x,a_i)$ in p_0 alternates maximally then is constantly true.
Maximality and indiscernability yields that p_0 is t-compressible (where $t = 2\,\mathrm{vc}^*(\phi) + 2$).
By relative local density, p_0 extends to a k-compressible type on A as required. □

4.4 Infinitary (p,q)

Proof of Theorem 1. Let k, n be sufficiently large (with n odd).
Let $S := \{k\text{-compressibles}\} \subseteq S_\phi(A)$.
Suppose $\mathrm{tp}_\phi(b/A)$ is not a rounded average of n elements of S.
Then if $S_0 \subseteq S$ with $|S_0| = n$,
there is $a \in A$ and $S_1 \subseteq_{>\frac{n}{2}} S_0$ s.t.
for all $p \in S_1$,
$p(x) \vDash \neg(\phi(x,a) \leftrightarrow \phi(b,a))$.
Let $C \subseteq \mathcal{U}^x$ contain a realisation of each element of S.
Taking n and N large enough,
by the (p,q)-theorem (with $\mathrm{p} = n$, $\mathrm{q} = \lceil \frac{n}{2} \rceil$),
$\{\bigvee_{i<N} \neg(\phi(c,y_i) \leftrightarrow \phi(b,y_i)) : c \in C\}$ is finitely satisfiable in A.
Completing this formula to $q(y_0, \ldots, y_{N-1}) \in S_{\phi^{\mathrm{opp}},A\text{-fs}}(\mathcal{U})$,
we contradict superdensity. □

5 Compressible models

T countable NIP.
Recall that compressible types are dense in any $S^x(A)$.
It follows easily that any A can be extended to a model $\mathcal{M} \vDash T$ which is **compressibly constructible** over A,
i.e. built transfinitely from A, where at the successor step we realise a compressible type over everything built so far.
When can we build such a model which is moreover **compressibly atomic** (c.a.) over A,
i.e. $\mathrm{tp}(b/A)$ is compressible for any tuple b from $\mathcal{M}$?
For A countable, this is easy once we note that compressibility is finitely transitive: if $\mathrm{tp}(b/A)$ and $\mathrm{tp}(c/Ab)$ are compressible, then so is $\mathrm{tp}(bc/A)$.
Being a little more careful, this kind of direct argument can also handle $|A| = \aleph_1$.
For general A, we need a new idea.

5.1 Rescoping

Theorem. *If $\mathrm{tp}(a/B)$ is compressible and $C \subseteq B$, then $\mathrm{tp}^B(a/C)$ is compressible.*

Main ingredient in the proof is Simon's decomposition (2020) of an arbitrary type as "compressible modulo a generically stable part".

Corollary. *For $C \subseteq B \subseteq A$,
if A is c.a. over B and B is c.a. over C,
then A is c.a. over C.*

Proof idea. $\mathrm{tp}^B(c/A)$ is compressible; now "compress the compression",
i.e. apply compressibility over A of the parameters from B to the formulas expressing the compression,
and deduce that we can replace those parameters by existentially quantified variables. □

Corollary. *If A is compressibly constructible over B, then A is c.a. over B.
Hence c.a. models exist over arbitrary parameter sets.*

6 Applications

6.1 Constraining stable parts

T countable NIP.

Corollary 1. *Suppose T is unstable and let $\mathcal{M} \vDash T$ be $\aleph_0$-saturated.
Let S be a stable definable set (i.e. any $S(x) \wedge \phi(x,y)$ is stable).
Then there exist arbitrarily large $\mathbb{N} \succ \mathcal{M}$ such that $S(\mathbb{N}) = S(\mathcal{M})$,
and more generally any generically stable type over $\mathcal{M}$ realised in $\mathbb{N}$ is already realised in $\mathcal{M}$.*

Proof idea. Via SOP, we can realise a compressible non-l-isolated type then extend to a compressible model;
this can't increase S, because compressibility implies l-isolation on S by stability.
Now iterate. □

Corollary 2. *Suppose S is a definable set such that the induced structure S_{ind} is stable.
Then the reduct functor from models of T to models of $T_0 := \mathrm{Th}(S_{\mathrm{ind}})$ is surjective.
It is also full, i.e. surjective on elementary embeddings.*

Proof idea. Similar to above; given $\mathcal{M}_0 \vDash T_0$, find a compressible model of T over it, see it doesn't increase S.
Fullness is a little more complicated. □

6.2 Valued fields

We also get a (somewhat) new proof (without explicit bounds) of the following result.

Theorem 2 (B, J-F Martin '21)**.**
*If K is a valued field with finite residue field (e.g. $\mathbb{F}_p(t)$),
then K is "qf-distal"; equivalently:
working in $K^{\mathrm{alg}} \vDash \mathrm{ACVF}$,
every type $\mathrm{tp}(b/A)$ with $Ab \subseteq K$ is compressible.
Szemeredi-Trotter results for such fields follow.*

Proof idea. More generally, if $A \subseteq M \vDash \mathrm{ACVF}$ and $k(M) = \mathrm{acl}^{\mathrm{eq}}(A) \cap k$,
then we can build a compressible construction sequence for M over A by alternating taking $\mathrm{acl}^{\mathrm{eq}}$ and adding a single new element from M;
by considering Swiss cheeses, one can see that such an extension is compressible. □