A survey of asymptotic classes and measurable structures

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1 Introduction

In this article we survey a body of results about classes of finite first order structures in which definable sets have a rather uniform asymptotic behaviour. Non-principal ultraproducts of such classes may have unstable theory, but will be supersimple of finite rank. In addition, they inherit a definable *measure* on definable sets.

The starting point for this work was the following result of [11] on definability in finite fields.

Theorem 1.1 ([11]) *Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language L_{rings} for rings, with $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_m)$. Then there is a positive constant C, and a finite set D of pairs (d, μ) with $d \in \{0, \dots, n\}$ and μ a non-negative rational number, such that for each finite field $\mathbb{F}_q$ and $\bar{a} \in \mathbb{F}_q^m$,*

$$\big||\varphi(\mathbb{F}_q^n, \bar{a})| - \mu q^d\big| \leq C q^{d-(1/2)} \qquad (*)$$

for some $(d, \mu) \in D$.

Furthermore, for each $(d, \mu) \in D$, there is a formula $\varphi_{(d,\mu)}(\bar{x})$ which defines in each finite field $\mathbb{F}_q$ the set of tuples $\bar{a}$ such that $(*)$ holds.

This result rests on the Lang-Weil estimates for the number of $\mathbb{F}$-rational points of an absolutely irreducible variety defined over the finite field $\mathbb{F}$. The proof uses partial quantifier elimination for pseudofinite fields, derivable from the paper of Ax [2] which introduced pseudofinite fields: any formula $\varphi(\bar{x})$ is a boolean combination of formulas $\exists y(g(\bar{x}, y) = 0)$, where $g \in \mathbb{Z}[\bar{X}, Y]$. (In fact, by adding a set C of constants to the language, the authors arrange that φ is equivalent to a *conjunction* of similar formulas, rather than an arbitrary boolean combination; there is an alternative effective proof in [20], based on Galois stratification.) The reason why μ may be a fraction (rather than an integer as in Lang-Weil), is that the existential quantifier ranges over a finite set. Oversimplifying considerably, the set defined by $\exists y(g(\bar{x}, y) = 0)$ is, for some k, the image of a k-to-1 projection map of the variety defined by $g(\bar{x}, y)$; thus, its measure is $\frac{1}{k}$ times that of $\{(\bar{x}, y) : g(\bar{x}, y) = 0\}$.

As a short application of Theorem 1.1, noted in [11], it follows that the finite field $\mathbb{F}_q$ is not uniformly definable in $\mathbb{F}_{q^2}$. Theorem 1.1 also yields that pseudofinite fields are supersimple of rank 1, with an associated definable and finitely additive notion of measure on the definable sets.

The present article is a survey of recent work stimulated by Theorem 1.1. The idea, initiated in [43], is to consider classes of finite structures in which definable sets satisfy the same kind of asymptotic behaviour as for finite fields. In the initial work of [43], only a 1-dimensional version was considered, and the error terms were as in Theorem 1.1. More recently, Elwes [18] has developed a theory of higher dimensional asymptotic classes where the error terms are weaker – see Definition 2.1 below.

There is also a notion of *measurable structure*: a supersimple structure of finite SU-rank equipped with the kind of measure function on definable sets which exists, by virtue of Theorem 1.1, in pseudofinite fields. Any ultraproduct of members of an asymptotic class will be measurable (Theorem 3.9 below) but there are also measurable structures which do not even have the finite model property, so cannot arise from asymptotic classes; see Theorem 3.12. Familiar supersimple structures such as the random graph, smoothly approximable structures, and pseudofinite fields, are all measurable.

One of the main results in this area is a theorem of Ryten [51], stemming from earlier work of Ryten and Tomašić [52], that for any natural number d, the collection of all finite simple groups of Lie rank at most d forms an asymptotic class (Theorem 6.1 below). This opens the possibility of variations of the Algebraicity Conjecture of Cherlin and Zilber that any simple group of finite Morley rank is an algebraic group. Namely, the following seems reasonable, and, unlike the Algebraicity Conjecture, incorporates the classes of *twisted* finite simple groups.

Conjecture 1.2 If G is a simple group with measurable theory, then G is a Chevalley group (possibly of twisted type) over a pseudofinite field.

In Section 2 below we give the current definition of *asymptotic class*, and some examples. Measure is introduced in Section 3, again with examples and discussion of the connection with asymptotic classes, and with Hrushovski's notion of *unimodular* theory. Smoothly approximable structures provide an important class of examples, and we describe in Section 4 how these fit into the framework. In Section 5, we survey results from [51] and [52] on measure in ACFA, and an extension of Theorem 1.1 to finite difference fields. This yields the above result on finite simple groups of fixed Lie rank, sketched in Section 6. We turn in Section 7 to measurable groups, and asymptotic classes of groups, of low dimension. The main results here, due to Elwes and Ryten, are that any 2-dimensional asymptotic class of groups consists of groups with a uniformly definable soluble subgroup of bounded index (Theorem 7.5), and an analogue of Hrushovski's result on groups of finite Morley rank in a stable theory which act transitively on a strongly minimal set. The paper concludes with a section on open questions.

If $\mathcal{C}$ is a class of finite structures in a language L, then the *asymptotic theory* of $\mathcal{C}$ is the collection of all sentences which hold in all but finitely many members of $\mathcal{C}$. Equivalently, it consists of those sentences which hold of any non-principal ultraproduct of members of $\mathcal{C}$. Notation is introduced locally, but we try to stick to the convention that for a formula $\varphi(\bar{x},\bar{y})$, $\bar{x}=(x_1,\ldots,x_n)$, $\bar{y}=(y_1,\ldots,y_m)$, with $\bar{y}$ the parameter variables. The algebraic closure of a field K is denoted $\tilde{K}$. We write $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ to mean that the sets A and B are independent over C (in the sense of model-theoretic non-forking).

Though this material is close to stability and simplicity, our intention is that little knowledge of simplicity theory is needed to follow this paper; the reader can refer to [48] and [60]. For details of the ranks mentioned (S_1-rank, D-rank, SU-rank), see [35]. For background on the model theory of finite and pseudofinite fields, we refer to the original paper [2], the survey [14], or to [21]. For background on ACFA, see [12]. For the structure of the finite simple groups of Lie type, see [10].

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2 Asymptotic classes

Definition 2.1 (Elwes, [18]) Let $N \in \mathbb{N}$, and let $\mathcal{C}$ be a class of finite L-structures, where L is a finite language. Then we say that $\mathcal{C}$ is an **N-dimensional asymptotic class** if the following hold.

(i) For every L-formula $\varphi(\bar{x},\bar{y})$ where $l(\bar{x})=n$ and $l(\bar{y})=m$, there is a finite set of pairs $D\subseteq(\{0,\ldots,Nn\}\times\mathbb{R}^{>0})\cup\{(0,0)\}$ and for each $(d,\mu)\in D$ a collection $\Phi_{(d,\mu)}$ of pairs of the form $(M,\bar{a})$ where $M\in\mathcal{C}$ and $\bar{a}\in M^m$, so that $\{\Phi_{(d,\mu)}:(d,\mu)\in D\}$ is a partition of $\{(M,\bar{a}): M\in\mathcal{C},\bar{a}\in M^m\}$, and

$$\big||\varphi(M^n,\bar{a})|-\mu|M|^{\frac{d}{N}}\big|=o(|M|^{\frac{d}{N}})$$

as $|M|\longrightarrow\infty$ and $(M,\bar{a})\in\Phi_{(d,\mu)}$.

(ii) Each $\Phi_{(d,\mu)}$ is $\emptyset$-definable, that is, $\{\bar{a}\in M^m:(M,\bar{a})\in\Phi_{(d,\mu)}\}$ is uniformly $\emptyset$-definable across $\mathcal{C}$.

We may write D_φ for D, and will call $\{\Phi_{(d,\mu)}:(d,\mu)\in D\}$ a *(definable) asymptotic partition*. We write $h(\varphi(M^n,\bar{a})):=(\mathrm{Dim}(\varphi(M^n,\bar{a})),\mathrm{Meas}(\varphi(M^n,\bar{a}))):=(d,\mu)$ where $(M,\bar{a})\in\Phi_{(d,\mu)}$, except that if $d=\mu=0$ we work with the convention that $\mathrm{Dim}(\varphi(M^n,\bar{a}))=-1$. We call $\mathcal{C}$ a *weak asymptotic class* when $\mathcal{C}$ satisfies the asymptotic criteria (i) for all φ, but the $\Phi_{(d,\mu)}$ are not assumed to be definable.

Remark 2.2 1. In this context the o-notation in (i) means the following: for every $\varepsilon>0$ there is $Q\in\mathbb{N}$ such that for all $M\in\mathcal{C}$ with $|M|>Q$ and all $\bar{a}\in M^m$ where $(M,\bar{a})\in\Phi_{d,\mu}$, we have

$$\big||\varphi(M^n,\bar{a})|-\mu|M|^{\frac{d}{N}}\big|<\varepsilon|M|^{\frac{d}{N}}.$$

2. For every member of an N-dimensional asymptotic class, the universe of the structure is viewed as being N-dimensional.

3. Unlike the presentation in [43] and [18], the definition covers all formulas $\varphi(\bar{x},\bar{y})$, not just those in which $\bar{x}$ is a singleton. It is a theorem (Theorem 2.3 below) that this is equivalent to the same condition just for formulas $\varphi(x,\bar{y})$.

4. The focus of [43] was on the case when $N=1$ (the case for finite fields).

5. The general theory below works equally well with tighter error terms: in 1 above, replace $=o(|M|^{\frac{d}{N}})$ by $<C|M|^{\frac{d}{N}-\frac{1}{2N}}$, where C is a constant depending on φ. Many of the known examples of asymptotic classes satisfy this constraint. However it is not clear that envelopes of smoothly approximable structures satisfy it. For example, if p,q are distinct primes, and M is the disjoint union of two infinite-dimensional vector spaces, one over $\mathbb{F}_p$, the other over $\mathbb{F}_q$, is M approximated by an asymptotic class satisfying the tighter error terms?

6. It is easy to create artificial examples of N-dimensional classes (for any N) in which the μ are irrational, or even transcendental. This

can be done just using sets with a unary predicate. We do not know of natural examples with transcendental μ.

7. It is immediate that any reduct of an asymptotic class is a weak asymptotic class. However, the definability clause (ii) can be lost under reducts. Elwes [18, Section 2.2] has shown that if $\mathcal{C}$ is an asymptotic class, and $\mathcal{D}$ is a class of finite structures uniformly interpretable in $\mathcal{C}$, then $\mathcal{D}$ is a weak asymptotic class. If $\mathcal{C}$ and $\mathcal{D}$ are uniformly bi-interpretable (without parameters), then $\mathcal{D}$ will also be an asymptotic class. This is relevant to Section 6 below (bi-interpretations between classes of simple groups and fields or difference fields).

The following theorem gives a more easily recognised criterion for being an asymptotic class. By ensuring that the condition is really one on one-variable definable sets, it gives an analogy to o-minimality, and related minimality notions.

Theorem 2.3 (Lemma 2.1.2 of [18]) *Suppose that $\mathcal{C}$ is a class of finite structures which satisfies Definition 2.1 (clauses (i) and (ii)) for $n = 1$, i.e. for definable sets in 1 variable. Then $\mathcal{C}$ is an N-dimensional asymptotic class.*

Sketch of the proof. The proof is inductive. For a definable subset X of M^{n+1} consider the projection $\pi : M^{n+1} \to M$ to the first variable, and apply the assumption to $\pi(X)$, and the inductive hypothesis to the fibres X_a $(a \in \pi(X))$. The definability clause (ii) is essential.

We draw attention to the following *non-example*. The class of all finite total orders is not an asymptotic class of any dimension; for, as a varies through a finite ordering, the formula $x < a$ defines an arbitrary proportion of the domain. In fact, by Proposition 3.9 (and Corollary 3.7) below, any nonprincipal ultraproduct of an asymptotic class is supersimple, so cannot interpret a partial order with an infinite chain (i.e. cannot have the *strict order property*).

Example 2.4 1. By Theorem 1.1, the collection of all finite fields forms a 1-dimensional asymptotic class. This class, when restricted to fixed characteristic, has an important expansion. For any fixed p, and positive integers m, n with $m \geq 1$, $n > 1$, and $(m, n) = 1$, there is a 1-dimensional asymptotic class $\mathcal{C}_{(m,n,p)}$ of difference fields, namely $\mathcal{C}_{(m,n,p)} = \{(\mathbb{F}_{p^{kn+m}}, \mathrm{Frob}^k) : k > 0\}$ – see Theorem 5.8. The automorphism here is not uniformly definable in the field. The classes $\mathcal{C}_{(m,n,p)}$

are significant for finite simple groups: $\mathcal{C}_{1,2,2}$ is uniformly parameter bi-interpretable with the classes of Suzuki groups ${}^2B_2(2^{2k+1})$ and the Ree groups ${}^2F_4(2^{2k+1})$, and $\mathcal{C}_{(1,2,3)}$ with the class of Ree groups ${}^2G_2(3^{2k+1})$. See Section 6.

2. It is shown in [43, Theorem 3.14] that the collection of all finite cyclic groups is a 1-dimensional asymptotic class. This follows from the partial quantifier elimination ('near model completeness') of Szmielew [55] for abelian groups. The multiplicative groups of finite fields form a subclass.

3. For any odd prime p, the class $\mathcal{C}_p$ of finite extraspecial p-groups of exponent p forms a 1-dimensional asymptotic class (see [43, Proposition 3.11]). Here, a group G is *extraspecial* if $G' = Z(G) = \Phi(G)$ (the Frattini subgroup), and G' is isomorphic to the cyclic group $\mathbb{Z}/p\mathbb{Z}$. An extraspecial group of exponent p is a central product of several copies of the unique non-Abelian group of order p^3 and exponent p, and a finite one will have order p^{2t+1} for some t. The groups in $\mathcal{C}_p$ are bounded-by-abelian (in fact, $(\mathbb{Z}/p\mathbb{Z})$-by-abelian), but not abelian-by-bounded, that is, they do not have an abelian normal subgroup of bounded index. The quotient $G/Z(G)$ carries definably the structure of a vector space over the field $\mathbb{F}_p$, equipped with an alternating bilinear form which comes from the commutator map. An infinite ultraproduct of groups in $\mathcal{C}_p$ will be ω-categorical, smoothly approximable (see Section 4), and supersimple of rank 1, but not stable. See also Theorem 7.3, and the remark after Lemma 7.4, for partial converses.

4. If q is a prime power with $q \equiv 1 \pmod 4$, then there is a graph P_q (known as a *Paley graph*) with vertex set the finite field $\mathbb{F}_q$, with vertices a, b joined if $a - b$ is a square. By [43], the class $\mathcal{C}$ of all Paley graphs is a 1-dimensional asymptotic class. Essentially, the reason is that by a theorem of Bollobás and Thomason [7] (see also [8, Ch.XIII.2]), if U, W are disjoint sets of vertices of P_q with $m := |U \cup W|$, and $v(U, W)$ is the number of vertices of P_q joined to everything in U and to nothing in W, then

$$|v(U,W) - 2^{-m}q| \leq \frac{1}{2}(m - 2 + 2^{-m+1})q^{\frac{1}{2}} + m/2.$$

It follows that an infinite ultraproduct of Paley graphs is elementarily equivalent to the random graph, so has quantifier elimination. This persists in sufficiently large finite Paley graphs, so corresponding asymptotic estimates hold for *all* formulas in one variable, and hence, by Theorem 2.3, for all formulas. The definability clause of Definition 2.1 is easily verified.

There are analogous classes, interpretable in finite fields, associated with other homogeneous structures. For example, if one considers primes $q \equiv 3 \pmod 4$, and defines $a \to b$ (on $\mathbb{F}_q$) whenever $a - b$ is a square, one obtains the *Paley tournaments*. These approximate the random tournament as above; the analogue of the Bollobás-Thomason result was proved in [22]. For more on this, see [56], and the results of Tomašić [58] mentioned at the end of Section 3.

There is an analogue of the Paley graphs for arity 3 (so a class of 3-hypergraphs), considered in [43, Example 3.6]. For $q \equiv 1 \pmod 4$ one considers a hypergraph on $\mathbb{F}_q$ where $\{a, b, c\}$ is an edge if a, b, c are distinct and $(a - b)(b - c)(a - c)$ is a square. The asymptotic theory of such hypergraphs is that of the generic homogeneous *two-graph* (a 3-hypergraph which is a reduct of the random graph), not the generic 3-hypergraph.

5. By Proposition 3.3.2 of [18], if $\mathcal{M}$ is a smoothly approximable structure, then $\mathcal{M}$ is the union of a chain of 'envelopes' which form an asymptotic class. A basic example is an infinite vector space over a finite field, obtained as a union of infinitely many finite dimensional vector spaces. This example is 1-dimensional but in general such classes will be of higher dimension. See Section 4. As a special case, consider the smoothly approximable structure consisting of a set equipped with an equivalence relation with infinitely many infinite classes. It is smooothly approximated by a chain of structures each of size t^2 (with t varying) equipped with an equivalence relation with t classes all of size t. The latter class of structures is an asymptotic class of dimention 2.

6. Suppose that $\mathcal{C}$ is a class of finite structures such that every infinite ultraproduct of members of $\mathcal{C}$ is strongly minimal. Then $\mathcal{C}$ is a 1-dimensional asymptotic class (Lemma 2.5 of [43]). This is an easy consequence of Theorem 2.3.

In particular, let d be a positive integer, and let $\mathcal{C}_d$ be the collection of all finite graphs of valency d whose automorphism group is transitive on the vertex set. Then $\mathcal{C}_d$ is a 1-dimensional asymptotic class. The reason is that any infinite ultraproduct M of members of $\mathcal{C}_d$ is itself vertex transitive: take the ultraproduct of 2-sorted structures, with a second sort for the automorphism group. It follows that M is strongly minimal, so the last paragraph applies.

In the next section, it will be shown that if $\mathcal{C}$ is an N-dimensional asymptotic class, then any infinite ultraproduct of members of $\mathcal{C}$ is supersimple of rank at most N (and, furthermore, *measurable*). A number of other model-theoretic properties of ultraproducts can be recognised directly from asymptotics. One such is the following. The main issue is to show, by an elementary argument with indiscernibles, that if the asymptotic conditions hold then $\varphi(x,\bar{y})$ is unstable in some ultraproduct.

Proposition 2.5 ([43]) *Let $\mathcal{C}$ be a 1-dimensional asymptotic class. Then some ultraproduct of members is unstable if and only if there is a formula $\varphi(x,\bar{y})$, and for each $k \in \mathbb{N}$ some $M \in \mathcal{C}$ and $\bar{a}_1,\ldots,\bar{a}_k \in M^{\ell(\bar{y})}$ with*

(a) $|\varphi(M,\bar{a}_i)| \geq k$ *for each* $i = 1,\ldots,k$, *and*
(b) $|\varphi(M,\bar{a}_i)\triangle\varphi(M,\bar{a}_j)| \geq k$ *for all distinct* $i,j \in \{1,\ldots,k\}$.

In Chapter 5 of [18], finitary criteria are given for ultraproducts to be ω-categorical, or all to be 1-based.

3 Measurable structures

It was already noted in [11, 4.10, 4.11] that, because of Theorem 1.1, pseudofinite fields are supersimple of S_1-rank 1, and that there is a definable 'measure' on the definable sets. Below, following Section 5 of [43], we generalise this.

Definition 3.1 An infinite L-structure M is *measurable* if there is a function $h : \mathrm{Def}(M) \to \mathbb{N}\times\mathbb{R}\cup\{(0,0)\}$ (we also write $h(X)$ as $(\mathrm{Dim}(X),\mathrm{Meas}(X))$ or $(\mathrm{Dim},\mathrm{Meas})(X)$) such that the following hold.

(1) For each L-formula $\varphi(\bar{x},\bar{y})$ there is a finite set $D \subset \mathbb{N}\times\mathbb{R}^{>0}\cup\{(0,0)\}$, so that for all $\bar{a}\in M^m$ we have $h(\varphi(M^n,\bar{a}))\in D$.
(2) If $\varphi(M^n,\bar{a})$ is finite then $h(\varphi(M^n,\bar{a})) = (0,|\varphi(M^n,\bar{a})|)$.
(3) For every L-formula $\varphi(\bar{x},\bar{y})$ and all $(d,\mu)\in D_\varphi$, the set $\{\bar{a}\in M^m : h(\varphi(M^n,\bar{a})) = (d,\mu)\}$ is $\emptyset$-definable.
(4) (Fubini) Let $X,Y \in \mathrm{Def}(M)$ and $f : X \to Y$ be a definable surjection. Then there are $r\in\omega$ and $(d_1,\mu_1),\ldots,(d_r,\mu_r)\in(\mathbb{N}\times\mathbb{R}^{>0})\cup\{(0,0)\}$ so that if $Y_i := \{\bar{y}\in Y : h(f^{-1}(\bar{y})) = (d_i,\mu_i)\}$, then $Y = Y_1\cup\ldots\cup Y_r$ is a partition of Y into non-empty disjoint definable sets. Let $h(Y_i) = (e_i,\nu_i)$ for $i\in\{1,\ldots,r\}$. Also let $c := \mathrm{Max}\{d_1+e_1,\ldots,d_r+e_r\}$, and suppose (without loss) that this maximum is attained by $d_1+e_1,\ldots,d_s+e_s$. Then $h(X) = (c,\mu_1\nu_1+\ldots+\mu_s\nu_s)$.

If $X \in \mathrm{Def}(M)$ and $h(X) = (d, \mu)$, we call d the *dimension* of X and μ the *measure* of X, and h the *measuring function.* We say that a complete theory T is *measurable* if it has a measurable model (see Remark 3.8 (1)).

Example 3.2 The basic motivating example of a measurable structure is a pseudofinite field. Essentially, this was shown in [11]. If $X \subset F^n$ is definable (F a pseudofinite field) then $\mathrm{Dim}(X)$ (which equals its S_1-rank or D-rank) is just the algebraic-geometric dimension of the Zariski closure of X in $\tilde{F}^n$. The measure of any absolutely irreducible variety in F will be 1. Measurability of any pseudofinite field was used in [28], in a new proof of the well-known fact that any almost simple algebraic group has a universal cover. The main point was that, by measure considerations, if F is a pseudofinite field and G_1, G are connected algebraic groups defined over F, and $f : G_1 \to G$ is an isogeny defined over F, then $|\,\mathrm{Ker}(f) \cap G_1(F)| = |G(F) : f(G_1(F))|$.

A key question is whether, conversely, every measurable field is pseudofinite (see Section 8). An algebraically closed field cannot be measurable – see the remark after Proposition 3.15.

Generalising the case of pseudofinite fields, Proposition 3.9 below shows that asymptotic classes yield measurable structures.

Definition 3.1 is slightly different from that in [43], since we do not specify in the definition that M has a supersimple theory. However, it follows from the next few lemmas, in particular Corollary 3.6, that indeed if M is measurable, then $\mathrm{Th}(M)$ is supersimple of finite rank. Similar arguments were communicated to the authors by Ryten, and can also be found in [32]. The computations below are with D-rank, but by [35, Section 6], in a supersimple theory in which D-rank is finite, D-rank, S_1-rank, and SU-rank all agree for formulas. See [35] for definitions of these ranks. In 3.3–3.6 below the sets are taken in an ambient structure M, which is assumed to be measurable.

Lemma 3.3 *Let $n \in \omega$, and $A_1, \ldots, A_n$ be definable sets, where* $\mathrm{Dim}(A_i) = d$ *for each i.*

(i) $\mathrm{Dim}(\bigcup_{i=1}^n A_i) = d$.

(ii) If, in addition we have $\mathrm{Dim}(A_{i_1} \cap A_{i_2}) < d$ *for each distinct* i_1, i_2*, then*

$$(\mathrm{Dim}, \mathrm{Meas})(\bigcup_{i=1}^{n} A_i) = (d, \sum_{i=1}^{n} \mathrm{Meas}(A_i)).$$

Proof. We proceed by induction on n. For $n = 2$, pick any distinct $a_1, a_2, a_3 \in M$, and define $f : A_1 \cup A_2 \to \{a_1, a_2, a_3\}$ by $f(\bar{x}) := a_1$ if $\bar{x} \in A_1 \backslash A_2$; $f(\bar{x}) := a_2$ if $\bar{x} \in A_2 \backslash A_1$; and $f(\bar{x}) = a_3$ if $\bar{x} \in A_1 \cap A_2$. Then (i) and (ii) are immediate from the Fubini condition.

Suppose now that both statements hold for $n = k - 1$. Then given $A_1, \ldots A_k$, we may first apply the inductive hypothesis to $A_1, \ldots A_{k-1}$, and derive statement (i) by applying the case $n = 2$ to $\bigcup_{i=1}^{k-1} A_i$ and A_k.

Now for (ii), we know by the inductive hypothesis that

$$(\mathrm{Dim}, \mathrm{Meas})(\bigcup_{i=1}^{k-1} A_i) = (d, \sum_{i=1}^{k-1} \mathrm{Meas}(A_i)).$$

Now $A_k \cap \bigcup_{i=1}^{k-1} A_i = \bigcup_{i=1}^{k-1}(A_k \cap A_i)$, and so by (i) we know that $\mathrm{Dim}(A_k \cap \bigcup_{i=1}^{k-1} A_i) < d$. Therefore we may apply the case $n = 2$ to find that $\mathrm{Meas}(\bigcup_{i=1}^{k} A_i) = \mathrm{Meas}(\bigcup_{i=1}^{k-1} A_i) + \mathrm{Meas}(A_k) = \sum_{i=1}^{k-1} \mathrm{Meas}(A_i) + \mathrm{Meas}(A_k) = \sum_{i=1}^{k} \mathrm{Meas}(A_i)$, as required. □

Corollary 3.4 $(\mathrm{Dim}, \mathrm{Meas})$ *is monotonic, that is, whenever* $A \subseteq B$ *are definable, then* $(\mathrm{Dim}, \mathrm{Meas})(A) \leq (\mathrm{Dim}, \mathrm{Meas})(B)$ *(under the lexicographic ordering).*

For convenience we assume that all the following occurs in the home sort.

Lemma 3.5 *Let* X *be a definable set,* $\varphi(\bar{x}, \bar{y})$ *an* L*-formula, and* $(\bar{b}_i : i \in \omega)$ *an indiscernible sequence where for each* $i \in \omega$ *we have* $\varphi(M^n, \bar{b}_i) \subseteq X$*. Suppose that* $\{\varphi(M^n, \bar{b}_i) : i \in \omega\}$ *is inconsistent. Then* $\mathrm{Dim}(X) > \mathrm{Dim}(\varphi(M^n, \bar{b}_i))$*.*

Proof. Suppose not. Suppose $\mathrm{Dim}(X) = \mathrm{Dim}(\varphi(M^n, b_i)) = d$. Then, as $\{\varphi(M^n, \bar{b}_i) : i \in \omega\}$ is inconsistent, by compactness there exists some minimal k such that $\mathrm{Dim}(\varphi(M^n, \bar{b}_1) \cap \ldots \cap \varphi(M^n, \bar{b}_{k+1}) \cap \varphi(M^n, \bar{b}_{k+2})) < d$.

For $i \geq 1$ define $A_i := \varphi(M^n, \bar{b}_1) \cap \ldots \cap \varphi(M^n, \bar{b}_k) \cap \varphi(M^n, \bar{b}_{k+i})$. Notice that by indiscernibility and the minimality of k, we have $\mathrm{Dim}(A_i) = d$, and for $i_1 \neq i_2$ also $\mathrm{Dim}(A_{i_1} \cap A_{i_2}) < d$. Say $\mathrm{Meas}(A_i) = \mu$. Thus by Lemma 3.3, for any $t \geq 1$ we have $(\mathrm{Dim}, \mathrm{Meas})(\bigcup_{i=1}^{t} A_i) = (d, t\mu)$. But then by Corollary 3.4 $\mathrm{Meas}(X) \geq t\mu$ for all t, which is clearly impossible. □

Corollary 3.6 *For any definable set* X*, we have* $D(X) \leq \mathrm{Dim}(X)$*.*

Proof. We proceed by induction on r, showing $(*)$: if $\mathrm{Dim}(X) \leq r$ then $D(X) \leq r$. The case $r = 0$ is automatic from clause (2) of Definition 3.1.

For the inductive step, assume $(*)$ below r, and suppose for a contradiction that $\mathrm{Dim}(X) = r$, and $D(X) \geq r + 1$.

By definition of D-rank, there is an indiscernible sequence $(\bar{b}_i : i \in \omega)$ and an L-formula $\varphi(\bar{x}, \bar{y})$ where $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is inconsistent, and for each $i \in \omega$ we have $D(\varphi(M^n, \bar{b}_i)) \geq r$ and $\varphi(M^n, \bar{b}_i) \subseteq X$. By Lemma 3.5, we have $\mathrm{Dim}(\varphi(M^n, \bar{b}_i)) < r$ for each i. It follows by the inductive hypothesis that $D(\varphi(M^n, \bar{b}_i)) < r$ for each i, which is a contradiction. □

Corollary 3.7 *If M is measurable, then M has a supersimple theory.*

Remark 3.8 1. It is immediate from the definition, and noted in [43], that measurability is a property of a *theory*; that is, if M is measurable and $M \equiv N$, then N is measurable.

2. Less obviously, if M is measurable, and we adjoin to M finitely many sorts from M^{eq}, then the resulting structure is measurable – see [43, Proposition 5.10]. In fact, the class of measurable theories is closed under bi-interpretability.

3. A measurable structure may have many different measuring functions. For example, for the random graph, there is a measure corresponding to any edge probability p, where $0 < p < 1$: we let p be the measure of the set of neighbours of a vertex [43, 5.12]. This is generalised in Theorem 3.10 below, and the remarks after it. If a theory is measurable, then, rather as with Theorem 2.3, the measuring function is determined by its restriction to definable sets in one variable.

4. It can happen that in a measurable structure M there are definable sets X_1 and X_2 with $\mathrm{Dim}(X_1) = \mathrm{Dim}(X_2)$ but $D(X_1) \neq D(X_2)$. For example, consider a structure (M, P, E) where P is a unary predicate picking out an infinite subset with infinite complement, and E is an equivalence relation partitioning $P(M)$ into infinitely many infinite classes, but with $\neg P(M)$ a single E-class. Then $D(P(M)) = 2$ and $D(\neg P(M)) = 1$, but we can artificially choose a dimension and measure such that $\mathrm{Dim}(P(M)) = \mathrm{Dim}(\neg P(M)) = 2$.

One can modify this example to produce a measurable structure in which D-rank is not definable. Indeed, consider the structure

$$(M, E, F_i)_{i \in \omega},$$

where E is an equivalence relation whose classes $\{X_i : i \in \mathbb{Z}\}$ are all infinite, and for each $i \in \omega$, F_i is an equivalence relation which agrees with E except on X_i, which it partitions into infinitely many infinite classes. There is a measuring function on M under which each X_i has dimension 2. But $D(X_i) = 1$ for $i < 0$ and $D(X_i) = 2$ for $i \geq 0$, and the latter are not uniformly definable.

5. If M is measurable with measuring function $(\mathrm{Dim}, \mathrm{Meas})$, and $S \subset M^n$ is definable, there is an induced finitely additive probability measure μ_S on the σ-algebra generated by the definable subsets of S: for definable $X \subset S$, put

$$\mu_S(X) = \begin{cases} \frac{\mathrm{Meas}(X)}{\mathrm{Meas}(S)} & \text{if } \mathrm{Dim}(X) = \mathrm{Dim}(S), \\ 0 & \text{if } \mathrm{Dim}(X) < \mathrm{Dim}(S). \end{cases}$$

If S is defined by the formula $\psi(\bar{x})$, we sometimes write it as μ_ψ.

6. Ben-Yaacov [3] defines, in a measurable theory, the relation $\bar{a} \mathop{\smile\hspace{-0.9em}\mid}^d_C \bar{b}$, to mean $\mathrm{Dim}(\bar{a}/C) = \mathrm{Dim}(\bar{a}/C \cup \{\bar{b}\})$, where, if $\bar{a} = (a_1, \ldots, a_n)$,

$$\mathrm{Dim}(\bar{a}/C) := \min\{\mathrm{Dim}(\varphi(M^n, \bar{c})) : \bar{c} \text{ in } C \text{ and } M \models \varphi(\bar{a}, \bar{c})\}.$$

He shows that this relation coincides with non-dividing, which, by simplicity, agrees with non-forking.

Proposition 3.9 ([18]) *Let $\mathcal{C} = \{M_i : i \in \omega\}$ be an N-dimensional asymptotic class, and $M = \prod_{i \in I} M_i/\mathcal{U}$ be an infinite ultraproduct of members of $\mathcal{C}$. Then M is measurable, with* $(\mathrm{Dim}, \mathrm{Meas})(M) = (N, 1)$.

Proof. We use the definability clause (ii) of Definition 2.1 to define $(\mathrm{Dim}, \mathrm{Meas})(\varphi(\bar{x}, \bar{a}))$ for any $\bar{a} \in M^m$, assigning the appropriate d, μ. Easy asymptotic arguments show that this is indeed a measure. □

It follows that all the classes of examples listed in Examples 2.4 yield corresponding examples of measurable structures. In particular, the random graph is measurable, as it is an ultraproduct of Paley graphs; and any infinite vertex transitive graph of finite valency is measurable.

We mention some further examples and constructions. The first comes from the generic predicate construction of [13].

Theorem 3.10 (5.11 of [43]) *Let T be a complete measurable theory over a language L with quantifier elimination, eliminating the quantifier $\exists^\infty$, such that for all $M \models T$ and $A \subset M$,* $\mathrm{acl}(A) = \mathrm{dcl}(A)$. *Let P be a unary predicate not in L, let $L' := L \cup \{P\}$, let S be a sort of T, and let $T_{P,S}$ be the model companion of the theory of L'-structures satisfying T,* *with P interpreted by a subset of S. Let T' be any completion of $T_{P,S}$. Then T' is measurable.*

In the proof, one uses the measuring function μ_T for T, fixes $p \in \mathbb{R}$ with $0 < p < 1$, and for $M \models T_{P,S}$, assigns to $P(M)$ the measure $p\mu_T(S)$. The partial quantifier elimination of [13, 2.6] makes this measure extendable to *all* definable sets. We expect that the assumption $\mathrm{acl}(A) = \mathrm{dcl}(A)$ is not needed, but it was assumed in [43] to avoid complications.

As noted in [43], it follows from Theorem 3.10 that for any $k \geq 2$, the universal homogeneous k-uniform hypergraph Γ_k (that is, a universal homogeneous structure with a single symmetric irreflexive k-ary relation) is measurable. It is not obvious that for $k \geq 2$ this structure is an ultraproduct of an asymptotic class, as there is no obvious analogue for the Paley graphs of Example 2.4 (4). However, Beyarslan in [6] has shown that for any $k \geq 2$, Γ_k is interpretable in a pseudofinite field, so it must at least be an ultraproduct of a *weak* asymptotic class.

The motivating example of a measurable structure is a pseudofinite field. Any pseudofinite field arises as the fixed field of the automorphism in a generic difference field, that is, in a model of ACFA. This, and the last theorem, suggests the following construction technique for measurable structures, given by Hrushovski in Proposition 11.1 of [30]. The definability of measure was not explicitly stated in [30], but uniqueness was, and as noted in [18, 3.4.2], definability follows from uniqueness by Beth's Theorem.

Theorem 3.11 ([30]) *Let D be any strongly minimal set over a language L, assume that* $\mathrm{Th}(D)$ *has the definable multiplicity property (DMP) and elimination of imaginaries, let σ be a generic automorphism of a sufficiently saturated model of* $\mathrm{Th}(D)$*, and let $K = \mathrm{Fix}(\sigma)$, an L-substructure. Then K is measurable, of dimension 1.*

We remark that under assumption (DMP), by [13, 3.11(2)], there is a generic automorphism (i.e. a model companion of the theory of expansions of D by automorphisms).

Recall Hrushovski's fusion construction [31], which, given two strongly minimal sets M_1, M_2 with (DMP) in disjoint languages L_1, L_2 respectively, yields a new strongly minimal set M in $L_1 \cup L_2$ whose reduct to each L_i is M_i. It is shown in [31] that this construction preserves the (DMP). Thus, there is a generic automorphism σ of M, and $\mathrm{Fix}(\sigma)$

will be measurable. As shown in Section 3.4 of [18], this yields the following, when applied to the fusion of two algebraically closed fields of characteristics p_1 and p_2.

Theorem 3.12 ([18]) *Let L_1, L_2 be disjoint languages for rings, and $L := L_1 \cup L_2$, and p_1, p_2 be distinct primes. Then there is a measurable structure M whose reduct to each language L_i is a pseudofinite field of characteristic p_i.*

Note that M is not elementarily equivalent to an ultraproduct of an asymptotic class, since there do not exist positive integers a_1, a_2 with $p_1^{a_1} = p_2^{a_2}$.

Example 3.13 ([43]) For any field F, any infinite vector space over F, in the language of F-modules, is a measurable structure. In the particular case when F is infinite, such a structure cannot be an ultraproduct of an asymptotic class. The point here, essentially, is that if M is *any* strongly minimal set with (DMP) and definable Skolem functions, then M is measurable, with dimension and measure equal to Morley rank and degree. In vector spaces over an infinite field, this is applied after naming a non-zero vector by a constant symbol.

The following definition is given in [26].

Definition 3.14 Let T be a complete theory. We say that T is *unimodular* if for any $M \models T$, definable sets X, Y in M^{eq}, and definable surjections $f_i : X \to Y$ such that f_i is k_i-to-1 (for $i = 1, 2$, and with k_i a positive integer), we have $k_1 = k_2$.

We say that M is unimodular if $\mathrm{Th}(M)$ is.

Clearly any theory with the finite model property (e.g. an ultraproduct of an asymptotic class) is unimodular. In fact, the following is almost immediate from the Fubini condition.

Proposition 3.15 *Let M be measurable. Then M is unimodular.*

In the main applications of measure seen by the authors so far, only unimodularity (plus finite rank supersimplicity with definability of some dimension or rank) is used.

It follows immediately from Proposition 3.15 that an algebraically closed field cannot be measurable. For example, in characteristic not equal to 2, if F is an algebraically closed field, then the identity map is

1-to-1, but the map $x \mapsto x^2$ is 2-to-1 on $F \setminus \{0\}$. More generally, we have the following theorem. Recall that a sufficiently saturated structure M with supersimple theory is said to be *1-based* if, for any subsets A, B of M^{eq}, $A \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)} B$.

Theorem 3.16 (Theorem 4.2.6 of [18]) *Any unimodular stable theory of finite U-rank (and hence any measurable stable theory) is 1-based.*

The main point in the proof is that by [26] (see also 2.4.15 and 5.3.2 of [48]), every minimal type of the theory will be 1-based. By the coordinatisation of such structures by minimal types, this is sufficient (see 2.5.8 of [48]).

We have the following easy observation.

Proposition 3.17 *Let M be ω-categorical. Then M is unimodular.*

Proof. Suppose that M is ω-categorical, and X, Y are definable sets in M^{eq} and $f_i : X \to Y$ are definable surjections with f_i k_i-to-1 (for $i = 1, 2$, and with $k_i \in \mathbb{N}$). By adding finitely many sorts to M, naming finitely many parameters, and adding dummy variables, we may suppose that X, Y are disjoint subsets of M, and X, Y and the f_i are $\emptyset$-definable. Let $a \in X$, and $D = \mathrm{acl}(a) \cap (X \cup Y)$. Then D is finite, by ω-categoricity, and is closed under the f_i and f_i^{-1}. Thus, f_i induces a k_i-to-1 map from $D \cap X$ onto $D \cap Y$ and it follows by counting that $k_1 = k_2$. □

Itay Ben-Yaacov [3] has investigated the connections between measurable theories and continuous model theory, and in particular, with measure algebras in an unbounded continuous logic. His main result is a measure-theoretic version of the independence theorem [35] for simple theories. A version was proved earlier by very different means, for pseudofinite fields, by Tomašić [59].

We mention also some work of Tomašić [58] which, in the context of pseudofinite fields, makes connections between measure and exponential sums, and ω-categoricity of reducts. It generalises the use of characters in the proof of Example 2.4 (4) for Paley graphs. See also the survey article by Szönyi [56] for combinatoiral applications of related results.

First, consider a measurable structure M with measuring function (d, μ), and a definable set S in M, with induced probability measure μ_S on the definable subsets of S, yielding a measure space $(S, \mathcal{M}_S, \mu_S)$, where $\mathcal{M}_S$ is the σ-algebra of subsets of S generated by the definable subsets. Then if $f : S \to \mathbb{C}$ is measurable with respect to this measure space (with $\mathbb{C}$ equipped with the Euclidean topology), we may form the integral $\int_S f d\mu_S$. In practice, Tomašić is only concerned with functions of the form $f = \Sigma_{i=1}^t \alpha_i \chi_{A_i}$, where the A_i are definable sets partitioning S and $\alpha_i \in \mathbb{C}$, and here, $\int_S f d\mu_S = \Sigma_{i=1}^t \alpha_i \mu_S(A_i)$. Tomašić investigates additive and multiplicative characters on a pseudofinite field F, and proves the following. As elsewhere in this paper, $\tilde{F}$ denotes the algebraic closure of the field F.

Theorem 3.18 ([58]) *Let X be an absolutely irreducible variety over a pseudofinite field F and let f be a rational function on X. Suppose either*

(i) χ is a multiplicative character of F of order $k > 1$ and f is not a k^{th} power of a rational function on $\tilde{X}$ (the corresponding variety over $\tilde{F}$), or

(ii) χ is a nontrivial additive character of F of the form $\chi_{a,1}$ (notation from [58]) and f is not of the form $g^p - g$ for any $g \in \tilde{F}(\tilde{X})$.

Then $\int_{X(F)} \chi \circ f = 0$.

Tomašić then considers reducts of a pseudofinite field F of the following form. Let X be an absolutely irreducible variety over F, let f be a regular function on $X \times X$, and let $\chi : X \to \mathbb{C}$ be a multiplicative character of order $k > 1$ (with the assumption of Theorem 3.18(i)). Given $x, y \in X(F)$, define the binary relation $R_j(x, y)$ to hold if $\chi(f(x, y)) = e^{2\pi ij/k}$. Then there is a binary structure $(X(F), R_0, \ldots, R_{k-1})$ interpretable in F. Ultraproducts of Paley graphs are really a special case, with $X(F) = F$, $f(x, y) = x - y$, and χ the quadratic character. Using Theorem 3.18, Tomašić shows that under certain conditions such reducts are ω-categorical, and their measures, and the corresponding structures in finite fields, share the 'equidistribution' properties of the theorem of Bollobás and Thomason in Example 2.4 (4). In particular, if $X(F) = F$, f is a symmetric polynomial defining a conic, and χ is a quadratic character, then (F, R_0) is ω-categorical.

We mention also work of E. Kowalski [36] on estimates for exponential sums over definable subsets of finite fields. It generalises the corresponding estimates for varieties (Weil, Deligne, and others) and the main theorem of [11].

4 Smoothly approximable structures

The class of *smoothly approximable* structures is a class of ω-categorical supersimple structures of finite rank which properly contains the class of ω-categorical ω-stable structures (so in particular the totally categorical structures). A deep structure theory is developed in [15], which includes, for example, proofs of the equivalence of smooth approximation, Lie coordinatisability, and other notions, such as 'strong 4-quasifiniteness'. In [15] there is also a proof of a version of quasi-finite axiomatisability, a Lachlan-style shrinking and stretching theory, results on definable groups (they must be finite-by-abelian-by-finite) and much else.

Following Definition 2.1.1 of [15], we say that a finite substructure N of a structure M is a *k-homogeneous substructure* of M if all $\emptyset$-definable relations on M induce $\emptyset$-definable relations of N, and for any pair $\bar{a}, \bar{b}$ of k-tuples from N, they have the same type in N if and only if they have the same type in M. An ω-categorical structure M is *smoothly approximated* if it is a union of a chain $(M_i : i \in \omega)$ of finite substructures, where for each i, M_i is an $|M_i|$-homogeneous substructure of M.

Smoothly approximated structures with primitive automorphism groups were classified in [33]. Based on this, a list of rank 1 *Lie geometries* is identified in [15, Section 2.1.2]. Typical examples are vector spaces (or their projective and affine versions) over finite fields, possibly equipped with bilinear forms, but there is a rather more mysterious example, the quadratic geometry. The sorts and languages to handle these geometries are chosen with care in [15] to ensure flexibility in handling slightly different automorphism groups (e.g. semilinear automorphisms, which involve field automorphisms), and quantifier elimination and weak elimination of imaginaries in certain cases. The authors define a *Lie coordinatised structure* to be one built by covering constructions from Lie geometries in a way indexed by a tree of finite height, and then prove that a countable structure is smoothly approximated if and only if it is Lie coordinatisable (i.e. bi-interpretable with a Lie cooordinatised structure).

It is easily seen that the Lie geometries arise as direct limits of 1-dimensional asymptotic classes. For example, given a finite field $\mathbb{F}_q$, let $\mathcal{C}$ be the collection of all finite dimensional vector spaces V over $\mathbb{F}_q$, equipped with a non-degenerate alternating bilinear form $\beta : V \times V \to \mathbb{F}_q$. We may code β by introducing a binary relation symbol R_a for each $a \in F_q$, with $R_a(x, y)$ whenever $\beta(x, y) = a$. Witt's Lemma, which says that partial isometries extend to total isometries, gives quantifier elimination, so we can reduce to considering the cardinalities of sets of the form $\{x : \beta(x, v_1) = a_1 \wedge \ldots \wedge \beta(x, v_r) = a_r\}$, where $a_1, \ldots, a_r \in \mathbb{F}_q$, and $v_1, \ldots, v_r \in V$ are linearly independent. Such a set has size $\frac{1}{q^r}|V|$. Thus, there is some resemblance with the way, in the random graph, 1-types over a finite set $\{x_1, \ldots, x_r\}$ are determined by edges/non-edges to each x_i, with all sets of adjacencies equiprobable (assuming edge probability $\frac{1}{2}$); but unlike in the Paley graphs, we get precise results on sizes of definable sets. The random graph is not smoothly approximable.

Theorem 4.1 (Section 3.3.2 of [18]) *(i) Let M be a smoothly approximable structure. Then M is the union of a sequence of substructures $(M_i : i < \omega)$ such that each M_i is a $|M_i|$-homogeneous substructure of M, and $\{M_i : i \in \omega\}$ is an asymptotic class.*

(ii) Every smoothly approximable structure is measurable.

The M_i are also called *envelopes* in [15]. Theorem 4.1(i) rests on rather precise information in [15, Proposition 5.2.2] on the sizes of definable sets in envelopes; these cardinalities are given by polynomials in certain dimensions. As a result, for smoothly approximated structures for which only one canonical projective geometry is involved in the coordinatisation, the asymptotics for the envelopes are much better than that required in Definition 2.1 – see [18, Proposition 3.3.5].

Part (ii) of Theorem 4.1 follows immediately from (i) and Proposition 3.9, since any smoothly approximable structure satisfies the asymptotic theory of any approximating chain $(M_i : i \in \omega)$ of envelopes.

One question throughout this paper concerns the extent to which the examples of asymptotic classes and measurable structures go beyond finite and pseudofinite fields, and structures interpretable in them. It is conceivable that any smoothly approximable structure is interpretable in a product of pseudofinite fields, in which case Theorem 4.1 does not provide new examples. Certainly, we have

Proposition 4.2 ([43]) *Let M be a smoothly approximable Lie geometry. Then M is interpretable in a pseudofinite field.*

As an example, consider an $\aleph_0$-dimensional vector space V over a finite field $\mathbb{F}_p$, equipped with a non-degenerate symmetric bilinear form $\beta : V \times V \to \mathbb{F}_p$. This is approximated by the family $(V_n : n \geq 1)$, where V_n is an n-dimensional vector space. We may identify V_n with $\mathbb{F}_{p^n}$, viewed as a vector space over $\mathbb{F}_p$. There is a trace map $\mathrm{Tr} : \mathbb{F}_{p^n} \to \mathbb{F}_p$, and we may put $\beta(x, y) = \mathrm{Tr}(xy)$ for any $x, y \in \mathbb{F}_{p^n}$. The trace map is uniformly definable (so definable in the limit). Indeed, its kernel has index p in $(\mathbb{F}_{p^n}, +)$, and is uniformly defined as $\{x^p - x : x \in \mathbb{F}_p^n\}$ (Hilbert's Theorem 90). To define the trace, we just specify its value on each of the p cosets of the kernel.

5 Measure and difference fields

The families of Suzuki and Ree finite simple groups are not uniformly interpretable in finite fields. However, as is clear from their constructions, they are uniformly interpretable in certain finite difference fields. In this and the next section, we describe work of Ryten, based on joint work of Ryten and Tomašić, which ensures that *all* families of finite simple groups of fixed Lie rank form asympotic classes.

Ryten's starting point is the following theorem of Hrushovski.

Theorem 5.1 ([31]) *Let $\tilde{K}$ be an algebraically closed field of characteristic p, let $t \in \mathbb{N}$, and $q = p^t$. Suppose $V(\bar{x})$ is an algebraic variety over $\tilde{K}$, and $W(\bar{x}\bar{z}) \subset V(\bar{x}) \times V^q(\bar{z})$ is an irreducible subvariety, where the polynomials defining V^q are the images under the automorphism $y \mapsto y^q$ of those defining V. Assume $\mathrm{Dim}(W) = \mathrm{Dim}(V) = d$, and that the projection $\pi_1 : W \to V$ is dominant of degree δ and $\pi_2 : W \to V^q$ is quasi-finite of purely inseparable degree δ'. Then there is a constant C depending on the total degree of W such that*

$$\Big||\{\bar{x}\bar{z} \in W(\tilde{K}) : \bar{z} = \bar{x}^q\}| - \frac{\delta}{\delta'} q^d\Big| \leq Cq^{d-\frac{1}{2}}.$$

From this, Hrushovski proves

Theorem 5.2 ([31]) *Any non-principal ultraproduct of difference fields $(\tilde{\mathbb{F}}_p, \mathrm{Frob}^k)$ is a model of ACFA.*

Using these results, Ryten and Tomašić prove the following.

Theorem 5.3 ([52]) *Let $\theta(\bar{x}, \bar{y})$ be a formula in the language of difference rings. Then there is a constant $C \in \mathbb{R}^+$ and a finite set D of pairs (d, μ) with $d \in \mathbb{Z} \cup \{\infty\}$ and $\mu \in \mathbb{Q}^+ \cup \{\infty\}$ such that in each difference field $(\tilde{\mathbb{F}}_p, \mathrm{Frob}^k)$, and for any $\bar{a} \in \tilde{\mathbb{F}}_p^m$,*

$$\big||\theta(\bar{x}, \bar{a})| - \mu p^{kd}\big| \leq Cp^{k(d-\frac{1}{2})}$$

holds for some $(d, \mu) \in D$.

Moreover, for each $(d, \mu) \in D$ there is a formula $\theta_{d,\mu}(\bar{y})$ such that for *each* $(\tilde{\mathbb{F}}_p, \mathrm{Frob}^k)$, *the above estimate holds for* $\theta(\tilde{\mathbb{F}}_p, \bar{a})$ *with* (d, μ) *if and only if* $(\tilde{\mathbb{F}}_p, \mathrm{Frob}^k) \models \theta_{d,\mu}(\bar{a})$.

The proof has a somewhat different presentation in Chapter 2 of [51]. The theorem is derived from Theorem 5.1 rather as Theorem 1.1 is derived from the Lang-Weil estimates. Formulas with quantifiers are handled using Theorem 5.2, together with the following partial quantifier elimination for ACFA.

Proposition 5.4 (1.5 and 1.6 of [12]) *If* $\theta(\bar{x}, \bar{y})$ *is a formula in the language of difference rings, then*

$$ACFA \models \theta(\bar{x}, \bar{y}) \Leftrightarrow \bigvee_{i-1}^{k} \exists t \theta_i(\bar{x}, \bar{y}, t),$$

where $\theta_i = \theta_i(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^\ell(\bar{x}), \bar{y}, \sigma(\bar{y}), \ldots, \sigma^\ell(\bar{y}), t, \sigma(t), \ldots, \sigma^\ell(t))$ *is a quantifier free formula in the language of rings, and for any* $(M, \sigma) \models ACFA$ *and* $(\bar{x}_0, \bar{y}_0, t_0) \in \theta_i(M)$, t_0 *is algebraic (in the sense of fields) over* $\{\sigma^i(\bar{x}_0), \sigma^i(\bar{y}_0) : 0 \le i \le \ell\}$.

Ryten and Tomašić also note that, by Theorem 5.3, in a model of ACFA, the family of *finite dimensional* sets is measurable, under a natural variant of Definition 3.1.

In [51], Ryten uses Theorem 5.3 to investigate an important class of finite difference fields. Fix a prime p, and integers m, n with $m \ge 1$, $n > 1$, and $(m, n) = 1$. Let $\mathcal{C}_{m,n,p}$ be the collection $\{(\mathbb{F}_{p^{kn+m}}, \mathrm{Frob}^k) : k > 0\}$ of finite difference fields. By Frob^k we understand its restriction to $\mathbb{F}_{p^{kn+m}}$, and notice that on this domain it is a solution in σ to the equation $\mathrm{Frob}^m \circ \sigma^n = \mathrm{id}$.

Consider the following conditions on a difference field (K, σ). Below, with m, n as above, if (K, σ) is a difference field of characteristic p with $\mathrm{Frob}^m \circ \sigma^n = \mathrm{id}$, we say that the extension of difference fields $(K, \sigma) \subseteq (L, \sigma')$ is *generic* if $\mathrm{Fix}(\mathrm{Frob}^m \circ \sigma'^n) = K$.

(1) K is a pseudofinite field of characteristic p.

(2) σ is an automorphism of K which satisfies $\mathrm{Frob}^m \circ \sigma^n = \mathrm{id}$.

(3) Suppose $U \subset \mathbb{A}^{nN}$ is an absolutely irreducible variety defined over K and let $\sigma(U)$ be the variety obtained from U by applying σ to the coefficients of the defining polynomials; let the variables of U be $(x_{11} \ldots x_{n1} \ldots x_{1N} \ldots x_{nN})$, those of $\sigma(U)$ be $(y_{11} \ldots y_{n1} \ldots y_{1N} \ldots y_{nN})$. Suppose $V \subset U \times \sigma(U)$ is an absolutely irreducible variety defined over K, whose definition includes the equations $y_{ij} = x_{i+1,j}$ and $y_{nj}^{p^m} = x_{1j}$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, N$. Suppose that V projects generically onto U and $\sigma(U)$, and that W is a K-algebraic set properly contained in V. Then there is a point $x \in V(K) \setminus W(K)$ such that $x = (a, b)$, where $a = (a_{ij:1 \leq i \leq n, 1 \leq j \leq N})$, $b = (b_{ij} : 1 \leq i \leq n, 1 \leq j \leq N)$, $a \in U$, $b \in \sigma(U)$, and $b_{ij} = \sigma(a_{ij})$ for each i, j.

(4) Let $K \subseteq L \subseteq H$ be a tower of finite field extensions. Suppose that $(K, \sigma) \subseteq (L, \sigma')$ is a generic extension of difference fields. Then there is an extension of difference fields $(L, \sigma') \subseteq (H, \sigma'')$ such that (H, σ'') is a generic extension of (K, σ).

As suggested by the referee, it may be more elegant to replace (4) by the statement:

(4') any lifting of σ to $\tilde{K}$ commutes with the action of $\mathrm{Aut}(\tilde{K}/K)$.

It can be shown that (1)–(4) (and (4')) are first order expressible. Furthermore, basic facts about ACFA yield the following.

Proposition 5.5 ([51]) *Let (M, τ) be a model of ACFA of characteristic p, let $K := \mathrm{Fix}(\mathrm{Frob}^m \circ \tau^n)$, and put $\sigma := \tau|_K$. Then $(K, \sigma) \models (1) - (4)$.*

By Proposition 5.5, conditions (1)–(4) axiomatise a theory, denoted $\mathrm{PSF}_{(m,n,p)}$. Ryten proves for $\mathrm{PSF}_{(m,n,p)}$ some results similar to those known for ACFA and PSF. The completions (and types) are described through the following theorem.

Theorem 5.6 ([51]) *Let (F, σ) and (E, τ) be models of $\mathrm{PSF}_{(m,n,p)}$ with a common substructure K (so $\sigma|_K = \tau|_K$). Then $(F, \sigma) \equiv_K (E, \tau)$ if and only if $(F \cap \tilde{K}, \sigma|_{F \cap \tilde{K}}) \cong_K (E \cap \tilde{K}, \tau|_{E \cap \tilde{K}})$.*

From this, near model completeness is proved: that is, for every formula $\varphi(\bar{x})$ there is a formula $\theta(\bar{x})$ which is a boolean combination of existential formulas (in fact, formulas of a rather specific type), such that $\mathrm{PSF}_{(m,n,p)} \vdash \varphi(\bar{x}) \leftrightarrow \theta(\bar{x})$.

Using Theorem 5.2, Ryten proves the following theorem.

Theorem 5.7 ([51]) *$\mathrm{PSF}_{(m,n,p)}$ is the asymptotic theory of the class $\mathcal{C}_{(m,n,p)}$ of finite difference fields.*

We show part of this, that every non-principal ultraproduct

$$(N, \sigma) = \prod_{i \in \mathbb{N}} (\mathbb{F}_{p^{nk_i+m}}, \mathrm{Frob}^{k_i})/\mathcal{U}$$

satisfies $\mathrm{PSF}_{(m,n,p)}$. Indeed, if $(M, \tau) = \prod_{i \in \mathbb{N}} (\tilde{\mathbb{F}}_p, \mathrm{Frob}^{k_i})/\mathcal{U}$, then

$(M,\tau) \models$ ACFA by Theorem 5.2. Also, $N = \mathrm{Fix}(\mathrm{Frob}^m \circ \sigma^n)$ and $\sigma = \tau|_N$, so $(N,\sigma) \models \mathrm{PSF}_{(m,n,p)}$ by Proposition 5.5.

Theorem 5.8 ([51]) *$\mathcal{C}_{(m,n,p)}$ is a 1-dimensional asymptotic class.*

To see this, let $\mathcal{D}_p := \{(\tilde{\mathbb{F}}_p, \mathrm{Frob}^k) : k \in \mathbb{N}\}$, a class of difference fields. There is a bijection $\mathcal{F} : \mathcal{D}_p \to \mathcal{C}_{(m,n,p)}$, with $\mathcal{F}((\tilde{\mathbb{F}}_p, \mathrm{Frob}^k)) = (\mathbb{F}_{p^{kn+m}}, \mathrm{Frob}^k)$. There is a corresponding function ${}^{\mathrm{Fix}}$ defined on the set of formulas of the language L_{diff} of difference rings, defined inductively. If $(K,\sigma) \in \mathcal{D}_p$ and $\mathcal{F}((K,\sigma)) = (M,\sigma) \in \mathcal{C}_{(m,n,p)}$, then for any L_{diff} formula $\varphi(y)$ and tuple $\bar{a}$ from M,

$$(M,\sigma) \models \varphi(\bar{a}) \Leftrightarrow (K,\sigma) \models \varphi^{\mathrm{Fix}}(\bar{a}).$$

From this and Theorem 5.2, the uniform asymptotic estimates for formulas in $\mathcal{C}_{(m,n,p)}$ are rapidly derived. The definability clause (Definition 2.1(ii)) requires a little more work.

Finally, Ryten shows that in a *pure* pseudofinite field (in fact, in any pure bounded PAC field) the only definable field automorphisms are powers of the Frobenius. Thus, $\mathrm{PSF}_{(m,n,p)}$ is a proper expansion of the theory of pseudofinite fields of characteristic p, and $\mathcal{C}_{(m,n,p)}$ is not interpretable in any class of pure finite fields.

In [51, Chapter 3], Ryten proves a number of other results about $\mathrm{PSF}_{(m,n,p)}$, indicating that its complexity is somewhere between that of PSF and ACFA. As for pseudofinite fields in [11], it is possible to add constants to the language to ensure, in the partial quantifier elimination, that only *positive* boolean combinations of existential formulas are used, so model-completeness in the expanded language is obtained. Model-theoretic independence is characterised algebraically. Elimination of imaginaries (over a language with constants for an elementary submodel) is proved.

6 Asymptotic classes of simple groups

There is a natural notion of *uniform parameter bi-interpretability* between two classes $\mathcal{C}$ and $\mathcal{D}$ of finite structures. We do not give it formally, but it requires a matching between $\mathcal{C}$ and $\mathcal{D}$, and that each element of $\mathcal{C}$ is bi-interpretable (not just mutually interpretable) with the corresponding element of $\mathcal{D}$, possibly using parameters, but with uniformity of the interpretation across the families.

The non-abelian finite simple groups, excluding the sporadics, are the alternating groups $\mathrm{Alt}(n)$, the classical groups of Lie rank n over the finite field $\mathbb{F}_q$, and certain exceptional groups, namely $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(g)$, $G_2(q)$, and the twisted groups ${}^3D_4(q)$, ${}^2E_6(q)$, the Ree groups ${}^2F_4(2^{2k+1})$, the Suzuki groups ${}^2B_2(2^{2k+1})$, and the Ree groups ${}^2G_2(2^{3k+1})$. We may regard $\mathrm{Alt}(n)$ as having Lie rank n. We emphasise that the twisted groups (including the classical unitary groups) do not arise over algebraically closed fields, since their definition depends on finite field extensions. They do not have finite Morley rank analogues.

Ryten shows that, for every family of non-abelian finite simple groups of fixed Lie rank, other than those of Suzuki and Ree groups, the family is uniformly parameter bi-interpretable (infact, *bi-definable*) with a class of finite fields. With some additional work, it follows that such a family is an asymptotic class. Care is needed here with the definability clause in Definition 2.1, because of the role of parameters in the bi-interpretation. For a given formula $\varphi(\bar{x}, \bar{y})$, the corresponding $\Phi_{(d,\mu)}$ should be definable *without parameters*.

For the Suzuki and Ree groups, the situation is more complicated. In fact, the classes of groups $\{{}^2F_4(2^{2k+1}) : k \in \mathbb{N}\}$, $\{{}^2B_2(2^{2k+1}) : k \in \mathbb{N}\}$, are uniformly parameter bi-interpretable with $\mathcal{C}_{(1,2,2)}$, and the class $\{{}^2G_2(3^{2k+1}) : k \in \mathbb{N}\}$ is uniformly parameter bi-interpretable with $\mathcal{C}_{(1,2,3)}$. A word of explanation is necessary here. The twisted group ${}^2E_6(q)$, for example, is the fixed point set inside $E_6(q^2)$ of a certain automorphism σ which is a product of a graph automorphism (an automorphism arising from the symmetry of the E_6 Dynkin diagram), and the Frobenius. All this data is definable in the field $\mathbb{F}_{q^2}$, with which ${}^2E_6(q)$ is bi-definable, uniformly in q. However, in the case ${}^2G_2(3^{2k+1})$, for example, the automorphism σ is a product of a graph automorphism and the automorphism $x \mapsto x^{3^k}$, a proper (unbounded) power of the Frobenius. So to define ${}^2G_2(3^{2k+1})$ uniformly, it is necessary to be able to define the automorphism Frob^k of $\mathbb{F}_{3^{2k+1}}$, and hence to work in $\mathcal{C}_{(1,2,3)}$.

It is hard to identify the right sources for the model-theoretic relationship between finite simple groups and fields. In work in his PhD thesis, not subsequently published, Thomas showed definability of the corresponding (pure) field in each finite simple group, uniformly across each class of groups. There are related results in, for example, [39]. The interpretation of the groups in the (difference) fields is pretty clear. For the Suzuki and Ree groups it seems to have been known to Hrushovski for a long time. Some consequences (e.g. decidability) are mentioned in the introduction of [31].

The above results, together with Theorem 5.8, yield the following theorem of Ryten.

Theorem 6.1 ([51]) *Any family of non-abelian finite simple groups of fixed Lie rank is an asymptotic class.*

We mention two further results, beyond the measurable/asymptotic class context. A group is *pseudofinite* if it is an infinite model of the theory of finite groups.

Theorem 6.2 ([44]) *Let G be a stable pseudofinite group. Then G has a definable soluble subgroup of finite index.*

The key point is that a non-principal ultraproduct of finite simple groups of fixed Lie rank, though supersimple, is not stable, as a pseudofinite field is interpretable in it. The proof of the theorem below rests on this, together with slightly delicate arguments with chain conditions, and basic facts about finite nilpotent groups. It is not possible here to strengthen 'soluble' to nilpotent: as noted independently by Khélif and Zilber, and not published, there is a metabelian pseudofinite stable group which is not nilpotent-by-finite.

Theorem 6.3 (Wilson [61]) *Every pseudofinite simple group is elementarily equivalent to a Chevalley group (possibly twisted) over a pseudofinite field.*

This builds on earlier work of Felgner. By Ryten's work, 'elementarily equivalent' can be strengthened to 'isomorphic' in Theorem 6.3. In [50], Point proves that an ultraproduct of simple Chevalley groups of fixed type (possibly twisted) is *isomorphic* to a Chevalley group over the ultraproduct of the fields, and is simple.

7 Groups of low dimension

Theorem 6.1 above yields immediately the converse to Conjecture 1.2: any Chevalley group (possibly twisted) over a pseudofinite field is measurable. In this section we sketch beginnings of a general structure theory for measurable groups, with a view to Conjecture 1.2. Many of the results are jointly due to Elwes and Ryten. They can be found in [18], and a joint paper [19] is in preparation.

The central definability result for groups of finite Morley rank is the

Zilber Indecomposability Theorem, which generalises the Indecomposability Theorem for algebraic groups. In the supersimple case, we have the following, proved first by Hrushovski [30] for S_1-theories, that is, supersimple theories of finite S_1-rank in which the S_1-rank is definable.

Theorem 7.1 (Wagner[60] Theorem 5.4.5) *Let G be a definable group in a supersimple theory of finite rank, and let X_i $(i \in I)$ be definable subsets of G. Then there is a definable subgroup H of G, and $n \in \omega$, and $i_1, \ldots, i_n \in I$, where*
(i) $H \leq X_{i_1}^{\pm 1} \cdot X_{i_2}^{\pm 1} \cdot \ldots \cdot X_{i_n}^{\pm 1}$
(ii) X_i/H *is finite for each* $i \in I$.

Proof. This follows from 5.4.5 and 5.5.4 of [60], by compactness. □

This is used repeatedly in arguments described below. It also yields, for example, that any non-abelian definably simple measurable group is simple, and that in a measurable theory, the derived subgroup of a definable group is always definable.

A second useful tool, already heavily used by Wagner for groups in supersimple theories, is Schlichting's Theorem [54], proved independently by Bergman and Lenstra [5]. A family of subgroups $\mathcal{H}$ is *uniformly commensurable* if there is $n \in \mathbb{N}$ bounding $|H : H \cap K|$ for all $H, K \in \mathcal{H}$. The version below is Theorem 4.2.4 of [60].

Theorem 7.2 ([54], [5]) *Let G be a group and $\mathcal{H}$ a uniformly commensurable family of subgroups. Then there is a subgroup N which is uniformly commensurable to $\mathcal{H}$ and is invariant under all automorphisms of G which stabilise $\mathcal{H}$ setwise. In particular, if $\mathcal{H}$ consists of all conjugates of some $H \leq G$, then N is normal in G.*

It is well-known that every rank 1 superstable group is abelian-by-finite. In the supersimple case, one expects 'abelian-by-finite' to be replaced by 'finite-by-abelian-by-finite', in view of the extraspecial groups considered in Example 2.4(3). However, this is an open question. The following result of Elwes and Ryten is proved in [18, 6.0.11]; it was proved under the stronger assumption of measurability in [43].

Theorem 7.3 ([18]) *If G is unimodular supersimple of rank 1, then G has a definable normal subgroup H of finite index, such that H has a finite central subgroup Z, with H/Z abelian.*

The proof is a counting argument with unimodularity. It uses the fact that if G is a BFC group (a group with a finite bound on the size of its conjugacy classes) then the derived subgroup G' is finite; see Theorem 3.1 of [47]. The key to the counting argument is the following lemma, proved in [43] under measurability; the proof under unimodularity is easily extracted from that of [18, 6.0.11]. It eliminates the possibility of measurable groups with finitely many non-identity conjugacy classes, all of full dimension.

Lemma 7.4 ([18], [19]) *Let G be a group defined in a supersimple unimodular theory of finite rank. Then there is $g \in G \setminus \{1\}$ such that $C_G(g)$ is infinite.*

In [43], a cruder argument is given to prove the asymptotic class analogue of Theorem 7.3. It states that any 1-dimensional asymptotic class of *finite* groups consists of groups which are bounded-by-abelian-by-bounded. The argument applies to any class of finite groups all of whose ultraproducts are supersimple of rank 1.

These results, and the analogue for superstable groups, suggest that any measurable 2-dimensional group should be soluble-by-finite. This question is open, but some progress in this direction was made in [19]. Using Theorems 7.3 and 7.1 (to replace any 1-dimensional normal subgroup by a definable one), Elwes and Ryten show that any counterexample interprets a simple group G of dimension and S_1-rank 2. Further information is obtained: for example, the group G must have conjugacy classes (in fact, infinitely many) of dimension 1, and it must also have at least 1, but at most finitely many, conjugacy classes of dimension 2. Such examples could not arise as ultraproducts of an asymptotic class, by the classification of finite simple groups. Thus, they prove the following; a proof can be found in [18] or [19].

Theorem 7.5 *Let $\mathcal{C}$ be a 2-dimensional asymptotic class of groups. Then there is $d \in \mathbb{N}$ such that each group G in $\mathcal{C}$ has a subgroup of index at most d which is soluble of derived length at most 4 (and uniformly definable in the class).*

Elwes and Ryten have also investigated measurable structures (G, X), where G is a group with a definable faithful action on X. The intended analogy is with [42], on the structure of primitive permutation groups of finite Morley rank (see also [9] in this volume). More specifically, they generalise the theorem of Hrushovski [25], that in a stable theory, a connected group of finite Morley rank acting definably and transitively on a strongly minimal set must have Morley rank at most 3, and either acts regularly (so is a strongly minimal group) or is of form $\mathrm{AGL}(1, F)$ or $\mathrm{PSL}(2, F)$ in the natural action (F an algebraically closed field).

A permutation group G on a set X is *primitive* if there is no proper non-trivial G-invariant equivalence relation on X, or equivalently if each stabiliser G_x ($x \in X$) is a maximal subgroup of G. If G, X, and the G-action on X are definable in some structure, we say that (G, X) is *definably primitive* if there is no *definable* proper non-trivial G-invariant equivalence relation on X, or equivalently if point stabilisers are definably maximal. Definable primitivity implies transitivity, as the orbit equivalence relation is definable. We shall say that (G, X) is a *measurable group action* if G, X, and an action of G on X are all definable in a measurable structure. It is an *asymptotic group action* if it is elementarily equivalent to an infinite ultraproduct of an asymptotic class of group actions.

Proposition 7.6 ([19]) *Let (G, X) be a measurable faithful group action, with G infinite, and suppose that G acts faithfully and definably primitively on X. If* $\mathrm{Dim}(X) < \mathrm{Dim}(G)$, *then G is primitive on X.*

The proof uses Theorems 7.2 and 7.1, and in fact 'measurable' can be weakened to 'supersimple, finite rank, and eliminates $\exists^\infty$'. To start the proof (and illustrate a useful argument), define $\sim$ on X, putting

$$x \sim y \Leftrightarrow |G_x : G_x \cap G_y| < \infty,$$

(where G_x denotes the stabiliser of $x \in X$). Then $\sim$ is a G-invariant equivalence relation on X, so is definable, so either there is a single $\sim$-class, or $\sim$-classes are singletons. The first case is easily eliminated. For if there is a single $\sim$-class, then (because measurable theories eliminate the quantifier $\exists^\infty$), there is a fixed upper bound on the indices $|G_x : G_x \cap G_y|$ over all $x, y \in X$. Thus, by Theorem 7.2, there is $N \triangleleft G$ uniformly commensurable to all the G_x, and the proof of 7.2 yields that N is definable. Since N is normal and non-trivial, it is transitive on X. However, as N is commensurable with G_x, the orbit of x under N is finite, a contradiction as X is infinite.

Theorem 7.7 ([19]) *Let (G, X) be an asymptotic group action which is faithful and definably primitive, with* $\mathrm{Dim}(X) = 1$. *Then* $\mathrm{Dim}(G) \leq 3$, *and one of the following holds.*

(i) $\mathrm{Dim}(G) = 1$*, and* G *has a definable subgroup* B *of finite index which is torsion-free divisible abelian and acts regularly on* X*. (In this situation, for the description we only require measurability, not an asymptotic group action.)*

(ii) $\mathrm{Dim}(G) = 2$*. In this case, there is a definable pseudofinite field* K*, and a definable infinite subgroup* T *of the multiplicative subgroup* K^**, such that* G *is isomorphic to* $K^+ \rtimes T$ *(a subgroup of* $\mathrm{AGL}(1, K)$*) in its natural action on* $(K, +)$*).*

(iii) $\mathrm{Dim}(G) = 3$*. Here* G *has a unique minimal normal subgroup* T*, which is definable and isomorphic to* $\mathrm{PSL}_2(K)$ *(*K *a pseudofinite field) and its action on* X *is its natural action on the projective line.*

A non-regular example in (i) would be $K^+ \rtimes \{+1, -1\}$, where K is a pseudofinite field in characteristic 0. For (ii), one might take K to be an ultraproduct of fields $\mathbb{F}_p$ ($p \equiv 1 \pmod 4$) and T to be the squares of K.

Sketch of the Proof. First, if $\mathrm{Dim}(G) = 1$, then G is finite-by-abelian-by-finite by Theorem 7.3, and in particular G has a subgroup B of finite index defined as the union of the finite conjugacy classes of G. As $B \lhd G$, B is transitive (the orbits of B would yield a definable G-invariant partition of X). Also the derived subgroup B' is finite, so trivial by definable primitivity, so B is abelian, and hence acts regularly on X. The remaining analysis in this case is easy.

If $\mathrm{Dim}(G) = 2$, then by Theorem 7.5, G is soluble-by-finite. A minimal normal subgroup A of G will be abelian, and transitive on X (by primitivity); as A is abelian it acts regularly on X. It follows that we may identify A with X and G with a semidirect product $A \rtimes G_1$, where G_1 is the stabiliser of the identity of A (in its action by conjugation). For the full description in (ii), a version of the Zilber Field Theorem is used.

Finally, suppose that $\mathrm{Dim}(G) \geq 3$. In this case one may apply the O'Nan-Scott Theorem, the standard reduction theorem for finite primitive permutation groups; see [40] for a careful account of it. This, and the fact that $\mathrm{Dim}(X) = 1$, reduces us to a situation where G has a unique minimal definable normal subgroup T, and T is simple and $T \leq G \leq \mathrm{Aut}(T)$. A further analysis, using extensive information about finite simple groups, reduces to the case when $T = \mathrm{PSL}(2, K)$ (K a pseudofinite field) in its action on the projective line. Tools used include Aschbacher's description of maximal subgroups of finite classical groups [1].

As with Theorem 7.5, it would be very nice to have a version of Theorem 7.7 for groups with measurable theories, but the above proof makes heavy use of finite group theory.

We mention one further result which can be proved under the assumption of measurability, but is open under just supersimplicity.

Theorem 7.8 ([43]) *Let G be an ω-saturated measurable group. Then G has an infinite abelian subgroup.*

In the proof, one looks at a counterexample G of minimal dimension. The exponent must be finite (as otherwise there are infinite cyclic subgroups), and by the minimality assumption, every definable subgroup is finite or of finite index. Let $N := \{g \in G : g^G \text{ is finite}\}$. Then N is finite or of finite index. But $N \neq \{1\}$, for by Lemma 7.4 there is $g \in G \setminus \{1\}$ with $C_G(g)$ infinite; then $|G : C_G(g)|$ is finite, so $g \in N$. If N is infinite, then we may assume that $N = G$, so $|G : C_G(g)|$ is finite for all $g \in G$, and it is then easy to construct an infinite abelian subgroup. Finally, if N is finite but non-trivial, apply similar arguments to G/N.

Wagner has noted that a similar proof yields that any measurable group has an infinite *definable* finite-by-abelian subgroup.

8 Further questions

If M is a 1-dimensional measurable structure, then it has SU-rank 1, and so algebraic closure gives a pregeometry. It is natural to ask whether this pregeometry satisfies the Zilber Trichotomy: trivial, locally modular (so 1-based) non-trivial, and field-like. A trivial geometry arises for the random graph or any vertex transitive graph of finite valency. Smoothly approximable Lie geometries are locally modular, and, except for a pure set, non-trivial. And pseudofinite fields (as well as ultraproducts of classes $\mathcal{C}_{m,n,p}$ and some examples arising from Theorem 3.12) are non locally modular.

It should be possible to construct a non locally modular 1-dimensional measurable structure which does not interpret an infinite group. Let M be the saturated, strongly minimal set constructed in [27], which is known to have (DMP), is not locally modular, and has no infinite interpretable group. Then M admits a generic automorphism σ. It seems that a variant of Theorem 3.12 for finite Morley rank, mentioned in Remark (viii) at the end of [30], should apply in M^{eq} (to ensure elimination of imaginaries), and yield that if $N = \mathrm{Fix}(\sigma)$ then the induced structure on N is measurable of dimension 1. We can ensure that N is not 1-based: for algebraic closure in N is the same as that induced from M, and (M, id) embeds in some model (M', σ') of the theory of generic automorphisms of models of $\mathrm{Th}(M)$, so that $N := \mathrm{Fix}(\sigma')$ contains M, so contains a witness to non 1-basedness. Also, an infinite definable group in N would yield, by a group configuration argument, an infinite definable group in M, which is impossible.

However, we ask

Question 1. If M is a 1-dimensional measurable structure which is a non-principal ultraproduct of an asymptotic class, and M is not 1-based, must M interpret an infinite field?

The question can also be asked for structures interpretable in PSF. In the same vein, we ask whether measurability can be used to sharpen the group configuration results of [4].

Question 2. Is every ω-categorical measurable structure 1-based?

For this, a key example is Hrushovski's construction in [29] of an ω-categorical non-locally modular supersimple rank 1 structure. It is unimodular, by Proposition 3.17, but we do not know whether it is measurable. There is more chance of a positive answer to Question 2 for structures interpretable in PSF. On Question 2, Elwes [18, 5.2.3] has obtained some partial results analogous to Proposition 8 of [26].

Question 3. Is every measurable field pseudofinite?

By the results of Ax [2], a field F is pseudofinite if and only if it is perfect, quasifinite (i.e. has absolute Galois group $\hat{\mathbb{Z}}$), and satisfies the PAC condition; the latter asserts that any absolutely irreducible variety defined over F has an F-rational point. It is straightforward that any supersimple field of finite rank is perfect. By an argument of Scanlon ([43, Theorem 5.18 and Appendix], see also [53]), any measurable field is quasifinite, that is, has a unique extension of degree n for each n. We do not know whether every measurable field satisfies the PAC condition. It was shown in [49] that any supersimple division ring is commutative and has absolute Brauer group, so the norm from any finite extension to the field is surjective. By [45], any generic elliptic or hyperelliptic curve defined over a supersimple field F has an F-rational point. If F is supersimple and has a unique quadratic extension (e.g. if F is measurable), then by [46] *any* elliptic curve defined over F has an F-rational point.

Halupczok [23] has investigated a weakening of measure, where, roughly speaking, the Fubini property is dropped but measure is assumed to be invariant under definable bijections. He has shown that for perfect PAC fields with procyclic absolute Galois group (i.e. with at most one extension of each finite degree) there is such a measure, but that in general there is no such measure.

Question 4. Is every measurable simple group a (possibly twisted) Chevalley group over a pseudofinite field?

Of course, Question 4 requires a positive answer to Question 3. Question 4 looks difficult, but it has a positive answer, in unpublished work of Hrushovski, for groups definable in Ryten's theories $\mathrm{PSF}_{(m,n,p)}$ of measurable pseudofinite difference fields. In fact, Hrushovski classifies infinite definable simple groups in ACFA, without use of the classification of finite simple groups. Dello Stritto [personal communication] has partial results which, modulo a positive answer to Question 3 and an analogue for measurable difference fields, are likely to answer Question 4 positively for groups with a BN pair of rank at least 2.

It would also be interesting to obtain, without use of the classification, structural results on asymptotic classes of finite simple groups.

Hrushovski (Appendix to [30]) has suggested that one might use results on measurable groups, in conjunction with a version of Theorem 3.11 for structures of finite Morley rank, to study groups of finite Morley rank.

Question 5. Is every 2-dimensional measurable group soluble-by-finite? Equivalently, is every non-abelian infinite measurable simple group of dimension at least 3?

Following Proposition 4.2, we ask:

Question 6. Is every smoothly approximable structure interpretable in a pseudofinite field?

It would be helpful to compare measurability with some similar notions in the literature. One such is unimodularity, and Proposition 3.17 suggests the following.

Question 7. Is every ω-categorical supersimple (finite rank) theory measurable?

An example to try would again be Hrushovski's construction from [29].

Question 8. (Macintyre) Is there a natural example of a measurable structure in which some measures are transcendental numbers?

We finish with some comments on other counting and measure principles.

There is a notion of measure investigated by Keisler in [34], and more recently in [32]. The authors of the latter consider a sort X in a sufficiently saturated model $\bar{M}$, and define a Keisler measure on X to be a finitely additive probability measure on $\mathrm{Def}_1(X)$, the collection of parameter-definable subsets of X. A type, for example, is just a zero-one measure. Here we are not requiring a measure on subsets of all powers of M, so there is no Fubini-like assumption. If M is measurable, then for any sort S, there is a definable Keisler measure on S: the two measures will agree for definable subsets of S of dimension $\mathrm{Dim}(S)$, and sets of lower dimension will have Keisler measure 0.

Keisler measure behaves particularly well for theories with the NIP (that is, theories without the independence property) so is orthogonal to the context of the present paper. For example, the authors show that if T has the NIP, and μ is a Keisler measure on X, then there are boundedly many $\sim_\mu$-classes of definable subsets of X, where $Y \sim_\mu Z$ whenever $\mu(Y \triangle Z) = 0$. They investigate existence and uniqueness of Keisler measures on groups.

Next, model-theoretic ideas of Euler characteristic were developed initially for o-minimal theories (see e.g. [17]) but more generally by Krajíček [37] and later Krajíček and Scanlon [38]. The latter define a *strong ordered Euler characteristic* on a structure M to be a function $\chi : \mathrm{Def}(X) \to R$, where R is a partially ordreed ring, the image of χ takes values amone the non-negative elements of R, and we have

(a) $\chi(X) = \chi(Y)$ if X are in definable bijection,

(b) $\chi(X \times Y) = \chi(X).\chi(Y)$,

(c) $\chi(X \cup Y) = \chi(X) + \chi(Y)$ if $X \cap Y = \emptyset$, and

(d) $\chi(E) = c\chi(B)$ if $s : E \to B$ is a definable function and $c = \chi(f^{-1}(b))$ for each $b \in B$.

The Euler characteristic is *non-trivial* if $0 < 1$ in R and the image of χ is not just $\{0\}$.

Unlike for our notion of measure, there is no associated dimension, and indeed, the analogue of (c) for measure only holds if $\mathrm{Dim}(X) = \mathrm{Dim}(Y)$. However, under this definition very similar counting arguments are available. For example, Scanlon (see the Appendix of [43]) has shown that any field with strong ordered Euler characteristic is perfect and quasifinite, and these conclusions follow by the same argument for measurable fields.

Other questions about measure, and variations on measure are suggested by Hrushovski at the end of [30]. For example, there is a suggestion to consider finitely additive measures into $\mathbb{Q}[T]$, which could incorporate the current notion of dimension (exponent of leading term) and measure (coefficient of leading term).

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