Asymptotic classes and measurable structures.

§ Asymptotic classes.

Def: Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. then the asymptotic theory of $\mathcal{C}$ is the set of all sentences which hold in all but finitely many members of $\mathcal{C}$.

Equivalently, it consists the sentences which hold in all non-principal ultraproducts of members of $\mathcal{C}$.

- Recap the measure and dimension in pseudofinite fields.

Thm: Let $\varphi(\bar{x};\bar{y}) \in \mathcal{L}_{ring}$ with $\bar{x} = (x_1, \cdots x_n)$. Then there is a constant C and a finite set of pairs $(d,\mu) \in \{0,1,\cdots,n\} \times \mathbb{Q}^{>0} \cup \{(0,0)\}$ s.t.

(i) for any finite fields $\mathbb{F}_q$ and $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$, there is $(d,\mu) \in D$, with

$$\left| |\varphi(\mathbb{F}_q^n, \bar{a})| - \mu q^d \right| < C q^{d-\frac{1}{2}}. \quad (*)$$

(ii) And for each $(d,\mu) \in D$, there is $\psi_{(d,\mu)}(\bar{y})$ $\emptyset$-def. s.t. for any $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$, $(*)$ holds iff $\mathbb{F}_q \models \psi_{(d,\mu)}(\bar{a})$.

Rmk: There is a notion of one-dimensional asymptotic class of finite structures, defined by (Macpherson and Steinhorn, 2008), which takes exactly this thm above as definition except $\mu \in \mathbb{R}^{>0}$.

- N-dimensional asymptotic classes. (Elwes, 2005)

Def. Let $N \in \mathbb{N}^{>0}$, $\mathcal{C}$ be a class of finite $\mathcal{L}$-str. where $\mathcal{L}$ is a finite language. We say that $\mathcal{C}$ is an N-dimensional asymptotic class if the following holds.

(i) For any $\mathcal{L}$-formula $\varphi(\bar{x};\bar{y})$, $\bar{x} = (x_1, \cdots, x_n)$, there is a finite set of pairs $D \subseteq \{0, \cdots, Nn\} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ and for each $(d,\mu) \in D$ a collection $\Phi(d,\mu)$ of pairs of the form $(M, \bar{a})$ where $M \in \mathcal{C}$ and $\bar{a} \in M^{|\bar{y}|}$, s.t. $\{\Phi(d,\mu), (d,\mu) \in D\}$ is a partition of $\{(M,\bar{a}) : M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$ and

$$\left| |\varphi(M^n, \bar{a})| - \mu |M|^{\frac{d}{N}} \right| = o(|M|^{\frac{d}{N}})$$

as $|M| \to \infty$ and $(M,\bar{a}) \in \Phi(d,\mu)$.

(ii) Each $\Phi(d,\mu)$ is $\emptyset$-definable, i.e. there are formulas $\psi_{(d,\mu)}(\bar{y})$ over $\emptyset$, such that

$(M,\bar{a}) \in \Phi(d,\mu)$ iff $M \models \psi_{(d,\mu)}(\bar{a})$.

Rmk: $o(|M|^{\frac{d}{N}})$ means

$\forall \varepsilon > 0$, $\exists N_\varepsilon > 0$, s.t. $\forall |M| > N_\varepsilon$ and $(M,\bar{a}) \in \Phi(d,\mu)$

$$\left| |\varphi(M^n, \bar{a})| - \mu |M|^{\frac{d}{N}} \right| < \varepsilon \cdot |M|^{\frac{d}{N}}.$$

- Examples.

(1) The class of finite fields.

one-dimensional asymptotic class.

(2) Heisenberg groups over finite fields.

$H_q := \left\{ \begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ a,b,c \in \mathbb{F}_q \right\}$

Note $|H_q| = q^3$. $\to$ dim 3 sets.

Let $\mathcal{C}_H := \{ H_q, \ q = p^n, p \text{ prime}, n \in \mathbb{N}^{>0} \}$.

It's 3-dimensional asymptotic class.

- $Z(H_q) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ c \in \mathbb{F}_q \right\}$

  $|Z(H_q)| = q$. $\to$ dim 1 sets.

- Let $a_q \in \mathbb{F}_q \setminus \{0\}$ def. $g_q := \begin{pmatrix} 1 & a_q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  $C_{H_q}(g_q) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a, c \in \mathbb{F}_q \right\}$

  $|C_{H_q}(g_q)| = q^2$. $\to$ dim 2 sets.

As H_q is definable from the field str. $\mathbb{F}_q$, H_q cannot have more def. sets than $\mathbb{F}_q$. Hence all def. sets of H_q has size approximately $\mu \cdot q^d$ and is indeed a 3-dimensional asymptotic class.

(3) Paley graphs.

Def. Let q be a prime power with $q \equiv 1 \pmod 4$. A Paley graph P_q has vertex set the finite field $\mathbb{F}_q$ and $E(a,b)$ iff $a-b$ is a squre.

Fact: The class of Paley graph is a one-dimensional asymptotic class.

Why? P_q is definable from the field structure $\mathbb{F}_q$.

[illegible]

Rmk: E is symmetric iff -1 is a squre in $\mathbb{F}_q$ iff $q = 2^n$ but $x \mapsto x^2$ is an automorphism.
or $q \equiv 1 \pmod 4$)

Random graph axiom:

Thm. (Bollbás and Thomason, 1981)
If U, W are disjoint set of vertices of P_q. Let $m := |U \cup W|$ and $\nu(U, W)$ be the set of vertices of P_q joined to everything in U and to nothing in W.
then $$\big|\, |\nu(U,W)| - 2^{-m} q \,\big| \le \frac{1}{2}(m - 2 + 2^{-m+1}) q^{\frac{1}{2}} + \frac{m}{2}.$$

Cor: The theory of random graph RG is the asymptotic theory of the class of Paley graphs.

Rmk: RG has quantifier elimination, hence all formulas with one-variable are disjunctions of formulas $\bigwedge_i xEy_i \wedge \bigwedge_j \neg xEz_j$.
Use Bollbás-Thomason and inclusion exclusion principal we can estimate all sets defined by formulas in one-variable.

(4) Finite cyclic groups.

$(\mathbb{Z}/n\mathbb{Z}, +) = (\{0, \cdots, n-1\}, + \pmod n)$

Fact: The class of finite cyclic groups is a one-dimensional asymptotic class.

Example: $\varphi(x) := \exists z (x = z + z)$

$|\varphi(\mathbb{Z}/n\mathbb{Z})| = n$ if n is odd

$\frac{n}{2}$ if n is even.

$\psi_{(1,\frac{1}{2})} := \exists y \neq 0 \; y+y=0$

$\psi_{(1,1)} := \neg \psi_{\frac{1}{2}}$.

- Properties of N-dimensional asymptotic classes

- Properties of $(d,\mu) \to (\dim, \mathrm{meas})$

1. Disjoint union

Sps $\varphi_1(\bar{x})$ get (d_1, μ_1) and

$\varphi_2(\bar{x})$ get (d_2, μ_2) and

$\varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ is inconsistent.

$\varphi(\bar{x}) := \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$ has

$$(\dim, \mathrm{meas}) = (d_1,\mu_1)+(d_2,\mu_2) := \begin{cases} (d_1, \mu_1) & \text{if } d_1 > d_2 \\ (d_2, \mu_2) & \text{if } d_1 < d_2 \\ (d_1, \mu_1+\mu_2) & \text{if } d_1 = d_2. \end{cases}$$

$$|\varphi(M^{|\bar{x}|})| = |\varphi_1(M^{|\bar{x}|})| + |\varphi_2(M^{|\bar{x}|})|$$
$$\approx \mu_1 |M|^{\frac{d_1}{N}} \pm \varepsilon |M|^{\frac{d_1}{N}} + \mu_2 |M|^{\frac{d_2}{N}} \pm \varepsilon |M|^{\frac{d_2}{N}}$$

2. Projection and fiber.

$\varphi(\bar{x}, \bar{z})$

Sps $\exists \bar{x}\, \varphi(\bar{x}, \bar{z})$ has (d, μ)

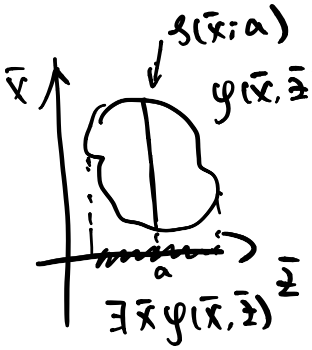
and $\forall \bar{z}$, when $\exists \bar{x}\, \varphi(\bar{x}, \bar{z})$ holds,

$S(\bar{x}; \bar{z}) := \varphi(\bar{x}, \bar{z})$ has (e, ν)

then $(\dim, \text{meas})(\varphi) = (\ell + d, \mu\cdot\nu)$

$(d,\mu)\cdot(\ell,\nu) := (\ell+d, \mu\cdot\nu)$

order (d,μ) lexicographically. We get an ordered semi-ring.



$$\mu|M|^{\frac{d}{N}} - \varepsilon|M|^{\frac{d}{N}} \le |\exists\bar{x}\,\varphi(\bar{x}, M^{|\bar{z}|})| \le \mu|M|^{\frac{d}{N}} + \varepsilon|M|^{\frac{d}{N}}$$

$$\nu|M|^{\frac{\ell}{N}} - \varepsilon|M|^{\frac{\ell}{N}} \le |\varphi(M^{|\bar{x}|}, a)| \le \nu|M|^{\frac{\ell}{N}} + \varepsilon|M|^{\frac{\ell}{N}}$$

$$\mu\cdot\nu|M|^{\frac{d+\ell}{N}} - \varepsilon(\mu+\nu-\varepsilon)|M|^{\frac{d+\ell}{N}} \le$$

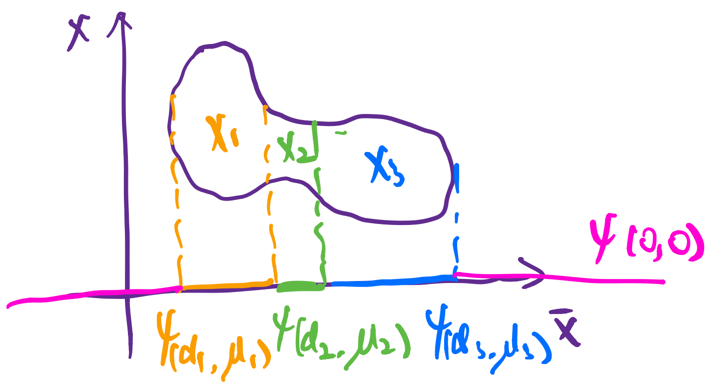
$$|\varphi(M^{|\bar{x}|+|\bar{y}|})| \le \mu\cdot\nu|M|^{\frac{d+\ell}{N}} + \varepsilon(\mu+\nu+\varepsilon)|M|^{\frac{d+\ell}{N}}$$

Thm. Sps. $\mathcal{C}$ is a class of finite structures which satisfies the definition of N-dimensional asymptotic class [clauses (i) and (ii)] for $n=1$. i.e, definable sets in 1-variable. Then $\mathcal{C}$ is an N-dimensional asymptotic class.

An easy case of the proof:

Assume $\varphi(x,\bar{x})$ with no parameter $\emptyset$-def. $\bar{x} = (x_1, \cdots x_n)$

Let $g(x;\bar{x}) := \varphi(x,\bar{x})$ by $n=1$ case, there are a finite set $D \subseteq \{0,\cdots,N\}\times\mathbb{R}^{>0} \cup \{(0,0)\}$ and $\emptyset$-def formulas $\psi_{d,\mu}(\bar{x})$, $(d,\mu)\in D$ which form a partition of $\bar{x}=\bar{x}$ and the fibers $\varphi(x;\bar{x})$ has constant (dim, meas) $=(d,\mu)$ for those $\bar{x}$ satisfies $\psi_{(d,\mu)}(\bar{x})$.



$$\varphi(x,\bar{x}) = \bigvee_{\substack{(d_i,\mu_i)\\ \in D}} \underbrace{\psi_{(d_i,\mu_i)}(\bar{x}) \wedge \varphi(x,\bar{x})}_{\varphi_i(x,\bar{x})}$$

Assume $\psi_{(d_i,\mu_i)}(\bar{x})$ has (dim, meas) $=(\ell_i,\nu_i)$

then $\varphi_i(x,\bar{x})$ has (dim, meas) $=(d_i+\ell_i,\ \mu_i\nu_i)$

and (dim, meas)$(\varphi) = \Big(\max_i d_i+\ell_i =: d,$

$$\sum_{d_j+\nu_j=d} \mu_j\nu_j\Big)$$

$\square$

Thm. Let $\mathcal{C}$ be an N-dimensional asymptotic class and M be an infinite ultraproduct of members of $\mathcal{C}$. Then $\mathrm{Th}(M)$ is supersimple of rank bounded by N.

§ Measurable structures.

- Generalisation of pseudofinite fields or of ultraproducts of members of N-dimensional asymptotic class.

Def. An infinite $\mathcal{L}$-structure M is measurable if there is a function $h: \mathrm{Def}(M) \to \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ (we write $h(X) = (\dim, \mathrm{meas})(X)$) such that the following hold.

1. For each $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ there is a finite set $D_\varphi \subset \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$, so that for any $\bar{a} \in M^{|\bar{y}|}$, we have $h(\varphi(M^n, \bar{a})) \in D_\varphi$
   h takes only finitely-many value for a def. family.
2. If $\varphi(M^n, \bar{a})$ is finite, then $h(\varphi(M^n, \bar{a})) = (0, |\varphi(M^n, \bar{a})|)$.
3. For every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and all $(d, \mu) \in D_\varphi$, the set $\{\bar{a} \in M^n,\ h(\varphi(M^n, \bar{a})) = (d, \mu)\}$ is $\emptyset$-def.
   definability of $(\dim, \mathrm{meas})$.
4. (Fubini) Let $X, Y \in \mathrm{Def}(M)$ and $f: X \to Y$ be a definable surjection. Then there are $r \in \omega$ and $(d_1, \mu_1), \cdots, (d_r, \mu_r) \in \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ s.t. if $Y_i := \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = (d_i, \mu_i)\}$, then $Y = Y_1 \cup Y_2 \cdots \cup Y_r$ is a partition of Y into non empty disjoint definable sets. (so far follows from 1. & 3.)

Let $h(Y_i) = (e_i, \nu_i)$ for $i \in \{1, \cdots, r\}$. Let

$C = \max_{1 \le i \le r} d_i + e_i$. Then

$$h(X) = \Big(C, \sum_{\substack{d_j + e_j = C \\ 1 \le j \le r}} \mu_j \nu_j .\Big)$$

We call $h(X) = (d, \mu)$, d the dimension of X, μ the measure of X and h the measuring function.

We say a complete theory T is measurable, if it has a measurable model.

- Examples :

1. Let $\mathcal{C}$ be an N-dimensional asymptotic class, and $M = \prod_{i \to \mathcal{U}} M_i$ with $M_i \in \mathcal{C}$, an infinite structure. Then M is measurable.

Pf. Given $\varphi(\bar{x};\bar{y})$, $\exists\, D \subseteq \{0, \cdots, N|\bar{x}|\} \times \mathbb{R}^{>0} \cup \{(0,0)\}$, and $\psi_{(d,\mu)}(\bar{y})$, $(d,\mu) \in D$ partition of $\bar{y} = \bar{y}$.

For any $\bar{a} \in M^{|\bar{y}|}$, $\exists!\, (d,\mu) \in D$ s.t. $M \models \psi_{(d,\mu)}(\bar{a})$.

Define $h(\varphi(M^{|\bar{x}|}, \bar{a})) := (d, \mu)$.

2. Let F be an infinite field. Consider the language $\mathcal{L}_F$ of F modules, $\mathcal{L}_F = (+, 0, (m_f)_{f \in F})$. Then any infinite vector space as $\mathcal{L}_F$-structure

is measurable.

If is strongly minimal, $h(x) =$ (Morley rank, Morley degree).

Rmks:

1. Being measurable is a property of the theory.
   if $M \equiv N$ and M measurable, then N is measurable.

2. A measurable structure can have different measuring functions.

   example: ① sps $h(X) = (d, \mu)$

   $$h^*(X) := (2d, \mu).$$

   ② Random graph.

   take any $p \in (0,1)$ and put $\mu(E(x,b)) = p$

   $\mu(\neg E(x,b)) = 1-p$. for all b.

   It determines a measuring function h_p.

3. If M is measurable, then the measuring function h is determined by its value on definable sets in 1-variable.

4. If M is measurable and H is interpretable by M. Then H is measurable.

§ Properties.

1. Disjoint union.

If $A = A_1 \sqcup A_2$ and $h(A_i) = (d_i, \mu_i)$, then

$$h(A) = \begin{cases} (d_1, \mu_1) & \text{if } d_1 > d_2 \\ (d_2, \mu_2) & \text{if } d_1 < d_2 \\ (d_1, \mu_1 + \mu_2) & \text{if } d_1 = d_2. \end{cases}$$

Proof. By Fubini.

$f: A \to \{a, b\}$ take $a \neq b \in M$.

by $f(A_1) = \{a\}$, $f(A_2) = \{b\}$.

$h(\{a\}) = (0,1)$ and $h(\{b\}) = (0,1)$.

By Fubini, we get the expression of $h(A)$.

Cor 1: h is monotic, if $A \subseteq B$ definable sets, then $h(A) \leq h(B)$ under lexicographic ordering.

Cor 2: $\dim(X) = 0$ iff X is finite.

2. Unimodularity.

Def. We say a theory T is unimodular if for any $M \models T$ and definable sets X, Y in M^{eq}, and definable surjections $f_i: X \to Y$ s.t. f_i is k_i-to-1, for $i = 1, 2$ and $k_i \in \mathbb{N}^{>0}$.

Then $k_1 = k_2$.

Note theory of ACF is not unimodular.

Prop: Let T be a measurable theory. then T is unimodular.

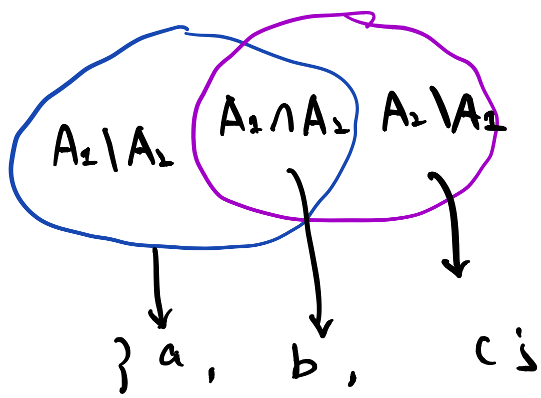
Pf: By the Fubini condition.

3. Let $A_1, A_2, \cdots, A_n$ be def. sets with $\dim(A_i) = d$ for each i. Then

(i) $\dim\left(\bigcup_{i\le n} A_i\right) = d$

(ii) If $\dim(A_i \cap A_j) < d$ for any $i \neq j$ then $\operatorname{meas}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \operatorname{meas}(A_i)$

Pf: $n=2$. (ii)



Supersimplicity of measurable structures.

Aim: Show that any measurable theory is supersimple of finite rank.

Lem. Let X be a def. set and $\varphi(\bar{x}, \bar{y})$ an $\mathcal{L}$-formula, $(\bar{b}_i, i \in \omega)$ an indiscernible sequence where for each $i \in \omega$ we have $\varphi(M^{|\bar{x}|}, \bar{b}_i) \subseteq X$. Sps $\{\varphi(M^{|\bar{x}|}, \bar{b}_i), i \in \omega\}$ is inconsistent. Then $\dim(X) > \dim(\varphi(M^{|\bar{x}|}, \bar{b}_i))$.

Pf: Suppose not. Then $\dim(X) = \dim(\varphi(M^{|\bar{x}|}, \bar{b}_i) = d$ by monotonicity. As $\{\varphi(M^{|\bar{x}|}, \bar{b}_i), i \in \omega\}$ is inconsistent, there is a minimal $k \geq 0$, s.t.

$$\dim(\varphi(M^{|\bar{x}|}, \bar{b}_1) \cap \cdots \cap \varphi(M^{|\bar{x}|}, \bar{b}_{k+1}) \cap \varphi(M^{|\bar{x}|}, \bar{b}_{k+2})) < d.$$

Let $A_i := \varphi(M^{|\bar{x}|}, \bar{b}_1) \cap \cdots \cap \varphi(M^{|\bar{x}|}, \bar{b}_k) \cap \varphi(M^{|\bar{x}|}, \bar{b}_{k+i})$

Note $A_i \cap A_j < d$ for $i \neq j$ by indiscernibility.

and $\dim(A_i) = d$ by minimality of k.

Sps $\mathrm{meas}(A_i) = \mu > 0$. Then for any $t \in \mathbb{N}$, consider $\bigcup_{i=1}^{t} A_i \subseteq X$, then

$$(\dim, \mathrm{meas})\left(\bigvee_{i=1}^{t} A_i\right) = (d, t \cdot \mu)$$

Hence $\mathrm{meas}(X) \geq t \cdot \mu$ for all t, contradiction.

Def. D-rank on formulas.

We work in a monster model of a complete theory T.

Let $\varphi(\bar{x})$ be a formula, defined over A.

Define the D-rank of $\varphi(\bar{x})$, be the least ordinal satisfying the following:

(if no such ordinal exists $D(\varphi(\bar{x})) := \infty$).

(i) $D(\varphi(\bar{x})) \geq 0$ if $\varphi(\bar{x})$ is consistent, else $D(\varphi(\bar{x})) = -\infty$.

(ii) $D(\psi(\bar{x})) \geq \alpha+1$ if $\exists$ $\mathcal{L}$-formula $\phi(\bar{x},\bar{y})$ and an infinite A-indiscernible sequence $(c_i, i<\omega)$ s.t. $\phi(\bar{x}, c_0) \vdash \psi(\bar{x})$ and $\{\phi(\bar{x}, c_i), i<\omega\}$ inconsistent and $D(\phi(x, c_0)) \geq \alpha$.

(iii) $D(\psi(\bar{x})) \geq \gamma$ for γ a limit ordinal, if $D(\psi(\bar{x})) \geq \alpha$ for all $\alpha < \gamma$.

Thm. For any def. set X in a measurable structure, $D(X) \leq \dim(X)$.
Hence measurable structures has super simple finite rank theory.

Pf. We want to show if $\dim(X) \leq r$ then $D(X) \leq r$ for all $r \in \mathbb{N}$.
Induction on r. $r=0$ follows by $\dim(x) = 0$ iff X is finite
Hence $D(X) = 0$. or $D(X) = -\infty$.
Sps IH for all $k<r$, and $\dim(X) = r$ but $D(X) \geq r+1$.

By definition, $\exists$ indiscernible $(b_i, i<\omega)$ / A and $\varphi(\bar{x}, \bar{y})$ where $\{\varphi(\bar{x}, b_i), i<\omega\}$ inconsistent s.t. $D(\varphi(\bar{x}, b_i)) \geq r$ and $\varphi(\bar{x}, b_i) \vdash \psi(\bar{x})$.

By Lemma above, $\dim(\varphi(\bar{x}, b_i)) < r$.

By IH, $D(\varphi(\bar{x}, b_i)) < r$ contradiction. $\square$.

Fact: In a complete theory.

$D(\varphi(\bar{x})) = SU(\varphi(\bar{x})) = S_1(\varphi(\bar{x}))$ if any of them is finite.

Rmks:

1. In general $D(X) \neq \dim(X)$
2. It can happen in a measurable structure M there are def. sets X_1, X_2 with $\dim(X_1) = \dim(X_2)$ but $D(X_1) \neq D(X_2)$.

Example: (M, P, E)

Unary predicate infinite & co infinite

Equivalence relation has infinitely many infinite classes on P but one class on $\neg P$.