Incidence bounds in positive characteristic via valuations and distality
Extended Edition!!

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This talk is based on joint work with **Jean-François Martin**.

Contents

1 Background

1.1 Incidence bounds in characteristic zero

- **Fact (Szemerédi-Trotter '83)** *There exists $C \in \mathbb{R}$ such that, given N points and N lines in $\mathbb{R}^2$, the number of incidences is bounded as*

  $$|\{(p,l) : p \in l\}| \leq C(N^{\frac{3}{2}-\frac{1}{6}}).$$

- This has been generalised in various ways. In particular:

  Fact (Elekes-Szabó '12) *If $(C_b)_{b\in B}$ is an algebraic family of distinct irreducible plane curves over a field K of characteristic 0, there are $C, \epsilon > 0$ such that given N points in K^2 and N curves in the family,*

  $$|\{(a,b) : a \in C_b\}| \leq C(N^{\frac{3}{2}-\epsilon}).$$

- Hrushovski: this indicates a certain *modularity* of the interaction between (pseudo)finite sets and field structure.

- In particular, abelian groups are the only source of relations on which finite sets "maximally accumulate":

  Fact (Elekes-Szabó '12 ($m = 3$), Raz-Sharir-de Zeeuw '18 ($m = 4$), B-Breuillard '20) *Let $V \subseteq \mathbb{A}^m$ be an irreducible affine variety over a field K of characteristic 0.*

  Exactly one of the following holds:

  (i) $\exists C, \epsilon > 0.\ \forall X_1, \ldots, X_m \subseteq_{\text{fin}} K.$

  $$|V(K) \cap (X_1 \times \ldots \times X_m)| \leq C \max(|X_i|)^{\dim V - \epsilon};$$

  (ii) OR: up to finite correspondences on co-ordinates and taking products,
  V is a subgroup of a power of a 1-dimensional algebraic group.

1.2 Incidence bounds in positive characteristic

- In positive characteristic, these bounds utterly fail:

  Remark *Let $K := \mathbb{F}_p^{\text{alg}}$. For any algebraic set $V \subseteq K^n$, there is $r > 0$ such that for arbitrarily large n,*

  $$|V(\mathbb{F}_{p^n})| \geq r(p^n)^{\dim V}.$$

  (This follows from the Lang-Weil estimates.)

- However, Hrushovski conjectures that the Zilber trichotomy applies:
  infinite pseudofinite fields should be the only obstruction to modularity.
  (Above it is $\prod_{n\to\mathcal{U}} \mathbb{F}_{p^n} \leq K^{\mathcal{U}}$.)

  (In the case of the sum-product theorem, this is true, i.e. failures of sum-product bounds are due to finite subfields (Bourgain-Katz-Tao, Tao-Vu, Tao, Hrushovski, Wagner).)

- As an extreme case, this conjecture suggests that for K with **finite** algebraic part $K \cap \mathbb{F}_p^{\text{alg}}$, e.g. $K = \mathbb{F}_p(t)$, the characteristic 0 results should go through.

1.3 Distality

- **Definition** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure.*
  - *Let $\phi(x;y)$ be an $\mathcal{L}$ formula, let $A, B \subseteq \mathcal{M}$.*
    An $\mathcal{L}$-formula $\zeta_\phi(x;z)$ is a **uniform strong honest definition (USHD)** *for ϕ on A over B if for any $a \in A^x$ and finite subset $B_0 \subseteq_{\text{fin}} B$ with $|B_0| \geq 2$, there is $d \in B_0^z$ such that*

    $$\text{tp}(a/B_0) \ni \zeta_\phi(x,d) \vdash \text{tp}_\phi(a/B_0)$$

    (where $\text{tp}_\phi(a/B_0) := \{\phi(x,b)^\epsilon \in \text{tp}(a/B_0) : b \in B_0^y\}$).
  - *If $A = \mathcal{M}$, we omit "on A".*
  - *$B \subseteq \mathcal{M}$ is* **distal in $\mathcal{M}$** *if every $\mathcal{L}$-formula $\phi(x;y)$ has a USHD on B over B.*

- Reduction to one variable:

  Fact *$B \subseteq \mathcal{M}$ is distal in $\mathcal{M}$ iff any $\mathcal{L}$-formula $\phi(x;y)$ with $|x| = 1$ has a USHD on B over B.*

  Proof idea. Given $a, b \in B$ and $\phi(x,y;z)$ and $B_0 \subseteq_{\text{fin}} B$,
  say $\text{tp}(a/bB_0) \ni \zeta(x,b,c) \vdash \text{tp}_{\phi(x;y,z)}(a/bB_0)$ with $c \in B_0^w$.

  Now say $\text{tp}(b/B_0) \ni \theta(y,c') \vdash \text{tp}_{\forall x.(\zeta(x,y,w)\to\phi(x,y,z))}(b/B_0)$.

  Then $\text{tp}(ab/B_0) \ni \zeta(x,y,c) \wedge \theta(y,c') \vdash \text{tp}^+_{\phi(x,y;z)}(ab/B_0)$.

  Now repeat with $\neg\phi$. Uniformity follows through. □

- **Example** *If $B = (b_i)_i \subseteq \mathcal{M}$ is a $\emptyset$-indiscernible sequence,*
  and there is an $\mathcal{L}$-formula $\theta_<$ with $\mathcal{M} \vDash \theta_<(b_i,b_j) \Leftrightarrow i < j$,
  then B is distal in $\mathcal{M}$.

- **Fact (Chernikov-Simon)** $\text{Th}(\mathcal{M})$ *is* **distal** *iff $\mathcal{M}$ is distal in $\mathcal{M}$.*

- Distality implies incidence bounds:

  Fact 1.1 (Chernikov-Galvin-Starchenko, Chernikov-Starchenko '20)
  Suppose $B \subseteq \mathcal{M}$ is distal in $\mathcal{M}$ and $\phi(x;y)$ is an $\mathcal{L}$-formula.

  Suppose $\phi(B;B)$ is $K_{d,s}$-free,
  i.e. there is no $X_0 \times Y_0 \subseteq \phi(B;B)$ with $|X_0| = d$ and $|Y_0| = s$.

  Let $\zeta_\phi(x,z)$ be a USHD for ϕ on B, and let $t := |z|$.

  Then there exists $C \in \mathbb{R}$ such that for $X_0 \subseteq_{\text{fin}} B^x$ and $Y_0 \subseteq_{\text{fin}} B^y$,

  $$|\phi(X_0;Y_0)| \leq C(|X_0|^{\frac{(t-1)d}{td-1}}|Y_0|^{\frac{td-t}{td-1}} + |X_0| + |Y_0|).$$

2 The result

2.1 Statements

- **Theorem 2.1** *Let k be a valued field with finite residue field.*
  Then k is distal in $k^{\text{alg}} \vDash \text{ACVF}$.

  (Note: a positive characteristic valued field with finite residue field is not distal as a structure, nor even NIP (by Kaplan-Scanlon-Wagner).)

- So by Fact 1.1:

  Corollary *Let k be a valued field with finite residue field.*

  Let $E \subseteq k^n \times k^m$ be quantifier-free definable in $\mathcal{L}_{\text{div}}(k)$.

  Suppose E is $K_{d,s}$-free, where $d, s \in \mathbb{N}$.

  Then there exist $t, C > 0$ such that for $A_0 \subseteq_{\text{fin}} k^n$ and $B_0 \subseteq_{\text{fin}} k^m$,

  $$|E \cap (A_0 \times B_0)| \leq C(|A_0|^{\frac{(t-1)d}{td-1}}|B_0|^{\frac{td-t}{td-1}} + |A_0| + |B_0|).$$

- In particular this yields a Szemerédi-Trotter-style result:

  Corollary *Suppose $(C_b)_{b\in B\subseteq k^m}$ is an algebraic family of distinct irreducible plane curves over a field k which admits a valuation with finite residue field.*

  Then $E := \{(a,b) : a \in C_b(k)\} \subseteq k^{2+m}$ is $K_{2,s}$-free for some $s \in \mathbb{N}$.

  So there exist $\epsilon_0, C > 0$ such that for $A_0 \subseteq_{\text{fin}} k^2$ and $B_0 \subseteq_{\text{fin}} k^m$,

  $$|E \cap (A_0 \times B_0)| \leq C(|A_0|^{1-\epsilon_0}|B_0|^{\frac{1}{2}(1+\epsilon_0)} + |A_0| + |B_0|)$$

  ($\epsilon_0 := \frac{1}{2t-1} > 0$), or in symmetric form:

  $$|E \cap (A_0 \times B_0)| \leq C'(\max(|A_0|, |B_0|))^{\frac{3}{2}-\epsilon}$$

  ($\epsilon := \frac{\epsilon_0}{2} > 0$, $C' := 3C$).

2.2 Fields admitting finite residue field

Example *The t-adic valuation on $\mathbb{F}_p(t)$ has residue field $\mathbb{F}_p$.*
Similarly, any finitely generated extension of $\mathbb{F}_p$, i.e. any function field over a finite field, admits a valuation with finite residue field.

However:

Proposition 2.2 *For any prime p, there exists an algebraic extension $L \geq \mathbb{F}_p(t)$ such that $L \cap \mathbb{F}_p^{\text{alg}} = \mathbb{F}_p$ but no valuation on L has finite residue field.*

Such an L can be built by recursively adjoining Artin-Schreier roots which force Artin-Schreier extensions of the residue fields of valuations on previously built fields; using the Artin-Schreier version of Kummer theory, one can always do this without extending the algebraic part.

3 Proof of Theorem 2.1

- Let $L \vDash \text{ACVF}$ and let $k \subseteq L$ be a subfield with $\text{res}(k)$ finite.
- We want to see that $k \subseteq L$ is distal in L.
- By reduction to 1 variable, it suffices to see:
  any $\phi(x,y)$ with $|x| = 1$ has a USHD over k.

3.1 Compressing balls

- By QE, for $a \in k$, $\phi(L,a)$ is a boolean combination of open and closed balls

  $$v(x-a') > \alpha \text{ or } v(x-a') \geq \alpha$$

  centred at points a' with $\deg(k(a')/k)$ bounded, say dividing d.

- Let $B_{k,d}$ be the set of balls (closed and open) centred at points of degree $|d$ field extensions of k within L.

  Using the bounded size of the residue field of such extensions, we obtain:

  Lemma *$x \in y$ has a USHD over $B_{k,d}$.*

  Proof.
  - Let $a \in L$ and $B_0 \subseteq_{\text{fin}} B_{k,d}$.
  - Then $x \in (b \setminus \bigcup_{i<s} b_i) \vdash \text{tp}_\in(a/B_0)$ where
    - b is smallest in B_0 such that $a \in b$ (or $b := L$).
    - $b_1, \ldots, b_s$ are the maximal proper subballs of b in B_0.
  - Sufficient to uniformly bound s.
  - By ultrametricity, joins of finitely many balls are joins of two.
    So we may assume B_0 is closed under $\vee$
    (where $b' \vee b'' :=$ smallest ball containing both).
  - Assume $s > 1$.
  - Say $p_i \in b_i$ is of degree $|d$ over k, and let $\alpha \in v(L)$ be the radius of b.
  - Now

    $$i \mapsto \lambda_i := \text{res}\left(\frac{p_i - p_1}{p_2 - p_1}\right)$$

    is an injection of $\{1, \ldots, s\}$ into $\text{res}(L)$:
    - If $i \neq j$ then $b_i \vee b_j = b$, so $v(p_i - p_j) = \alpha$.
    - Now suppose $\lambda_i = \lambda_j$. Then $\text{res}(\frac{p_i-p_j}{p_2-p_1}) = 0$,
      so $v(p_i - p_j) > v(p_2 - p_1) = \alpha$, so $i = j$.
  - Say $\text{res}(k) = \mathbb{F}_q$.
  - Since each λ_i is in the residue field of an extension of k of degree $|d^3$, by the valuation inequality

    $$\lambda_i \in \mathbb{F}_{q^{d^3}}.$$

  - So $s \leq q^{d^3}$.

  □

- But this is not enough on its own.

  To show that $\phi(x,y)$ has a USHD over k: given $C \subseteq_{\text{fin}} k$ we have to determine $\text{tp}_\phi(a/C)$ using only C as parameters – but the balls involved will generally not be defined over C!

- So we need to be more careful with the QE.

3.2 Compressing cheeses

- $\phi(L,a)$ has a unique-up-to-permutation "Swiss cheese decomposition" as a finite union of disjoint cheeses,

  $$\phi(L,a) = \bigcup_i (b_i \setminus \bigcup_j b_{ij}),$$

  where each cheese is a ball b_i minus a finite union of disjoint proper subballs b_{ij},
  and no b_i is equal to any hole $b_{i'j}$.

- Moreover:

  Fact (Uniform Swiss Cheese Decomposition) *There are N and d depending only on ϕ such that for all $a \in L^y$, $\phi(L,a)$ has Swiss cheese decomposition involving $\leq N$ balls, where each ball contains a point in a degree $|d$ field extension of the subfield generated by a.*

- Increasing N and allowing the empty ball, we can assume the form of the decomposition is constant, given by a Boolean term $D(x_1, \ldots, x_N)$.

- Let $X \subseteq B^N$ be the set defined by the inclusion relations required for $D(b_1, \ldots, b_N)$ to be a Swiss cheese decomposition.

- So $D : X \to$ [codes for subsets] is definable with boundedly finite fibres, and

  $$D(X \cap (B_{k,d})^N) \supseteq \{\ulcorner\phi(L,c)\urcorner : c \in k^y\}.$$

3.3 Collapsing USHDs

- We conclude by a general elementary model theoretic lemma on USHDs:

  Lemma *Let f be definable with boundedly finite fibres. Let $C \subseteq \text{im}(f)$.*

  If $\psi(x, f(z))$ has a USHD over $f^{-1}(C)$,
  then $\psi(x,y)$ has a USHD over C.

  (Explicitly, to conclude:
  - apply this Lemma to $x \in D(\overline{z})$, which has a USHD over $B_{k,d}$ by ball compression;
  - this yields that $x \in w$ has a USHD over

    $$\{\ulcorner\phi(L,c)\urcorner : c \in k^y\},$$

  - hence $\phi(x,y)$ has a USHD over k.)

  Proof idea. Given a and $C_0 \subseteq_{\text{fin}} C$, we have $\zeta(x, \widetilde{d}) \ni \text{tp}(a/f^{-1}(C_0))$ implying all instances of ψ over C_0. Then find $C_1 \subseteq C_0$ bounded by the fibre size such that if $f(\widetilde{d'}) = f(\widetilde{d})$ and $\zeta(x, \widetilde{d'})$ implies the instances over C_1, then it implies all. Then "exists such a $\widetilde{d'}$ over $f(\widetilde{d})$" is a USHD. □

- **Remark** *Following the proof gives a bound on the exponent in the distal cell decompositions, hence in the incidence bound for $|x| = 1$, of $t = 2(q^{d^3} + 1)$ where $\text{res}(k) = \mathbb{F}_q$.*

  For $|x| > 1$ a bound can in theory be computed, but it involves QE in ACVF (for the reduction to one variable).

  For the Szemerédi-Trotter case $\{((x,y),(a,b)) : y = ax + b\}$, we get $t = 4(q+1)$, giving an exponent in the symmetric form of $\frac{3}{2} - \frac{1}{16(q+1)-2}$.

  This uses linear QE (Weispfenning): if an $\mathcal{L}_{\text{div}}$ qf-formula $\psi(x, \overline{y}, \overline{z})$ is linear in $x, \overline{y}$, i.e. each polynomial has degree 1 in x and each y_i, then $\exists x.\psi(x, \overline{y}, \overline{z})$ is equivalent modulo ACVF *to a qf-formula linear in $\overline{y}$.*

  Question: in the Szemerédi-Trotter case with $\mathbb{F}_p(t)$, could there be an exponent which doesn't depend on p? Worst lower bound I know is $\frac{4}{3}$, with a similar "rectangular grid" argument as in the char 0 case:

  $$\begin{aligned}|\{((x,y),(a,b)) \in (\mathbb{F}_p[t]_{<n} \times \mathbb{F}_p[t]_{<2n})^2 : y = ax+b\}| &= |\mathbb{F}_p[t]_{<n} \times \mathbb{F}_p[t]_{<n} \times \mathbb{F}_p[t]_{<2n}| \\ &= p^{4n} \\ &= (p^{3n})^{\frac{4}{3}} \\ &= |\mathbb{F}_p[t]_{<n} \times \mathbb{F}_p[t]_{<2n}|^{\frac{4}{3}}.\end{aligned}$$

4 Elekes-Szabó consequences

As in the characteristic 0 case, these incidence bounds yield "modularity of coherence", and hence Elekes-Szabó bounds.

- Let k_0 be a field admitting a valuation with finite residue field.

- For $r \geq 1$, let

  $$k_r := \{a \in (k_0)^{\text{alg}} : \deg(k_0(a)/k_0) \leq r\}.$$

- From the proof for k_0, we get that each k_r is also distal in $(k_0)^{\text{alg}} \vDash \text{ACVF}$.

- This is sufficient for the arguments of one direction of the 1-dimensional case of the main result of B-Breuillard to go through:

  Theorem *Let $\mathcal{U}$ be a non-principal ultrafilter on ω.*
  Let $k' := ((k_0)^{\mathcal{U}})^{\text{alg}} \leq ((k_0)^{\text{alg}})^{\mathcal{U}} =: L$, so $k' = \bigcup_{r\in\omega}(k_r)^{\mathcal{U}}$.
  Let $\xi \in \mathbb{N}^{\mathcal{U}} \setminus \mathbb{N}$.

  $$\boldsymbol{\delta}(\prod_{i\to\mathcal{U}} X_i) := \text{st}\log_\xi \lim_{i\to\mathcal{U}} |X_i| \in \mathbb{R} \cup \{\infty\}.$$

  Equip L with the structure generated by countably many internal relations, including each $(k_r)^{\mathcal{U}}$, such that $\boldsymbol{\delta}$ is continuous.
  For $\overline{a} \in L^{<\omega}$,

  $$\boldsymbol{\delta}(\overline{a}) = \boldsymbol{\delta}(\text{tp}(\overline{a})) = \inf_{\phi\in\text{tp}(\overline{a})} \boldsymbol{\delta}(\phi(L)).$$

  $P \subseteq L$ is **coherent** *if $\boldsymbol{\delta}(\overline{a}) = \text{trd}(\overline{a})$ for all $\overline{a} \in P^{<\omega}$.*

  Write acl^0 for field-theoretic algebraic closure.

  Let $P \subseteq k'$ be coherent, and let $\text{ccl}(P) := \{a \in \text{acl}^0(P) : \boldsymbol{\delta}(a) = \text{trd}(a)\} \subseteq k'$.

  Then $\text{ccl}(P)$ is coherent, and $\text{acl}^0|_{\text{ccl}(P)}$ is a modular pregeometry.

  Fact (Evans-Hrushovski) *Let Π be a modular subgeometries of dimension $3 \leq n < \omega$ of the geometry of algebraic closure on an algebraically closed field L (of any characteristic) over an algebraically closed subfield $C_0 \leq L$.*

  Then there exists a 1-dimensional algebraic group G over C_0 and generic $\overline{x} \in G^n$ over C_0 such that Π embeds in the projective geometry $\{\text{acl}_{C_0}(\sum_i e_i x_i) : e_i \in \text{End}(G),\ \overline{e} \neq 0\}$.

  One deduces:

  Theorem *Let k_0 be a field admitting a valuation with finite residue field. Let $V \subseteq \mathbb{A}^m$ be an irreducible affine variety over k_0.*

  At least one of the following holds:

  (i) $\exists C, \epsilon > 0.\ \forall X_1, \ldots, X_m \subseteq_{\text{fin}} k_0.$

  $$|V(k_0) \cap (X_1 \times \ldots \times X_m)| \leq C \max(|X_i|)^{\dim V - \epsilon};$$

  (ii) Up to finite correspondences on co-ordinates and taking products,
  V is a subgroup of a power of a 1-dimensional algebraic group.

  Remark 4.1 *In characteristic 0 there is a higher dimensional ("coarse general position") version of this. That doesn't immediately go through, but only because it needs a positive characteristic version of higher dimensional Evans-Hrushovski, which remains to be proven. (The $m = 3$ case probably goes through directly, but I haven't checked.)*

  Getting a version with "exactly one" in place of "at least one" would also take some more work.