Metrizability of universal minimal flows * (Blaise)

Recall: $G \curvearrowright X$ continuous right action

$G \curvearrowright (X, x_0)$ with $G x_0 \subseteq X$ dense

given $Y \subseteq X$ closed & G-invariant, we get a subflow by restriction

A **minimal** flow is a flow with no subflow ($\Leftrightarrow$ all orbits are dense).

A minimal flow (X, G) is **universal** if for any other minimal flow Y, there is a continuous $X \longrightarrow Y$. (cfr. with the universal minimal ambit).

$\Rightarrow$ any minimal subflow of the universal minimal G-ambit is an universal minimal flow.

FACT. all universal minimal flows are isomorphic.

Recall that for a model M,

$$\Sigma^{m} = \{ tp^{full}(\sigma(\bar{m})) \mid \sigma \in G^* \}$$

where $m = (M, G)$ in the full language, $(M^*, G^*) \succ (M, G)$ monster model. For any $\bar{a} \in M$,

$$\Sigma^{m}_{\bar{a}} = \{ tp^{full}(\sigma(\bar{a})) \mid \sigma \in G^* \}$$

gives a projection $\pi_{\bar{a}} : \Sigma^{m} \to \Sigma^{m}_{\bar{a}}$ of the limit

$$\varprojlim \pi_{\bar{a}} =: h : \Sigma^{m} \to \varprojlim \Sigma^{m}_{\bar{a}}.$$

FACT. $\Sigma^{m}_{\bar{a}} \cong \beta(\bar{a} \cdot G)$.

§I. Metrizability

DEF. we say that M has **separately finite embedding Ramsey degree**, witnessed by $(k_{\bar{a}})_{\bar{a} \in M}$, if for any $B \subseteq M$ finite and $\bar{a} \in B$, for any $c : \binom{M}{\bar{a}} \to n \in \omega$, there is $B' \in \binom{M}{B}$ such that $c\left[\binom{B'}{\bar{a}}\right] \leq k_{\bar{a}}$.

LEMMA. if M has SFERD, witnessed by $(k_{\bar{a}})_{\bar{a} \in M}$, then for every finite $A \subseteq B$ and all $r \in \mathbb{N}$ there is $B \subseteq C \subseteq M$ finite such that for any $(c_{\bar{a}})_{\bar{a} \in A}$, where $c_{\bar{a}} : \binom{C}{\bar{a}} \to r$, there is $B' \in \binom{C}{B}$ such that

$$c_{\bar{a}}\left[\binom{B'}{\bar{a}}\right] \leq k_{\bar{a}}.$$

<u>Proof:</u> (by induction)

$\forall \bar{a}_1, \dots \bar{a}_n$, $\forall B \ni \bar{a}_1, \dots \bar{a}_n$ we want a finite $B \subseteq C$ s. that $\forall r \in \mathbb{N}$, $\forall c_{\bar{a}_i} : \binom{C}{\bar{a}_i} \to r$, there is $B' \in \binom{C}{B}$ s. that $c_{\bar{a}_i}\left[\binom{B'}{\bar{a}_i}\right] \leq k_{\bar{a}_i}$.

For $n = 1$, this is just definition of SFERD.

Suppose $\bar{a}_1, \dots \bar{a}_{n+1} \in B$, $r \in \mathbb{N}$

- apply to $\bar{a}_{n+1}$ first: $C_{n+1} \supseteq B$ s. that for any $c_{\bar{a}_{n+1}} : \binom{C_{n+1}}{\bar{a}_{n+1}} \to r$ there is $B' \in \binom{C}{B}$ such that $c_{\bar{a}_{n+1}}\left[\binom{B'}{\bar{a}_{n+1}}\right] \leq k_{\bar{a}_{n+1}}$.

- apply to $\bar{a}_1, \dots \bar{a}_n$: there is C finite, $C_{n+1} \subseteq C$ such that for all $c_{\bar{a}_i} : \binom{C}{\bar{a}_i} \to r$ there is C'_{n+1},

$C'_{n+1} \in \binom{C}{C_{n+1}}$, such that $C_{\bar{a}_i}\left[\binom{C_{n+1}}{a_i}\right] \leq k_{\bar{a}_i}$.

$\blacksquare$

THEOREM. for M countable, $G = \mathrm{Aut}(M)$, then the universal minimal G-flow is metrizable iff M has SFERD.

__Proof:__ assume SFERD witnessed by $(k_{\bar{a}})_{\bar{a} \in M}$.

Take $\underbrace{\{\varphi_1(\bar{x}_1), \dots \varphi_n(\bar{x}_n)\}}_{=:\Delta}$ full formulae, $A \subseteq M$ finite.

Let

$$C_{\bar{a}}(\bar{a}')_i = \begin{cases} 1 & \text{if } M \models \varphi_i(a') \\ -1 & \text{if } M \models \neg\varphi_i(a') \\ 0 & \text{otherwise} \end{cases}$$

defining $C_{\bar{a}} : \binom{M}{a} \to \{-1, 0, 1\}^n$.

Apply Lemma with $A = B$: let C be finite with $A' \in \binom{C}{A}$ s.that $\forall \bar{a} \in A$, $C_{\bar{a}}\left[\binom{A'}{C}\right] \leq k_{\bar{a}}$.

Let $\sigma_{\bar{\varphi},A} \in G$ be such that $\sigma_{\bar{\varphi},A}(A) = A'$.

For any $g_1, \dots g_m \in G$ such that $g_i(\bar{a}) \in A$, the tuples $(\sigma_{\varphi,A}(g_i(\bar{a})))\,|\, i \leq m$ have at most $k_{\bar{a}}$ Δ-types.

$\Rightarrow$ (by compactness) $\sigma \in G^*$ such that

$$\#\{tp_{\Delta'}(\sigma(g(\bar{a}))) \mid g \in G\} \leq k_{\bar{a}},\ \bar{a} \in M$$

for all Δ'.

If we let $X_{\bar{a}} = \{tp^{full}(\sigma(g(\bar{a}))) \mid g \in G\}$, then $\# X_{\bar{a}} \leq k_{\bar{a}}$

Now recall $tp^{full}(\sigma(\bar{m})) \cdot G := \{ tp^{full}(\sigma(g(\bar{m}))) \mid g \in G \}$.

Then $\pi_{\bar{a}}[tp^{full}(\sigma(\bar{m})) \cdot G] = X_{\bar{a}}$.

$$\Rightarrow h : tp^{full}(\sigma(\bar{m})) \cdot G \longrightarrow \underbrace{\varprojlim X_{\bar{a}}}_{\text{profinite}} \overset{\text{closed}}{\subseteq} \varprojlim \Sigma^{m}_{\bar{a}}$$

$\Downarrow$

Compact, Hausdorff & II countable

$\Rightarrow h[tp^{full}(\sigma(\bar{m})) \cdot G] \subseteq \hat{X} = \varprojlim X_{\bar{a}}$ is dense

$\Rightarrow \hat{X}$ is metrizable