NIP formulas in continuous logic

<u>Setting</u>: M metric structure (in particular a complete bdd metric space), $G = \mathrm{Aut}(M)$.

* If M is separable, it is called <u>ω-categorical</u> if unique separable model of $Th(M)$ up to isomorphism.

Recall (continuous Ryll-Nardzewski):

M separable. TFAE

- M is ω-cat.
- $M^\omega /\!/ G$ is compact ($G \curvearrowright M$ approximately oligomorphic)
- All uniformly continuous G-invariant bdd functions $h : M^\omega \to \mathbb{R}$ are given by formulas.

<u>Terminology</u>: M <u>$\emptyset$-saturated</u> if M realizes all $p \in S_\omega(\emptyset)$.

<u>Fact 1</u>: M ω-cat $\Rightarrow$ M $\emptyset$-saturated.

Fix countable ordinals α, β, variables $(x_i)_{i<\alpha} = x$ and $y = (y_i)_{i<\beta}$ and a formula $f(x;y) : M^\alpha \times M^\beta \to \mathbb{R}$. Moreover, let $A \subseteq M^\alpha$ and $B \subseteq M^\beta$ be $\emptyset$-(type)-definable sets. If $\tilde{M} \succ M$, we write $\tilde{f}$ and $\tilde{A}, \tilde{B}$ for the interpretations in $\tilde{M}$.

Prop. 2 (Char of NIP formulas)

Let M be $\emptyset$-saturated and f, A, B as above. TFAE

(1) There are no $r \neq s$ in $\mathbb{R}$ and $(a_i)_{i<\omega}$ from A and $(b_I)_{I\subseteq\omega}$ from $\tilde{B}$ s.th. $\forall i\in\omega\ \forall I\subseteq\omega$:

$$\tilde{f}(a_i; b_I) = \begin{cases} r & \text{if } i\in I \\ s & \text{if } i\notin I. \end{cases}$$

[$\hat{=}$ no infinite set is shattered, in classical discrete logic]

(2) For any indiscernible $(a_i)_{i<\omega}$ in A and $b\in\tilde{B}$ $\lim_i \tilde{f}(a_i, b)$ exists. [$\hat{=}$ finite alternation]

(3) For any sequence $(a_i)_{i<\omega}$ in A there is a subsequence $(a_{i_j})_{j<\omega}$ s.th. $\lim_j f(a_{i_j}, b)$ exists for all $b\in\tilde{B}$.

$f(x,y)$ is said to be NIP on $A\times B$ if these equivalent conditions hold.

Pf: (3) $\Rightarrow$ (1): Trivial, since any subsequence $(a_{i_j})_{j<\omega}$ of a counterex to (1) yields $r = \lim_j (f(a_{i_j}, b_J) = s$ whenever $J\subseteq\omega$ intersects $\{i_j \mid j\in\omega\}$ in an infinite coinfinite set.

(2) $\Rightarrow$ (3): By compactness, for every $\varepsilon>0$ there is a finite set Δ of formulas and $\delta>0$ s.th. whenever $(a_i)_{i\in\omega}$ is Δ-δ-indiscernible in A (i.e. $|g(a_{i_1},\dots,a_{i_k}) - g(a_{j_1},\dots,a_{j_k})| \leq \delta$ for any $g\in\Delta$ and $i_1<\dots i_k$ and $j_1<\dots j_k$ from ω) and $b\in\tilde{B}$, there is $N\in\omega$ s.th.

$|\tilde{f}(a_i, b) - \tilde{f}(a_j, b)| < \varepsilon$ for all $i, j \geq N$.

③

Now let (Δ_n, δ_n) correspond to $\varepsilon_n = \frac{1}{n}$.

- Set $(a_i^0)_{i<\omega} := (a_i)_{i<\omega}$.
- Given $n \in \omega$ and assuming $(a_i^n)_{i<\omega}$ is a subsequence of $(a_i)_{i<\omega}$, by Ramsey we find a $\Delta_{n+1}, \delta_{n+1}$-indiscernible subsequence $(a_i^{n+1})_{i<\omega}$ of $(a_i^n)_{i<\omega}$.

Diagonalize! Set $a_{i_j} := a_j^j$

By construction $(a_{i_j})_{j<\omega}$ is as we wanted, i.e. $\lim_j \tilde{f}(a_{i_j}, b)$ exists for all $b \in \tilde{B}$.

①⇒②: If $f(a_i, b)$ does not converge, for $(a_i)_{i<\omega}$ indiscernible in A and $b \in \tilde{B}$, we find $i_0 < j_0 < i_1 < j_1 < \ldots$ in ω and $r \neq s$ in $\mathbb{R}$ s.th. $\tilde{f}(a_{i_k}, b) \xrightarrow[k\to\infty]{} r$ and $\tilde{f}(a_{j_k}, b) \xrightarrow[k\to\infty]{} s$.

~~Replacing~~ Passing to a subsequence, WMA $\tilde{f}(a_{2k}, b) \to r$ and $\tilde{f}(a_{2k+1}, b) \to s$.

M $\emptyset$-saturated $\Rightarrow$ WMA $\tilde{f}(a_{2k}, b) = r$ and $\tilde{f}(a_{2k+1}, b) = s$ $\forall k$.

Now let $I \subseteq \omega$ be arbitrary. Set $\tau: \omega \to \omega$, $\tau(n) := \begin{cases} 2n & \text{if } n \in I \\ 2n+1 & \text{else} \end{cases}$

$\sigma(a_i) := a_{\tau(i)}$ defines a partial elementary map, extend it to $\tilde{\sigma} \in \mathrm{Aut}(\tilde{M})$. Let $b_I := \tilde{\sigma}^{-1}(b)$. Then $\tilde{f}(a_i, b_I) = \begin{cases} r & \text{if } i \in I \\ s & \text{if } i \notin I \end{cases}$ by construction

RK 3: ③ in Prop. 2 is equivalent to the following:

Let $B^* := \{ tp(b/M) \in S_y(M) \mid b \in \tilde{B} \text{ for } \tilde{M} \succ M \}$. Then

$\{ \tilde{f}_a|_{B^*} : a \in A \}$ is a sequentially precompact subset

of $\mathbb{R}^{B^*}$, where $\tilde{f}_a|_{B^*}$ is the function on B^* induced by $\tilde{f}(a, y)$.

[I.e., every sequence admits a convergent subsequence.]

~~[illegible]~~

Link to tame functions

Let X be a G-space

$\mathcal{C}(X) := \{ f : X \to \mathbb{R} \text{ continuous + bdd} \}$

$\exists$ Def^n of a "tame" function $f \in \mathcal{C}(X)$ (we don't give it)

Fact 4: If X is compact, $f \in \mathcal{C}(X)$ is tame iff $G \cdot f$ is sequentially precompact in $\mathbb{R}^X$.

Here, $(g \cdot f)(x) := f(g^{-1} \cdot x)$

Prop 5: M ω-sat, $f(x,y)$ formula, $B \subseteq M^\beta$ (M-) definable, $a \in M^\alpha$, then $\tilde{f}_a \in \mathcal{C}(B^*)$ is tame iff $f(x,y)$ is NIP on $[a] \times B$.

Note that $[a]$ is closed and G-invariant, so (M) definable. As $G \cdot a$ is dense in $[a]$, $\{ \tilde{f}_{a'}|_{B^*} : a' \in [a] \}$ is sequentially precompact iff $\{ \tilde{f}_{g \cdot a}|_{B^*} : g \in G \}$ is, so we conclude by Remark 3 + Prop 2.

Now let $B = M$ (or $B = M^{\beta}$ for some ctble β).

The previous proposition, characterizing tame functions, applies to the set of fns

$$Def(M) := \{ f(a,y) : M \to \mathbb{R} \mid f(x,y) \text{ formula}, a \in M^{\omega} \} \subseteq \mathcal{C}(X)$$

Given a G-space X, we set

$$RUC(X) := \{ f \in \mathcal{C}(X) \mid \forall \varepsilon > 0 \; \exists U \text{ nbhd of } 1 \in G : \| g \cdot f - f \| < \varepsilon \; \forall g \in U \}$$

If X is a metric space,

$$RUC_u(X) := RUC(X) \cap \{ f : X \to \mathbb{R} \mid f \text{ unif cont} \}$$

Lemma 6: M ω-cat metric structure, $G = Aut(M)$. Then

$$Def(M) = RUC_u(M)$$

Pf sketch: "$\subseteq$" holds in any metric structure (easy exercise)

ad "$\supseteq$": Let $h \in RUC_u(M)$. Let $a \in M^{\omega}$ enumerate a dense subset of M. Define $f : G \cdot a \times M \to \mathbb{R}$ via

$$f(ga, b) := (g \cdot h)(b) = h(g^{-1} b)$$

Since a is dense in M, f is well-def. Moreover, f is bdd, unif cont + G-invariant (for diagonal G-action on $G \cdot a \times M$).

[Exercise: use $h \in RUC_u(M)$ + a dense in M]

$\rightsquigarrow$ f extends to $\tilde{f} : [a] \times M \to \mathbb{R}$ unif cont bdd G-invar

$\rightsquigarrow$ $\tilde{f}$ factors through $\bar{f} : ([a] \times M) /\!/ G \to \mathbb{R}$ unif cont + bdd

Tietze $\rightsquigarrow$ $\bar{f}$ extends to $\bar{F} : (M^{\omega} \times M) /\!/ G \to \mathbb{R}$ unif cont + bdd

$\rightsquigarrow$ $F = \bar{F} \circ \pi : M^{\omega} \times M \to \mathbb{R}$ unif cont + bdd + G-invar

Since M is ω-cat, $F(x,y)$ is a formula, and by construction $h(y) = F(a,y)$, so $h \in Def(M)$ ∎