Descriptive set-theoretic dichotomy theorems

Benjamin D. Miller
Universität Münster

Luminy
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Introduction

Within the last few years, it has become clear that many descriptive set-theoretic dichotomy theorems can be seen as consequences of a small handful of graph-theoretic dichotomy theorems.

This has led to classical proofs of many theorems which previously relied on sophisticated machinery from mathematical logic.

Here we give a detailed summary of the new arguments.

Part I

The G_0 dichotomy

Definition

A *digraph* on X is an irreflexive set $G \subseteq X \times X$.

The *restriction* of G to $Y \subseteq X$ is given by $G \upharpoonright Y = G \cap (Y \times Y)$.

Definition

Suppose that $R \subseteq \prod_{i \in n} X_i$.

A sequence $(Y_i)_{i \in n}$ is *R-independent* if $R \cap \prod_{i \in n} Y_i = \emptyset$.

A set $Y \subseteq X$ is *G-independent* if (Y, Y) is G-independent.

Definition

An $(I\text{-})$*coloring* of G is a function $c \colon X \to I$ with the property that for all $i \in I$, the set $c^{-1}(\{i\})$ is G-independent.

A *homomorphism* from $R \subseteq X \times X$ to $S \subseteq Y \times Y$ is a function $\varphi \colon X \to Y$ which sends R-related points to S-related points.

A *homomorphism* from $(R_i)_{i \in I}$ to $(S_i)_{i \in I}$ is a function which is a homomorphism from R_i to S_i for all $i \in I$.

A *reduction* from $R \subseteq X \times X$ to $S \subseteq Y \times Y$ is a homomorphism from (R, R^c) to (S, S^c). An *embedding* is an injective reduction.

Example

The *digraph on* 2^ω *associated with* $S \subseteq 2^{<\omega}$ is given by

$$G_S = \{(s^\frown 0^\frown x, s^\frown 1^\frown x) \mid s \in S \text{ and } x \in 2^\omega\}.$$

Definition

A set $S \subseteq 2^{<\omega}$ is *dense* if $\forall r \in 2^{<\omega} \exists s \in S \ (r \sqsubseteq s)$.

Lemma 1

Suppose that $B \subseteq 2^\omega$ is a non-meager set with the Baire property and $S \subseteq 2^{<\omega}$ is dense. Then B is not G_S-independent.

Proof of Lemma 1

Fix $r \in 2^{<\omega}$ such that B is comeager in $\mathcal{N}_r$.

Fix $s \in S$ such that $r \sqsubseteq s$.

Then $(s^\frown 0^\frown x, s^\frown 1^\frown x) \in G_S \upharpoonright B$ for comeagerly many $x \in 2^\omega$. ⊠

Lemma 2

Suppose that κ is an aleph, $S \subseteq 2^{<\omega}$ is dense, and c is a κ-coloring of G_S. Then $(c \times c)^{-1}(\leq)$ does not have the Baire property.

Proof of Lemma 2

Set $R = (c \times c)^{-1}(\leq)$ and $E = (c \times c)^{-1}(\Delta(\kappa))$.

Proof of Lemma 2 (continued)

If R has the Baire property, then Kuratowski-Ulam yields a least $\alpha \in \kappa$ for which $c^{-1}(\leq^{\alpha})$ is non-meager and has the Baire property.

Then the E-class $C = c^{-1}(\{\alpha\})$ is non-meager.

By Lemma 1, there exists $(x, y) \in G_S \upharpoonright C$, a contradiction. ⊠

Lemma 3

Suppose that κ is an aleph, $S \subseteq 2^{<\omega}$ is dense, and the family of subsets of 2^ω with the Baire property is closed under κ-length unions. Then there is no κ-coloring of G_S with respect to which pre-images of singletons have the Baire property.

Proof of Lemma 3

Suppose that c is a κ-coloring of G_S with respect to which pre-images of singletons have the Baire property.

Then $(c \times c)^{-1}(\leq)$ has the Baire property.

But this directly contradicts Lemma 2. $\boxtimes$

Definition (Kechris-Solecki-Todorcevic)

Fix sequences $s_n \in 2^n$ such that the set $S = \{s_n \mid n \in \omega\}$ is dense.

Define $G_0 = G_0(2^\omega) = G_S$.

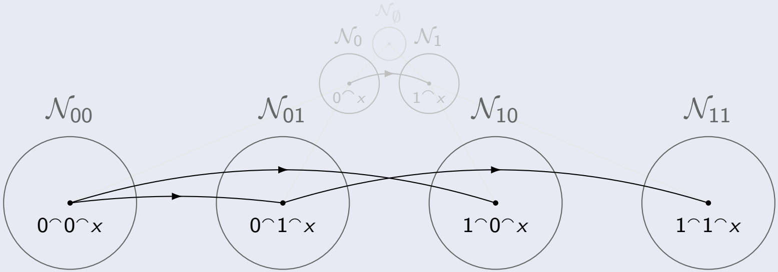
Alternatively, let $G_0(2^n)$ be the digraph on 2^n given recursively by

$$G_0(2^{n+1}) = (G_0(2^n) \otimes 2) \cup \{(s_n{}^\frown 0, s_n{}^\frown 1)\},$$

where $G_0(2^n) \otimes 2 = \{(s^\frown i, t^\frown i) \mid i \in 2 \text{ and } (s,t) \in G_0(2^n)\}$. Then

$$G_0(2^\omega) = \bigcup_{n \in \omega} \{(s^\frown x, t^\frown x) \mid (s,t) \in G_0(2^n) \text{ and } x \in 2^\omega\}.$$

I. The G_0 dichotomy
The canonical obstruction



Definition

A set $A \subseteq X$ is *weakly κ-Souslin* if it is the continuous image of a κ^+-Borel subset of κ^ω.

Definition

For the purposes of these talks, we will say that an aleph κ is *good* if any two disjoint weakly κ-Souslin subsets of a Hausdorff space can be separated by a κ^+-Borel set.

Our arguments in the classical case $\kappa = \omega$ generalize word-for-word to the case of good alephs.

In order to obtain generalizations to odd projective pointclasses under AD, one must work with a different notion.

Definition

For the purposes of these talks, we will say that an aleph κ is *nice* if any two disjoint weakly $(<\kappa)$-Souslin subsets of a Hausdorff space can be separated by a κ-Borel set.

Question

Does ZF imply that all alephs are nice?

Lemma 4

Suppose that κ is a good aleph, $n \in \omega$, $(X_i)_{i \in n}$ is a sequence of Hausdorff spaces, $R \subseteq \prod_{i \in n} X_i$ is weakly κ-Souslin, and $(A_i)_{i \in n}$ is an R-independent sequence of weakly κ-Souslin sets. Then there is an R-independent sequence $(B_i)_{i \in n}$ of κ^+-Borel sets such that $A_i \subseteq B_i$ for all $i \in n$.

Proof of Lemma 4

We will recursively construct κ^+-Borel sets $B_i \subseteq X_i$ such that $(B_i)_{i \in m}{}^\frown(A_i)_{i \in n \setminus m}$ is R-independent for all $m \in n$.

Suppose that $m \in n$ and we have already found $(B_i)_{i \in m}$.

Proof of Lemma 4 (continued)

Set $P_m = \prod_{i \in m} B_i \times X_m \times \prod_{i \in n \setminus (m+1)} A_i$.

Set $Q_m = \prod_{i \in m} B_i \times \mathrm{proj}_{X_m}(R) \times \prod_{i \in n \setminus (m+1)} A_i$.

Define $A'_m = \mathrm{proj}_{X_m}(R \cap P_m) = \mathrm{proj}_{X_m}(R \cap Q_m)$.

Then $A_m \cap A'_m = \emptyset$ and both of these sets are weakly κ-Souslin.

Fix a κ^+-Borel set $B_m \subseteq X_m$ separating A_m from A'_m. ⊠

Lemma 5

Suppose that κ is a good aleph, X is a Hausdorff space, G is a weakly κ-Souslin digraph on X, and $A \subseteq X$ is G-independent and weakly κ-Souslin. Then there is a G-independent, κ^+-Borel set $B \subseteq X$ such that $A \subseteq B$.

Proof of Lemma 5

By Lemma 4, there is a G-independent pair (B_0, B_1) of κ^+-Borel subsets of X such that $A \subseteq B_0$ and $A \subseteq B_1$.

Clearly the set $B = B_0 \cap B_1$ is as desired. ⊠

Theorem 6 (Kanovei, Kechris-Solecki-Todorcevic, Louveau)

Suppose that κ is an aleph, X is a Hausdorff space, and G is a κ-Souslin digraph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of G.
2. There is a continuous homomorphism from G_0 to G.

Proof of Theorem 6

We will prove the special case of the theorem for good κ.

Before discussing the proof, we first note a standard reduction.

Lemma 7

It is sufficient to handle the special case that $X = \kappa^\omega$.

Proof of Lemma 7

We can clearly assume that $G \neq \emptyset$, so $\mathrm{proj}_X(G) \neq \emptyset$, thus there is a continuous surjection $\varphi \colon \kappa^\omega \to \mathrm{proj}_X(G)$. Set $H = (\varphi \times \varphi)^{-1}(G)$.

If there is a κ^+-Borel κ-coloring of H, then Lemma 5 allows us to produce a κ^+-Borel κ-coloring of G.

If $\psi \colon 2^\omega \to \kappa^\omega$ is a continuous homomorphism from G_0 to H, then $\varphi \circ \psi$ is a continuous homomorphism from G_0 to G. ⊠

I. The G_0 dichotomy

The main theorem

The idea behind the proof

We will try to build a continuous homomorphism φ from G_0 to G.

Fix a tree ⚘ on $\kappa \times (\kappa \times \kappa)$ such that $G = \mathrm{proj}_{\kappa^\omega \times \kappa^\omega}[⚘]$.

When successful, our strategy will also produce continuous functions $\psi_k \colon 2^\omega \to \kappa^\omega$ verifying our success, in the sense that

$$(\psi_k(x), (\varphi(s_k{}^\frown 0^\frown x), \varphi(s_k{}^\frown 1^\frown x))) \in [⚘]$$

for all $k \in \omega$ and $x \in 2^\omega$.

The idea behind the proof (continued)

The functions φ and ψ_k will be of the form

$$\varphi(x) = \bigcup_{n \in \omega} \varphi_n(x \upharpoonright n)$$

and

$$\psi_k(x) = \bigcup_{k \in n \in \omega} \psi_{k,n}(x \upharpoonright (n - (k+1))),$$

where $\varphi_n \colon 2^n \to \kappa^n$ and $\psi_{k,n} \colon 2^{n-(k+1)} \to \kappa^n$ for $k \in n \in \omega$, and $(\varphi_n)_{n \in \omega}$ and $(\psi_{k,n})_{n \in \omega}$ are increasing.

The idea behind the proof (continued)

There are of course many possible choices of $(\varphi_n, (\psi_{k,n})_{k \in n})$.

We will consider only those which are restrictions of homomorphisms $\varphi'_n \colon 2^n \to \kappa^\omega$ from $G_0(2^n)$ to G and verifiers $\psi'_{k,n} \colon 2^{n-(k+1)} \to \kappa^\omega$.

The inability to extend such a $(\varphi_n, (\psi_{k,n})_{k \in n})$ to another such pair $(\varphi_{n+1}, (\psi_{k,n+1})_{k \in n+1})$ will yield a G-independent, κ^+-Borel set.

The idea behind the proof (continued)

By removing these sets, we obtain a derivative on κ^ω.

If the derivative succeeds in eventually cutting out the entire space before stage κ^+, then we will have our desired coloring.

Otherwise, we will be able to construct $(\varphi_n, (\psi_{k,n})_{k \in n})$ for $n \in \omega$, and thereby obtain the desired homomorphism.

Definition

An *approximation* is a triple of the form $a = (n^a, \varphi^a, (\psi^a_k)_{k \in n^a})$, where $n^a \in \omega$, $\varphi^a \colon 2^{n^a} \to \kappa^{n^a}$, and $\psi^a_k \colon 2^{n^a - (k+1)} \to \kappa^{n^a}$.

We say that an approximation a is *extended* by an approximation b if for all $k \in n^a$, the following conditions are satisfied:

1. $n^a \leq n^b$.
2. $\forall r \in 2^{n^a} \forall s \in 2^{n^b} \; (r \sqsubseteq s \implies \varphi^a(r) \sqsubseteq \varphi^b(s))$.
3. $\forall r \in 2^{n^a - (k+1)} \forall s \in 2^{n^b - (k+1)} \; (r \sqsubseteq s \implies \psi^a_k(r) \sqsubseteq \psi^b_k(s))$.

If $n^b = n^a + 1$, then we say that b is a *one-step extension* of a.

Fix a κ-length well-ordering of the set of all approximations.

Definition

A *configuration* is a triple of the form $\gamma = (n^\gamma, \varphi^\gamma, (\psi_k^\gamma)_{k \in n^\gamma})$, where $n^\gamma \in \omega$, $\varphi^\gamma \colon 2^{n^\gamma} \to \kappa^\omega$, and $\psi_k^\gamma \colon 2^{n^\gamma - (k+1)} \to \kappa^\omega$, such that

$$(\psi_k^\gamma(s), (\varphi^\gamma(s_k {}^\frown 0 {}^\frown s), \varphi^\gamma(s_k {}^\frown 1 {}^\frown s))) \in [\text{🌿}]$$

for all $k \in n^\gamma$ and $s \in 2^{n^\gamma - (k+1)}$.

This simply says that φ^γ is a homomorphism from $G_0(2^{n^\gamma})$ to G, and moreover, that this fact is verified by $(\psi_k^\gamma)_{k \in n^\gamma}$.

Definition

We say that a configuration γ is *compatible* with an approximation a if the following conditions are satisfied:

1. $n^a = n^\gamma$.
2. $\forall s \in 2^{n^a} \ (\varphi^a(s) \sqsubseteq \varphi^\gamma(s))$.
3. $\forall k \in n^a \forall s \in 2^{n^a-(k+1)} \ (\psi^a_k(s) \sqsubseteq \psi^\gamma_k(s))$.

We say that γ is *compatible* with a set $Y \subseteq \kappa^\omega$ if $\varphi^\gamma[2^{n^\gamma}] \subseteq Y$.

We use $\Gamma(a, Y)$ to denote the family of all configurations which are compatible with both a and Y.

Definition

We say that an approximation a is Y*-terminal* if $\Gamma(b, Y) = \emptyset$ for all one-step extensions b of a.

We use $T(Y)$ to denote the family of all such approximations.

Define $A(a, Y) \subseteq Y$ by $A(a, Y) = \{\varphi^\gamma(s_{n^a}) \mid \gamma \in \Gamma(a, Y)\}$.

Lemma 8

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$, and $A(a, Y)$ is not G-independent. Then a is not Y-terminal.

Proof of Lemma 8

Fix configurations $\gamma_0, \gamma_1 \in \Gamma(a, Y)$ with $(\varphi^{\gamma_0}(s_{n^a}), \varphi^{\gamma_1}(s_{n^a})) \in G$.

Fix $x \in \kappa^\omega$ such that $(x, (\varphi^{\gamma_0}(s_{n^a}), \varphi^{\gamma_1}(s_{n^a}))) \in [\text{🌿}]$.

Proof of Lemma 8 (continued)

Let γ denote the configuration given by:

1. $n^\gamma = n^a + 1$.
2. $\forall i \in 2 \forall s \in 2^{n^a} \ (\varphi^\gamma(s^\frown i) = \varphi^{\gamma_i}(s))$.
3. $\forall i \in 2 \forall k \in n^a \forall s \in 2^{n^a-(k+1)} \ (\psi_k^\gamma(s^\frown i) = \psi_k^{\gamma_i}(s))$.
4. $\psi_{n^a}^\gamma(\emptyset) = x$.

Let b denote the approximation given by:

1. $n^b = n^\gamma$.
2. $\forall s \in 2^{n^b} \ (\varphi^b(s) = \varphi^\gamma(s) \upharpoonright n^b)$.
3. $\forall k \in n^b \forall s \in 2^{n^b-(k+1)} \ (\psi_k^b(s) = \psi_k^\gamma(s) \upharpoonright n^b)$.

Proof of Lemma 8 (continued)

Clearly γ is compatible with b.

Clearly b is a one-step extension of a.

It follows that a is not Y-terminal. ⊠

Lemma 9

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is Y-terminal. Then there is a G-independent, κ^+-Borel subset $B(a, Y)$ of κ^ω such that $A(a, Y) \subseteq B(a, Y)$.

Proof of Lemma 9

Lemma 8 ensures that $A(a, Y)$ is G-independent.

The desired result therefore follows from Lemma 5. ⊠

Definition

Set $Y' = Y \setminus \bigcup_{a \in T(Y)} B(a, Y)$.

Lemma 10

There is a κ^+-Borel κ-coloring of $G \upharpoonright (Y \setminus Y')$.

Proof of Lemma 10

Define $c(y) = \min\{a \in T(Y) \mid y \in B(a, Y)\}$ for $y \in Y \setminus Y'$.

As $c^{-1}(\{a\}) \subseteq B(a, Y)$ for all $a \in T(Y)$, it follows that c is a coloring of $G \upharpoonright (Y \setminus Y')$. ⊠

Definition

Recursively define a sequence $(Y_\alpha)_{\alpha \in \kappa^+}$ of subsets of κ^ω by

$$Y_\alpha = \begin{cases} \kappa^\omega & \text{if } \alpha = 0, \\ Y'_\beta & \text{if } \alpha = \beta + 1, \text{ and} \\ \bigcap_{\beta \in \alpha} Y_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Since there are only κ-many approximations, there exists $\alpha \in \kappa^+$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

Lemma 11

Suppose that the trivial approximation is Y_α-terminal. Then there is a κ^+-Borel κ-coloring of G.

Proof of Lemma 11

Note first that $Y_{\alpha+1} = \emptyset$, thus $\kappa^\omega = \bigcup_{\beta \leq \alpha} Y_\beta \setminus Y_{\beta+1}$.

As all $G \restriction (Y_\beta \setminus Y_{\beta+1})$ admit κ^+-Borel κ-colorings, so does G. ⊠

Lemma 12

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is not Y'-terminal. Then there is a one-step extension of a which is not Y-terminal.

Proof of Lemma 12

Fix a one-step extension b of a for which $\Gamma(b, Y') \neq \emptyset$.

Fix a configuration $\gamma \in \Gamma(b, Y')$.

Then $\varphi^\gamma(s_{n^b}) \in Y'$, thus b is not Y-terminal. ⊠

Lemma 13

Suppose that the trivial approximation is not Y_α-terminal. Then there is a continuous homomorphism from G_0 to G.

Proof of Lemma 13

By Lemma 12, there are approximations $a_n = (n, \varphi_n, (\psi_{k,n})_{k \in n})$ that are not Y_α-terminal, and each of which is extended by the next.

As promised earlier, we define $\varphi \colon 2^\omega \to \kappa^\omega$ and $\psi_k \colon 2^\omega \to \kappa^\omega$ by

$$\varphi(x) = \bigcup_{n \in \omega} \varphi_n(x \upharpoonright n) \text{ and } \psi_k(x) = \bigcup_{k \in n \in \omega} \psi_{k,n}(x \upharpoonright (n - (k+1))).$$

Proof of Lemma 13 (continued)

It remains to show that if $k \in \omega$ and $x \in 2^\omega$, then

$$(\psi_k(x), (\varphi(s_k \frown 0 \frown x), \varphi(s_k \frown 1 \frown x))) \in [\text{⚘}].$$

It is enough to show that every open neighborhood U of the pair $(\psi_k(x), (\varphi(s_k \frown 0 \frown x), \varphi(s_k \frown 1 \frown x)))$ contains a point of $[\text{⚘}]$.

Towards this end, fix $n \in \omega$ sufficiently large that $k \in n$ and

$$\mathcal{N}_{\psi_{k,n}(s)} \times (\mathcal{N}_{\varphi_n(s_k \frown 0 \frown s)} \times \mathcal{N}_{\varphi_n(s_k \frown 1 \frown s)}) \subseteq U,$$

where $s = x \upharpoonright (n - (k+1))$.

Proof of Lemma 13 (continued)

Our choice of a_n ensures the existence of $\gamma \in \Gamma(a_n, Y_\alpha)$.

Then $(\psi^\gamma(s), (\varphi^\gamma(s_k{}^\frown 0^\frown s), \varphi^\gamma(s_k{}^\frown 1^\frown s))) \in [\text{🌿}] \cap U$. $\boxtimes$ $\boxtimes$

Part II

Applications of the G_0 dichotomy

Theorem 14 (Mansfield, Souslin)

Suppose that κ is an aleph, X is a Hausdorff space, and $A \subseteq X$ is κ-Souslin. Then at least one of the following holds:

1. The cardinality of A is at most κ.
2. There is a continuous injection of 2^ω into A.

Proof of Theorem 14

Define $G = \Delta(A)^c$.

If there is a κ-coloring of G, then the cardinality of A is at most κ.

Proof of Theorem 14 (continued)

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi \colon 2^\omega \to A$ from G_0 to G.

Define $E = (\varphi \times \varphi)^{-1}(\Delta(A))$.

Then E is an equivalence relation on 2^ω with the Baire property which is disjoint from G_0.

Lemma 15

The equivalence relation E is meager.

Proof of Lemma 15

By Kuratowski-Ulam, it is enough to show each E-class is meager.

Suppose that C is a non-meager E-class.

By Lemma 1, there exists $(x, y) \in G_0 \upharpoonright C$.

But this contradicts the fact that E is disjoint from G_0. $\boxtimes$

Proof of Theorem 14 (continued)

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into E.

It follows that $\varphi \circ \psi$ is a continuous injection of 2^ω into A. ☒

Theorem 16 (Feng)

Suppose that κ is an aleph, X is a κ-Souslin Hausdorff space, and G is an open graph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of G.
2. There is a continuous embedding of $\Delta(2^\omega)$ into G^c.

Proof of Theorem 16

By Theorem 6, we can assume there is a continuous homomorphism $\varphi \colon 2^\omega \to X$ from G_0 to G.

Proof of Theorem 16 (continued)

Define $H = (\varphi \times \varphi)^{-1}(G)$.

Then H is an open graph intersecting all non-empty open squares.

Lemma 17

There is a continuous embedding ψ of $\Delta(2^\omega)$ into H^c.

Proof of Lemma 17

We will find a strictly increasing sequence of natural numbers k_n and an increasing sequence of functions $\psi_n \colon 2^n \to 2^{k_n}$ such that

$$\forall n \in \omega \forall s, t \in 2^n \; (s \neq t \implies \mathcal{N}_{\psi_n(s)} \times \mathcal{N}_{\psi_n(t)} \subseteq H).$$

Suppose that we have already found ψ_n.

Proof of Lemma 17 (continued)

For each $s \in 2^n$, fix $(x_s, y_s) \in H \upharpoonright \mathcal{N}_{\psi_n(s)}$.

Fix $k_{n+1} > k_n$ such that $\mathcal{N}_{x_s \upharpoonright k_{n+1}} \times \mathcal{N}_{y_s \upharpoonright k_{n+1}} \subseteq H$ for all $s \in 2^n$.

Define $\psi_{n+1}(s) = x_s \upharpoonright k_{n+1}$. ⊠

Clearly $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into G^c. ⊠

Theorem 18 (Lusin-Novikov)

Suppose that κ is an aleph, X and Y are Hausdorff spaces, and $R \subseteq X \times Y$ is κ-Souslin. Then at least one of the following holds:

1. The set R is the union of κ-many relatively κ^+-Borel graphs of partial functions.
2. There is a continuous injection of 2^ω into some vertical section of R.

Proof of Theorem 18

Define $G = \{((x_0, y_0), (x_1, y_1)) \in R \times R \mid x_0 = x_1 \text{ and } y_0 \neq y_1\}$.

If there is a κ^+-Borel κ-coloring of G, then R is the union of κ-many relatively κ^+-Borel graphs of partial functions.

Proof of Theorem 18 (continued)

By Theorem 6, we can assume there is a continuous homomorphism $\varphi\colon 2^\omega \to R$ from G_0 to G.

Set $\varphi_X = \mathrm{proj}_X \circ \varphi$ and $\varphi_Y = \mathrm{proj}_Y \circ \varphi$.

Then φ_X is a continuous homomorphism from E_0 to $\Delta(X)$.

Let x denote the constant value of φ_X.

Proof of Theorem 18 (continued)

Define $E = (\varphi_Y \times \varphi_Y)^{-1}(\Delta(Y))$.

By Lemma 15, the equivalence relation E is meager.

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into E.

It follows that $\varphi_Y \circ \psi$ is a continuous injection of 2^ω into R_x. ⊠

Definition

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\varphi\colon \omega^\omega \to X$, the set $\varphi^{-1}(B)$ has the Baire property.

Definition

A set $B \subseteq X$ is *weakly ω-universally Baire* if for every continuous function $\varphi\colon 2^\omega \to X$, the set $\varphi^{-1}(B)$ has the Baire property.

Question

Does $\mathtt{ZFC}$ imply that there is a weakly ω-universally Baire set which is not ω-universally Baire?

Theorem 19 (Silver, Harrington-Shelah)

Suppose that κ is an aleph, X is a Hausdorff space, and E is a weakly ω-universally Baire, co-κ-Souslin equivalence relation on X. Then at least one of the following holds:

1. The equivalence relation E has at most κ-many classes.
2. There is a continuous embedding of $\Delta(2^\omega)$ into E.

Proof of Theorem 19

Define $G = E^c$.

If there is a κ-coloring of G, then E has at most κ-many classes.

Proof of Theorem 19 (continued)

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from G_0 to G.

Define $F = (\varphi \times \varphi)^{-1}(E)$.

By Lemma 15, the equivalence relation F is meager.

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into F.

Then $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into E. ☒

Theorem 20 (Louveau)

Suppose that κ is an aleph, X is a Hausdorff space, and R is a weakly ω-universally Baire, co-κ-Souslin quasi-order on X. Then at least one of the following holds:

1. The equivalence relation $\equiv_R$ has at most κ-many classes.
2. There is a continuous embedding of $\Delta(2^\omega)$ or $R_{\mathrm{lex}}(2^\omega)$ into R.

Proof of Theorem 20

Define $G = R^c$.

If there is a κ-coloring of G, then $\equiv_R$ has at most κ-many classes.

Proof of Theorem 20 (continued)

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from G_0 to G.

Define $S = (\varphi \times \varphi)^{-1}(R)$.

If there is a non-empty open square in which S is meager, then Mycielski yields a continuous embedding ψ of $\Delta(2^\omega)$ into S.

Then $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into R.

Proof of Theorem 20 (continued)

So suppose that S is non-meager in every non-empty, open square.

By Lemma 15, the equivalence relation $\equiv_S$ is meager.

Lemma 21

There is a continuous embedding ψ of $R_{\mathrm{lex}}(2^\omega)$ into S.

Proof of Lemma 21

We will find a strictly increasing sequence of natural numbers k_n, an increasing sequence of functions $\psi_n \colon 2^n \to 2^{k_n}$, extensions $u_{s,i}$ of $\psi_n(s)$, and decreasing sequences $(U_{m,s})_{m \in \omega}$ of dense, open subsets of $\mathcal{N}_{u_{s,0}} \times \mathcal{N}_{u_{s,1}}$ with $\bigcap_{m \in \omega} U_{m,s} \subseteq <_S$, such that

$$\mathcal{N}_{\psi_n(r \frown 0 \frown s)} \times \mathcal{N}_{\psi_n(r \frown 1 \frown t)} \subseteq U_{n,r}$$

for all $m \in n \in \omega$, $r \in 2^m$, and $s, t \in 2^{n-(m+1)}$.

Suppose that we have already found ψ_n, as well as u_s, v_s, and $(U_{m,s})_{m \in \omega}$ for all $s \in 2^{<n}$.

For each $s \in 2^n$, fix extensions $u_{s,i}$ of $\psi_n(s)$ such that $<_S$ is comeager in $\mathcal{N}_{u_s} \times \mathcal{N}_{v_s}$, as well as decreasing sequences $(U_{m,s})_{m \in \omega}$ of dense, open subsets of $\mathcal{N}_{u_s} \times \mathcal{N}_{v_s}$ with $\bigcap_{m \in \omega} U_{m,s} \subseteq <_S$.

Define $\psi'_{n+1} \colon 2^{n+1} \to 2^{<\omega}$ by $\psi'_{n+1}(s \frown i) = u_{s,i}$.

Obtain ψ_{n+1} by fixing an enumeration of the pairs of length n of the form $(r^\frown 0^\frown s, r^\frown 1^\frown t)$, and recursively extending $\psi'_{n+1}(r^\frown 0^\frown s)$ and $\psi'_{n+1}(r^\frown 1^\frown t)$ so that $\mathcal{N}_{\psi_{n+1}(r^\frown 0^\frown s)} \times \mathcal{N}_{\psi_{n+1}(r^\frown 1^\frown t)} \subseteq U_{n,r}$. ⊠

Cleary $\varphi \circ \psi$ is a continuous embedding of $R_{\mathrm{lex}}(2^\omega)$ into R. ⊠

II. Applications of the G_0 dichotomy

The perfect set theorem for linear quasi-orders

Theorem 22 (Friedman-Shelah)

Suppose that κ is an aleph, X is a Hausdorff space, and R is a linear, weakly ω-universally Baire, co-κ-Souslin quasi-order on X. Then at least one of the following holds:

1. There is an R-dense set of cardinality κ.
2. There is a continuous embedding of 2^ω into a pairwise disjoint set of non-empty, open R-intervals.

Proof of Theorem 22

Set $I = \{(x, y) \in X \times X \mid (x, y)_R \neq \emptyset\}$.

Define $G = \{((x_0, y_0), (x_1, y_1)) \in I \times I \mid [x_0, y_0]_R \cap [x_1, y_1]_R = \emptyset\}$.

Proof of Theorem 22 (continued)

If there is a κ-coloring of G, then the family of all closed R-intervals with non-empty interiors can be written as the union of κ-many intersecting families.

Under $\mathtt{AC}_\kappa$, this is easily seen to be equivalent to the existence of an R-dense set of cardinality κ.

Proof of Theorem 22 (continued)

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to I$ from G_0 to G.

Define $H = (\varphi \times \varphi)^{-1}(G)$.

Lemma 23

The relation H^c is meager.

Proof of Lemma 23

Note first that $H^c = \bigcup_{i,j \in 2} H_{ij}$, where

$$H_{ij} = \{(x_0, x_1) \in 2^\omega \times 2^\omega \mid \varphi_i(x_j) \in [\varphi_0(x_{1-j}), \varphi_1(x_{1-j})]_R\}.$$

By symmetry, it is sufficient to show that H_{00} is meager.

By Kuratowski-Ulam, it is enough to show that if $(H_{00})_{x_0}$ has the Baire property, then it is meager.

Proof of Lemma 23 (continued)

If it is non-meager, then Lemma 1 yields $(x_1, x_2) \in G_0 \restriction (H_{00})_{x_0}$.

Then $(\varphi(x_1), \varphi(x_2)) \notin G$, a contradiction. ⊠

Proof of Theorem 22 (continued)

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into H^c.

Then $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into G^c. ⊠

Theorem 24 (Friedman-Harrington-Kechris)

Suppose that κ is an aleph, X is a Hausdorff space, and d is a quasi-metric on X such that for all $\epsilon > 0$, the set $d^{-1}[0, \epsilon)$ is ω-universally Baire and co-κ-Souslin. Then one of the following holds:

1. There is a d-dense set of cardinality at most κ.
2. There is a continuous embedding of $\Delta(2^\omega)$ into $d^{-1}[0, \epsilon)$, for some $\epsilon > 0$.

Proof of Theorem 24

For each $n \in \omega \setminus \{0\}$, define $G_n = d^{-1}[1/n, \infty)$.

Proof of Theorem 24 (continued)

If each G_n has a κ-coloring, then there is a basis of size at most κ.

Under $\mathtt{AC}_\kappa$, this is easily seen to be equivalent to the existence of a d-dense set of cardinality at most κ.

Proof of Theorem 24 (continued)

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from G_0 to some G_n.

Define $e\colon 2^\omega \to \mathbb{R}$ by $e(x, y) = d(\varphi(x), \varphi(y))$.

Lemma 25

The set $e^{-1}[0, 1/2n)$ is meager.

Proof of Lemma 25

By Kuratowski-Ulam, it is enough to show that if $\mathcal{B}_e(x, 1/2n)$ has the Baire property, then it is meager.

Suppose that $\mathcal{B}_e(x, 1/2n)$ is non-meager.

By Lemma 1, there exists $(y, z) \in G_0 \upharpoonright \mathcal{B}_e(x, 1/2n)$.

Then $e(y, z) < 1/n$, thus $(\varphi(y), \varphi(z)) \notin G_n$, a contradiction. ⊠

Proof of Theorem 24 (continued)

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into the relation $e^{-1}[0, 1/2n)$.

It follows that $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into the relation $d^{-1}[0, 1/2n)$. ⊠

Theorem 26 (Dougherty-Jackson-Kechris)

Suppose that κ is an aleph, X is a Hausdorff space, and E is a weakly ω-universally Baire, κ-Souslin equivalence relation on X. Then at least one of the following holds:

1. There are κ-many κ^+-Borel partial E-transversals covering X.
2. There is a continuous injection of 2^ω into some E-class.
3. There is a continuous embedding of E_0 into E.

Proof of Theorem 26

Define $G = E \setminus \Delta(X)$.

Proof of Theorem 26 (continued)

If there is a κ^+-Borel κ-coloring of G, then there is a family of κ-many κ^+-Borel partial transversals of E which cover X.

By Theorem 6, we can assume there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from G_0 to G.

Define $D = (\varphi \times \varphi)^{-1}(\Delta(X))$.

By Lemma 15, the equivalence relation D is meager.

Proof of Theorem 26 (continued)

If the equivalence relation $F = (\varphi \times \varphi)^{-1}(E)$ is non-meager, then Kuratowski-Ulam yields a non-meager F-class C.

Mycielski gives a continuous embedding ψ of $\Delta(2^\omega)$ into $D \upharpoonright C$.

Then $\varphi \circ \psi$ is a continuous injection of 2^ω into $\varphi[C]$.

Otherwise, F is a meager equivalence relation containing E_0.

Lemma 27

There is a continuous embedding ψ of $(\Delta(2^\omega), E_0)$ into (D, F).

Proof of Lemma 27

Fix a decreasing sequence of dense, open sets $U_n \subseteq D^c$ such that $F \cap \bigcap_{n \in \omega} U_n = \emptyset$.

It is enough to construct $k_n \in \omega$ and $u_{i,n} \in 2^{k_n}$ such that

$$\forall n \in \omega \forall s, t \in 2^n \ (\mathcal{N}_{\psi_{n+1}(s \frown 0)} \times \mathcal{N}_{\psi_{n+1}(t \frown 1)} \subseteq U_n),$$

where $\psi_n \colon 2^n \to 2^{\sum_{m \in n} k_m}$ is given by $\psi_n(s) = \bigoplus_{m \in n} u_{s(m),m}$.

Proof of Lemma 27 (continued)

Suppose that we have found k_m and $u_{i,m}$ for all $i \in 2$ and $m \in n$.

Fix an enumeration $(s_k, t_k)_{k \leq \ell}$ of $2^n \times 2^n$.

Recursively construct increasing sequences $(u_{i,k,n})_{k \leq \ell}$ such that

$$\forall k \leq \ell \ (\mathcal{N}_{\psi_n(s_k) \frown u_{0,k,n}} \times \mathcal{N}_{\psi_n(t_k) \frown u_{1,k,n}} \subseteq U_n).$$

Set $u_{i,n} = u_{i,\ell,n}$ and $k_n = |u_{0,n}| = |u_{1,n}|$. ⊠

Clearly $\varphi \circ \psi$ is a continuous embedding of E_0 into E. ⊠

Definition

Let F_0 denote the equivalence relation on 2^ω given by

$$xF_0y \iff (\mathrm{parity}(x \upharpoonright n))_{n\in\omega} E_0 (\mathrm{parity}(y \upharpoonright n))_{n\in\omega},$$

where $\mathrm{parity}(s) = \sum_{i\in n} s(i) \pmod 2$ for $n \in \omega$ and $s \in 2^n$.

Theorem 28 (Louveau)

Suppose that κ is an aleph, X is a Hausdorff space, E is a weakly ω-universally Baire, κ-Souslin equivalence relation on X, and F is a weakly ω-universally Baire, co-κ-Souslin equivalence relation on X of index two below E. Then at least one of the following holds:

1. There is a cover of X with κ-many κ^+-Borel partial transversals of E over F.
2. There is a continuous embedding of (E_0, F_0) into (E, F).

Proof of Theorem 28

Define $G = E \setminus F$.

Proof of Theorem 28 (continued)

If there is a κ^+-Borel κ-coloring of G, then there are κ-many κ^+-Borel partial transversals of E over F which cover X.

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from G_0 to G.

Define $E' = (\varphi \times \varphi)^{-1}(E)$ and $F' = (\varphi \times \varphi)^{-1}(F)$.

Proof of Theorem 28 (continued)

By Lemma 15, the equivalence relation F' is meager.

Kuratowski-Ulam then implies that E' is meager.

Observe that $F_0 \subseteq F'$ and $E_0 \setminus F_0 \subseteq E' \setminus F'$.

Set $D' = (\varphi \times \varphi)^{-1}(\Delta(X))$.

Lemma 29

There is a continuous embedding ψ of the triple $(\Delta(2^\omega), E_0, F_0)$ into the triple (D', E', F').

Proof of Lemma 29

Fix a decreasing sequence of dense, open sets $U_n \subseteq (D')^c$ such that $E' \cap \bigcap_{n \in \omega} U_n = \emptyset$.

We construct $k_n \in \omega$ and $u_{i,n} \in 2^{k_n}$ with differing parities such that

$$\forall n \in \omega \forall s, t \in 2^n \ (\mathcal{N}_{\psi_{n+1}(s \frown 0)} \times \mathcal{N}_{\psi_{n+1}(t \frown 1)} \subseteq U_n),$$

where $\psi_n \colon 2^n \to 2^{\sum_{m \in n} k_m}$ is given by $\psi_n(s) = \bigoplus_{m \in n} u_{s(m),m}$.

Proof of Lemma 29 (continued)

Suppose that we have found k_m and $u_{i,m}$ for all $i \in 2$ and $m \in n$.

Fix an enumeration $(s_k, t_k)_{k \leq \ell}$ of $2^n \times 2^n$.

Recursively construct increasing sequences $(u_{i,k,n})_{k \leq \ell}$ such that

$$\forall k \leq \ell \ (\mathcal{N}_{\psi_n(s_k)^\frown u_{0,k,n}} \times \mathcal{N}_{\psi_n(t_k)^\frown u_{1,k,n}} \subseteq U_n).$$

Proof of Lemma 29 (continued)

If $\mathrm{parity}(u_{0,\ell,n}) \neq \mathrm{parity}(u_{1,\ell,n})$, then set $u_{i,n} = u_{i,\ell,n}$.

Otherwise, set $u_{i,n} = u_{i,\ell,n} {}^\frown i$.

Define $k_n = |u_{0,n}| = |u_{1,n}|$. ⊠

Clearly $\varphi \circ \psi$ is a continuous embedding of (E_0, F_0) into (E, F). ⊠

Part III

The hypergraph G_0 dichotomy

Definition

A $(\leq d)$-*dimensional dihypergraph* on X is a set $G \subseteq X^d$ of non-constant sequences.

The *restriction* of G to $Y \subseteq X$ is given by $G \restriction Y = G \cap Y^d$.

A set $Y \subseteq X$ is G-*independent* if $G \restriction Y = \emptyset$.

An (I-)*coloring* of G is a function $c \colon X \to I$ with the property that for all $i \in I$, the set $c^{-1}(\{i\})$ is G-independent.

Example

The *dihypergraph on d^ω associated with $S \subseteq d^{<\omega}$* is given by

$$G_S = \{(s^\frown i^\frown x)_{i \in d} \mid s \in S \text{ and } x \in d^\omega\}.$$

Definition

A set $S \subseteq d^{<\omega}$ is *dense* if $\forall r \in d^{<\omega} \exists s \in S \ (r \sqsubseteq s)$.

Lemma 30

Suppose that $d \in \omega \setminus 2$, $B \subseteq d^\omega$ is a non-meager set with the Baire property, and $S \subseteq d^{<\omega}$ is dense. Then B is not G_S-independent.

Proof of Lemma 30

Fix $r \in d^{<\omega}$ such that B is comeager in $\mathcal{N}_r$.

Fix $s \in S$ such that $r \sqsubseteq s$.

Then $(s^\frown i^\frown x)_{i \in d} \in G_S \upharpoonright B$ for comeagerly many $x \in d^\omega$. ⊠

Lemma 31

Suppose that $d \in \omega \setminus 2$, κ is an aleph, $S \subseteq d^{<\omega}$ is dense, and c is a κ-coloring of G_S. Then the set $(c \times c)^{-1}(\leq)$ does not have the Baire property.

Proof of Lemma 31

Set $R = (c \times c)^{-1}(\leq)$ and $E = (c \times c)^{-1}(\Delta(\kappa))$.

Proof of Lemma 31 (continued)

If R has the Baire property, then Kuratowski-Ulam yields a least $\alpha \in \kappa$ for which $c^{-1}(\leq^\alpha)$ is non-meager and has the Baire property.

Then the E-class $C = c^{-1}(\{\alpha\})$ is non-meager.

By Lemma 30, there exists $(x_i)_{i \in d} \in G_S \upharpoonright C$, a contradiction. ⊠

Lemma 32

Suppose that $d \in \omega \setminus 2$, κ is an aleph, $S \subseteq d^{<\omega}$ is dense, and the family of subsets of d^ω with the Baire property is closed under κ-length unions. Then there is no κ-coloring of G_S with respect to which pre-images of singletons have the Baire property.

Proof of Lemma 32

Suppose that c is a κ-coloring of G_S with respect to which pre-images of singletons have the Baire property.

Then $(c \times c)^{-1}(\leq)$ has the Baire property.

But this directly contradicts Lemma 31. ⊠

Definition

Fix sequences $s_n \in d^n$ such that the set $S = \{s_n \mid n \in \omega\}$ is dense.

Define $G_0(d^\omega) = G_S$.

Lemma 33

Suppose that $d \in \omega \setminus 2$, κ is a good aleph, X is a Hausdorff space, G is a weakly κ-Souslin, $(\leq d)$-dimensional dihypergraph on X, and $A \subseteq X$ is G-independent and weakly κ-Souslin. Then there is a G-independent, κ^+-Borel set $B \subseteq X$ such that $A \subseteq B$.

Proof of Lemma 33

By Lemma 4, there is a G-independent sequence $(B_i)_{i \in d}$ of κ^+-Borel subsets of X such that $A \subseteq B_i$ for all $i \in d$.

Clearly the set $B = \bigcap_{i \in d} B_i$ is as desired. ⊠

Theorem 34 (Louveau)

Suppose that $d \in \omega \setminus 2$, κ is an aleph, X is a Hausdorff space, and G is a κ-Souslin, $(\leq d)$-dimensional dihypergraph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of G.
2. There is a continuous homomorphism from $G_0(d^\omega)$ to G.

Proof of Theorem 34

We will prove the special case of the theorem for good κ.

Lemma 35

It is sufficient to handle the special case that $X = \kappa^\omega$.

Proof of Lemma 35

We can clearly assume that $G \neq \emptyset$, so $\mathrm{proj}_X(G) \neq \emptyset$, thus there is a continuous surjection $\varphi\colon \kappa^\omega \to \mathrm{proj}_X(G)$. Set $H = (\varphi^d)^{-1}(G)$.

If there is a κ^+-Borel κ-coloring of H, then Lemma 33 allows us to produce a κ^+-Borel κ-coloring of G.

If $\psi\colon d^\omega \to \kappa^\omega$ is a continuous homomorphism from $G_0(d^\omega)$ to H, then $\varphi \circ \psi$ is a continuous homomorphism from $G_0(d^\omega)$ to G. ⊠

Definition

An *approximation* is a triple of the form $a = (n^a, \varphi^a, (\psi^a_k)_{k \in n^a})$, where $n^a \in \omega$, $\varphi^a \colon d^{n^a} \to \kappa^{n^a}$, and $\psi^a_k \colon d^{n^a-(k+1)} \to \kappa^{n^a}$.

We say that an approximation a is *extended* by an approximation b if for all $k \in n^a$, the following conditions are satisfied:

1. $n^a \leq n^b$.
2. $\forall r \in d^{n^a} \forall s \in d^{n^b} \ (r \sqsubseteq s \implies \varphi^a(r) \sqsubseteq \varphi^b(s))$.
3. $\forall r \in d^{n^a-(k+1)} \forall s \in d^{n^b-(k+1)} \ (r \sqsubseteq s \implies \psi^a_k(r) \sqsubseteq \psi^b_k(s))$.

If $n^b = n^a + 1$, then we say that b is a *one-step extension* of a.

Fix a κ-length well-ordering of the set of all approximations.

Proof of Theorem 34 (continued)

Fix a tree $\mathcal{T}$ on $\kappa \times \kappa^d$ such that $G = \mathrm{proj}_{(\kappa^\omega)^d}[\mathcal{T}]$.

Definition

A *configuration* is a triple of the form $\gamma = (n^\gamma, \varphi^\gamma, (\psi_k^\gamma)_{k \in n^\gamma})$, where $n^\gamma \in \omega$, $\varphi^\gamma \colon d^{n^\gamma} \to \kappa^\omega$, and $\psi_k^\gamma \colon d^{n^\gamma - (k+1)} \to \kappa^\omega$, such that

$$(\psi_k^\gamma(s), (\varphi^\gamma(s_k {}^\frown i {}^\frown s))_{i \in d}) \in [\mathcal{T}]$$

for all $k \in n^\gamma$ and $s \in d^{n^\gamma - (k+1)}$.

Definition

We say that a configuration γ is *compatible* with an approximation a if the following conditions are satisfied:

1. $n^a = n^\gamma$.
2. $\forall s \in d^{n^a}\ (\varphi^a(s) \sqsubseteq \varphi^\gamma(s))$.
3. $\forall k \in n^a \forall s \in d^{n^a-(k+1)}\ (\psi^a_k(s) \sqsubseteq \psi^\gamma_k(s))$.

We say that γ is *compatible* with a set $Y \subseteq \kappa^\omega$ if $\varphi^\gamma[d^{n^\gamma}] \subseteq Y$.

We use $\Gamma(a, Y)$ to denote the family of all configurations which are compatible with both a and Y.

Definition

We say that an approximation a is *Y-terminal* if $\Gamma(b, Y) = \emptyset$ for all one-step extensions b of a.

We use $T(Y)$ to denote the family of all such approximations.

Define $A(a, Y) \subseteq Y$ by $A(a, Y) = \{\varphi^\gamma(s_{n^a}) \mid \gamma \in \Gamma(a, Y)\}$.

Lemma 36

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$, and $A(a, Y)$ is not G-independent. Then a is not Y-terminal.

Proof of Lemma 36

Fix configurations $\gamma_i \in \Gamma(a, Y)$ with $(\varphi^{\gamma_i}(s_{n^a}))_{i \in d} \in G$.

Fix $x \in \kappa^\omega$ such that $(x, (\varphi^{\gamma_i}(s_{n^a}))_{i \in d}) \in [🌿]$.

Proof of Lemma 36 (continued)

Let γ denote the configuration given by:

1. $n^\gamma = n^a + 1$.
2. $\forall i \in d \forall s \in d^{n^a} \ (\varphi^\gamma(s ^\frown i) = \varphi^{\gamma_i}(s))$.
3. $\forall i \in d \forall k \in n^a \forall s \in d^{n^a-(k+1)} \ (\psi_k^\gamma(s ^\frown i) = \psi_k^{\gamma_i}(s))$.
4. $\psi_{n^a}^\gamma(\emptyset) = x$.

Let b denote the approximation given by:

1. $n^b = n^\gamma$.
2. $\forall s \in d^{n^b} \ (\varphi^b(s) = \varphi^\gamma(s) \upharpoonright n^b)$.
3. $\forall k \in n^b \forall s \in d^{n^b-(k+1)} \ (\psi_k^b(s) = \psi_k^\gamma(s) \upharpoonright n^b)$.

Proof of Lemma 36 (continued)

Clearly γ is compatible with b.

Clearly b is a one-step extension of a.

It follows that a is not Y-terminal. ⊠

Lemma 37

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is Y-terminal. Then there is a G-independent, κ^+-Borel subset $B(a, Y)$ of κ^ω such that $A(a, Y) \subseteq B(a, Y)$.

Proof of Lemma 37

Lemma 36 ensures that $A(a, Y)$ is G-independent.

The desired result therefore follows from Lemma 33. ⊠

Definition

Set $Y' = Y \setminus \bigcup_{a \in T(Y)} B(a, Y)$.

Lemma 38

There is a κ^+-Borel κ-coloring of $G \restriction (Y \setminus Y')$.

Proof of Lemma 38

Define $c(y) = \min\{a \in T(Y) \mid y \in B(a, Y)\}$ for $y \in Y \setminus Y'$.

As $c^{-1}(\{a\}) \subseteq B(a, Y)$ for all $a \in T(Y)$, it follows that c is a coloring of $G \restriction (Y \setminus Y')$. ⊠

Definition

Recursively define a sequence $(Y_\alpha)_{\alpha \in \kappa^+}$ of subsets of κ^ω by

$$Y_\alpha = \begin{cases} \kappa^\omega & \text{if } \alpha = 0, \\ Y'_\beta & \text{if } \alpha = \beta + 1, \text{ and} \\ \bigcap_{\beta \in \alpha} Y_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Since there are only κ-many approximations, there exists $\alpha \in \kappa^+$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

Lemma 39

Suppose that the trivial approximation is Y_α-terminal. Then there is a κ^+-Borel κ-coloring of G.

Proof of Lemma 39

Note first that $Y_{\alpha+1} = \emptyset$, thus $\kappa^\omega = \bigcup_{\beta \leq \alpha} Y_\beta \setminus Y_{\beta+1}$.

As all $G \restriction (Y_\beta \setminus Y_{\beta+1})$ admit κ^+-Borel κ-colorings, so does G. ⊠

Lemma 40

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is not Y'-terminal. Then there is a one-step extension of a which is not Y-terminal.

Proof of Lemma 40

Fix a one-step extension b of a for which $\Gamma(b, Y') \neq \emptyset$.

Fix a configuration $\gamma \in \Gamma(b, Y')$.

Then $\varphi^\gamma(s_{n^b}) \in Y'$, thus b is not Y-terminal. ⊠

Lemma 41

Suppose that the trivial approximation is not Y_α-terminal. Then there is a continuous homomorphism from G_0 to G.

Proof of Lemma 41

By Lemma 40, there are approximations $a_n = (n, \varphi_n, (\psi_{k,n})_{k \in n})$ that are not Y_α-terminal, and each of which is extended by the next.

Define $\varphi \colon d^\omega \to \kappa^\omega$ and $\psi_k \colon d^\omega \to \kappa^\omega$ by

$$\varphi(x) = \bigcup_{n \in \omega} \varphi_n(x \restriction n) \text{ and } \psi_k(x) = \bigcup_{k \in n \in \omega} \psi_{k,n}(x \restriction (n - (k+1))).$$

Proof of Lemma 41 (continued)

It remains to show that if $k \in \omega$ and $x \in d^\omega$, then

$$(\psi_k(x), (\varphi(s_k \frown i \frown x))_{i \in d}) \in [\text{🌿}].$$

It is enough to show that every open neighborhood U of the pair $(\psi_k(x), (\varphi(s_k \frown i \frown x))_{i \in d})$ contains a point of $[\text{🌿}]$.

Towards this end, fix $n \in \omega$ sufficiently large that $k \in n$ and

$$\mathcal{N}_{\psi_{k,n}(s)} \times \prod_{i \in d} \mathcal{N}_{\varphi_n(s_k \frown i \frown s)} \subseteq U,$$

where $s = x \upharpoonright (n - (k+1))$.

Proof of Lemma 41 (continued)

Our choice of a_n ensures the existence of $\gamma \in \Gamma(a_n, Y_\alpha)$.

Then $(\psi^\gamma(s), (\varphi^\gamma(s_k {}^\frown i {}^\frown s))_{i \in d}) \in [\text{🌿}] \cap U$. ⊠ ⊠

Part IV

Applying the hypergraph dichotomy

Theorem 42 (Kunen-Miller-van Engelen)

Suppose that $d \in \omega \setminus 2$, κ is an aleph, X is a Hausdorff space, $A \subseteq X$ is analytic, and X is equipped with a vector space structure for which the set $D \subseteq X^{\leq d}$ of dependent sequences is weakly ω-universally Baire and co-κ-Souslin. Then at least one of the following holds:

1. There is a cover of A with κ-many translates of $(\leq d)$-dimensional, κ^+-Borel subsets of X.
2. There is a continuous embedding of the set of non-injective sequences in $(2^\omega)^{d+1}$ into $A^{d+1} \cap D$.

Proof of Theorem 42

For each $\ell \leq d$, set $G_\ell = \{(x_i)_{i \leq \ell} \in A^{\ell+1} \mid (x_i - x_\ell)_{i \in \ell} \notin D\}$.

If there is a κ^+-Borel κ-coloring of G_d, then we obtain the covering.

By Theorem 34, we can assume that there is a continuous homomorphism $\varphi\colon (d+1)^\omega \to X$ from $G_0((d+1)^\omega)$ to G_d.

For each $\ell \leq d$, set $H_\ell = (\varphi^\ell)^{-1}(G_\ell)$.

Lemma 43

Suppose that $\ell \leq d$. Then H_ℓ^c is meager.

Proof of Lemma 43

By Kuratowski-Ulam, it is enough to show that if $\ell \in d$, $x \in H_\ell$, and $(H_{\ell+1})_x$ has the Baire property, then $(H_{\ell+1})_x$ is comeager.

Suppose that $(H_{\ell+1})_x$ is not comeager.

Then there exists $(x_i)_{i \in d+1} \in G_0((d+1)^\omega) \restriction (H_{\ell+1})_x^c$.

Then $(\varphi(x_i))_{i \in d+1} \notin G$, a contradiction. ⊠

Proof of Theorem 42 (continued)

By Mycielski, there is a continuous embedding ψ of the set of non-injective sequences in $(2^\omega)^d$ into D_d.

Then $\varphi \circ \psi$ is a continuous embedding of the set of non-injective sequences in $(2^\omega)^d$ into D. ⊠

Part V

The sequential G_0 dichotomy

Definition

A set $Y \subseteq X$ is $(G^n)_{n \in \omega}$-*independent* if it is G^n-independent for some $n \in \omega$.

An (I-)*coloring* of $(G^n)_{n \in \omega}$ is a function $c \colon X \to I$ with the property that for all $i \in I$, the set $c^{-1}(\{i\})$ is $(G^n)_{n \in \omega}$-independent.

Suppose that $(d_n)_{n \in \omega} \in (\omega \setminus 2)^\omega$ and $f \colon \omega \times \omega \to \omega$ is a bijection.

Example

Associated with each set $S \subseteq \bigcup_{n \in \omega} \prod_{m \in n} d_{f_0(m)}$ are the sets

$$S^k = \{s \in S \cap \prod_{m \in n} d_{f_0(m)} \mid n \in \omega \text{ and } f_0(n) = k\}$$

and the dihypergraphs

$$G_S^k = \{(s^\frown i^\frown x)_{i \in d} \mid s \in S^k \text{ and } x \in \prod_{n \in \omega} d_{f_0(n)}\}.$$

Definition

A set $S \subseteq \bigcup_{n\in\omega} \prod_{m\in n} d_{f_0(m)}$ is *dense* if

$$\forall k \in \omega \forall n \in \omega \forall r \in \prod_{m\in n} d_{f_0(m)} \exists s \in S^k \ (r \sqsubseteq s).$$

Definition

Fix $s_n \in \prod_{m\in n} d_{f_0(m)}$ such that the set $S = \{s_n \mid n \in \omega\}$ is dense.

Define $G_0^k(\prod_{n\in\omega} d_{f_0(n)}) = G_S^k$.

Define also $G_0^k = G_0^k(2^\omega)$.

Theorem 44

Suppose that κ is an aleph, X is a Hausdorff space, and G^k is a κ-Souslin, $(\leq d_k)$-dimensional dihypergraph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of G.
2. There is a continuous homomorphism from the ω-sequence $(G_0^k(\prod_{n \in \omega} d_{f_0(n)}))_{k \in \omega}$ to the ω-sequence $(G^k)_{k \in \omega}$.

Proof of Theorem 44

We will prove the special case of the theorem for good κ.

Lemma 45

It is sufficient to handle the special case that $X = \kappa^\omega$.

Proof of Lemma 45

We can clearly assume that every G^k is non-empty, thus so too is every set of the form $\mathrm{proj}_X(G^k) \neq \emptyset$.

Fix a continuous surjection $\varphi\colon \kappa^\omega \to \bigcup_{k \in \omega} \mathrm{proj}_X(G^k)$.

Set $H^k = (\varphi^d)^{-1}(G^k)$.

Proof of Lemma 45 (continued)

If there is a κ^+-Borel κ-coloring of $(H^k)_{k\in\omega}$, then Lemma 5 allows us to produce a κ^+-Borel κ-coloring of $(G^k)_{k\in\omega}$.

If $\psi\colon \prod_{n\in\omega} d_{f_0(n)} \to \kappa^\omega$ is a continuous homomorphism from $(G_0^k(\prod_{n\in\omega} d_{f_0(n)}))_{k\in\omega}$ to $(H^k)_{k\in\omega}$, then $\varphi \circ \psi$ is a continuous homomorphism from $(G_0^k(\prod_{n\in\omega} d_{f_0(n)}))_{k\in\omega}$ to $(G^k)_{k\in\omega}$. ⊠

Definition

An *approximation* is a triple $a = (n^a, \varphi^a, (\psi^a_k)_{k \in n^a})$, where $n^a \in \omega$, $\varphi^a \colon \prod_{m \in n^a} d_{f_0(m)} \to \kappa^{n^a}$, and $\psi^a_k \colon \prod_{m \in n^a \setminus (k+1)} d_{f_0(m)} \to \kappa^{n^a}$.

We say that an approximation a is *extended* by an approximation b if φ^a and $(\psi^a_k)_{k \in n^a}$ are extended by φ^b and $(\psi^b_k)_{k \in n^a}$.

If $n^b = n^a + 1$, then we say that b is a *one-step extension* of a.

Fix a κ-length well-ordering of the set of all approximations.

Proof of Theorem 44 (continued)

Fix trees $\mathcal{T}^k$ on $\kappa \times \kappa^d$ such that $G^k = \mathrm{proj}_{(\kappa^\omega)^d}[\mathcal{T}^k]$.

Definition

A *configuration* is a triple $\gamma = (n^\gamma, \varphi^\gamma, (\psi_k^\gamma)_{k \in n^\gamma})$, where $n^\gamma \in \omega$, $\varphi^\gamma \colon \prod_{m \in n^\gamma} d_{f_0(m)} \to \kappa^\omega$, and $\psi_k^\gamma \colon \prod_{m \in n^\gamma \setminus (k+1)} d_{f_0(m)} \to \kappa^\omega$, with

$$(\psi_k^\gamma(s), (\varphi^\gamma(s_k \frown i \frown s))_{i \in d_{f_0(k)}}) \in [\mathcal{T}^{f_0(k)}]$$

for all $k \in n^\gamma$ and $s \in \prod_{m \in n^\gamma \setminus (k+1)} d_{f_0(m)}$.

Definition

We say that a configuration γ is *compatible* with an approximation a if the following conditions are satisfied:

1. $n^a = n^\gamma$.
2. $\forall s \in \prod_{m \in n^a} d_{f_0(m)} \ (\varphi^a(s) \sqsubseteq \varphi^\gamma(s))$.
3. $\forall k \in n^a \forall s \in \prod_{m \in n^a \setminus (k+1)} d_{f_0(m)} \ (\psi^a_k(s) \sqsubseteq \psi^\gamma_k(s))$.

We say that γ is *compatible* with $Y \subseteq \kappa^\omega$ if $\varphi^\gamma[\prod_{m \in n^\gamma} d_{f_0(m)}] \subseteq Y$.

We use $\Gamma(a, Y)$ to denote the family of all configurations which are compatible with both a and Y.

Definition

We say that an approximation a is *Y-terminal* if $\Gamma(b, Y) = \emptyset$ for all one-step extensions b of a.

We use $T(Y)$ to denote the family of all such approximations.

Define $A(a, Y) \subseteq Y$ by $A(a, Y) = \{\varphi^\gamma(s_{n^a}) \mid \gamma \in \Gamma(a, Y)\}$.

Lemma 46

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$, and $A(a, Y)$ is not $(G^k)_{k \in \omega}$-independent. Then a is not Y-terminal.

Proof of Lemma 46

Fix configurations $\gamma_i \in \Gamma(a, Y)$ with $(\varphi^{\gamma_i}(s_{n^a}))_{i \in d_{f_0(n^a)}} \in G^{f_0(n^a)}$.

Fix $x \in \kappa^\omega$ such that $(x, (\varphi^{\gamma_i}(s_{n^a}))_{i \in d_{f_0(n^a)}}) \in [\text{🌿}^{f_0(n^a)}]$.

Proof of Lemma 46 (continued)

Let γ denote the configuration given by:

1. $n^\gamma = n^a + 1$.
2. $\forall i \in d_{f_0(n^a)} \forall s \in \prod_{m \in n^a} d_{f_0(m)} \, (\varphi^\gamma(s^\frown i) = \varphi^{\gamma_i}(s))$.
3. $\forall i \in d_{f_0(n^a)} \forall k \in n^a \forall s \in \prod_{m \in n^a \setminus (k+1)} d_{f_0(m)}$
   $(\psi_k^\gamma(s^\frown i) = \psi_k^{\gamma_i}(s))$.
4. $\psi_{n^a}^\gamma(\emptyset) = x$.

Proof of Lemma 46 (continued)

Let b denote the approximation given by:

1. $n^b = n^\gamma$.
2. $\forall s \in \prod_{m \in n^a} d_{f_0(m)} \ (\varphi^b(s) = \varphi^\gamma(s) \upharpoonright n^b)$.
3. $\forall k \in n^b \forall s \in \prod_{m \in n^a \setminus (k+1)} d_{f_0(m)} \ (\psi_k^b(s) = \psi_k^\gamma(s) \upharpoonright n^b)$.

Proof of Lemma 46 (continued)

Clearly γ is compatible with b.

Clearly b is a one-step extension of a.

It follows that a is not Y-terminal. ⊠

Lemma 47

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is Y-terminal. Then there is a $(G^k)_{k \in \omega}$-independent, κ^+-Borel subset $B(a, Y)$ of κ^ω such that $A(a, Y) \subseteq B(a, Y)$.

Proof of Lemma 47

Lemma 46 ensures that $A(a, Y)$ is $(G^k)_{k \in \omega}$-independent.

The desired result therefore follows from Lemma 33. ⊠

Definition

Set $Y' = Y \setminus \bigcup_{a \in T(Y)} B(a, Y)$.

Lemma 48

There is a κ^+-Borel κ-coloring of $G \restriction (Y \setminus Y')$.

Proof of Lemma 48

Define $c(y) = \min\{a \in T(Y) \mid y \in B(a, Y)\}$ for $y \in Y \setminus Y'$.

As $c^{-1}(\{a\}) \subseteq B(a, Y)$ for all $a \in T(Y)$, it follows that c is a coloring of $G \restriction (Y \setminus Y')$. ⊠

Definition

Recursively define a sequence $(Y_\alpha)_{\alpha \in \kappa^+}$ of subsets of κ^ω by

$$Y_\alpha = \begin{cases} \kappa^\omega & \text{if } \alpha = 0, \\ Y'_\beta & \text{if } \alpha = \beta + 1, \text{ and} \\ \bigcap_{\beta \in \alpha} Y_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Since there are only κ-many approximations, there exists $\alpha \in \kappa^+$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

Lemma 49

Suppose that the trivial approximation is Y_α-terminal. Then there is a κ^+-Borel κ-coloring of $(G^k)_{k \in \omega}$.

Proof of Lemma 49

Note first that $Y_{\alpha+1} = \emptyset$, thus $\kappa^\omega = \bigcup_{\beta \leq \alpha} Y_\beta \setminus Y_{\beta+1}$.

As all of the sequences $(G^k)_{k \in \omega} \upharpoonright (Y_\beta \setminus Y_{\beta+1})$ admit κ^+-Borel κ-colorings, so too does $(G^k)_{k \in \omega}$. ⊠

Lemma 50

Suppose that a is an approximation, $Y \subseteq \kappa^\omega$ is κ^+-Borel, and a is not Y'-terminal. Then there is a one-step extension of a which is not Y-terminal.

Proof of Lemma 50

Fix a one-step extension b of a for which $\Gamma(b, Y') \neq \emptyset$.

Fix a configuration $\gamma \in \Gamma(b, Y')$.

Then $\varphi^\gamma(s_{n^b}) \in Y'$, thus b is not Y-terminal. ⊠

Lemma 51

Suppose that the trivial approximation a_0 is not Y_α-terminal. Then there is a continuous homomorphism from the sequence $(G_0^k(\prod_{n\in\omega} d_{f_0(n)}))_{k\in\omega}$ to the sequence $(G^k)_{k\in\omega}$.

Proof of Lemma 51

By Lemma 50, there are approximations $a_n = (n, \varphi_n, (\psi_{k,n})_{k\in n})$ that are not Y_α-terminal, and each of which is extended by the next.

Define $\varphi\colon \prod_{n\in\omega} d_{f_0(n)} \to \kappa^\omega$ and $\psi_k\colon \prod_{n\in\omega\setminus(k+1)} d_{f_0(n)} \to \kappa^\omega$ by

$$\varphi(x) = \bigcup_{n\in\omega} \varphi_n(x \restriction n) \text{ and } \psi_k(x) = \bigcup_{k\in n\in\omega} \psi_{k,n}(x \restriction (n-(k+1))).$$

Proof of Lemma 51 (continued)

It remains to show that if $k \in \omega$ and $x \in \prod_{n \in \omega \setminus (k+1)} d_{f_0(n)}$, then

$$(\psi_k(x), (\varphi(s_k {}^\frown i {}^\frown x))_{i \in d_{f_0(k)}}) \in [\mathbb{T}^{f_0(k)}].$$

It is enough to show that every open neighborhood U of the pair $(\psi_k(x), (\varphi(s_k {}^\frown i {}^\frown x))_{i \in d_{f_0(k)}})$ contains a point of $[\mathbb{T}^{f_0(k)}]$.

Proof of Lemma 51 (continued)

Towards this end, fix $n \in \omega$ sufficiently large that $k \in n$ and

$$\mathcal{N}_{\psi_{k,n}(s)} \times \prod_{i \in d_{f_0(k)}} \mathcal{N}_{\varphi_n(s_k \frown i \frown s)} \subseteq U,$$

where $s = x \upharpoonright (n - (k + 1))$.

Our choice of a_n ensures the existence of $\gamma \in \Gamma(a_n, Y_\alpha)$.

Then $(\psi^\gamma(s), (\varphi^\gamma(s_k \frown i \frown s))_{i \in d_{f_0(k)}}) \in [\text{🌿}^{f_0(k)}] \cap U$. ⊠ ⊠

Part VI

Applications of the sequential G_0 dichotomy

Theorem 52

Suppose that κ is an aleph, X is a Hausdorff space, and $(E^n)_{n\in\omega}$ is a sequence of ω-universally Baire, co-κ-Souslin equivalence relations on X. Then at least one of the following holds:

1. There is a cover of X with κ-many equivalence classes.
2. There is a continuous embedding of $\Delta(2^\omega)$ into $\bigcup_{n\in\omega} E^n$.

Proof of Theorem 52

Define $G^n = (E^n)^c$.

If there is a κ-coloring of $(G^n)_{n\in\omega}$, then there is a cover of X with κ-many equivalence classes.

Proof of Theorem 52 (continued)

By Theorem 44, we can assume that there is a continuous homomorphism $\varphi\colon 2^\omega \to X$ from $(G_0^n)_{n\in\omega}$ to $(G^n)_{n\in\omega}$.

Define $F^n = (\varphi \times \varphi)^{-1}(E^n)$.

Essentially by Lemma 15, each F^n is meager.

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into the union $\bigcup_{n\in\omega} F^n$.

Then $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into $\bigcup_{n\in\omega} E^n$. ⊠

Theorem 53

Suppose that κ is an aleph and X is a Hausdorff space equipped with a vector space structure for which the set $D \subseteq X^{<\omega}$ of dependent sequences is ω-universally Baire and co-κ-Souslin. Then at least one of the following holds:

1. There is a basis for X of cardinality at most κ.
2. There is a continuous embedding of the set of non-injective sequences in $(2^\omega)^{<\omega}$ into D.

Proof of Theorem 53

Set $G^n = X^{n+2} \setminus D$.

If there is a κ^+-Borel κ-coloring of $(G^n)_{n\in\omega}$, then there is a covering of X by κ-many finite-dimensional sets, thus there is a basis of cardinality at most κ.

By Theorem 44, we can assume that there is a continuous homomorphism φ from $(G_0^n(\prod_{n\in\omega} f_0(n)+2))_{n\in\omega}$ to $(G^n)_{n\in\omega}$.

For each $\ell \in \omega$, set $D_\ell = (\varphi^\ell)^{-1}(D)$.

Lemma 54

Suppose that $\ell \in \omega \setminus 1$. Then D_ℓ is meager.

Proof of Lemma 54

By Kuratowski-Ulam, it is enough to show that if $\ell \in \omega \setminus 1$, $x \in D_\ell^c$, and $(D_{\ell+1})_x$ has the Baire property, then $(D_{\ell+1})_x$ is meager.

Suppose that $(D_{\ell+1})_x$ is non-meager.

Then there exists $(x_i)_{i \in \ell+1} \in G_0^{\ell+1}(\prod_{n \in \omega} f_0(n) + 2) \restriction (D_{\ell+1})_x$.

Then $(\varphi(x_i))_{i \in \ell+1} \notin G^{\ell+1}$, a contradiction. ⊠

Proof of Theorem 53 (continued)

By Mycielski, there is continuous embedding ψ of the set of non-injective sequences in $(2^\omega)^{<\omega}$ into $\bigcup_{\ell\in\omega} D_\ell$.

Then $\varphi \circ \psi$ is a continuous embedding of the set of non-injective sequences in $(2^\omega)^{<\omega}$ into D. ⊠

Theorem 55 (Hjorth)

Suppose that κ is an aleph, X is a Hausdorff space, and G is an acyclic, κ-Souslin graph on X such that $E_G \setminus d_G^{-1}(n)$ is ω-universally Baire for all $n \in \omega$. Then at least one of the following holds:

1. There are κ-many κ^+-Borel sets such that every E_G-class intersects one of them in a singleton.
2. There is a continuous embedding of E_0 into E_G.

Proof of Theorem 55

We will establish the special case of the theorem for good κ.

Set $G^n = E_G \setminus d_G^{-1}(n)$.

Suppose first that there is a κ^+-Borel κ-coloring of $(G^n)_{n \in \omega}$.

Then there is a cover with κ-many κ^+-Borel sets of finite diameter.

Lemma 56

Suppose that $B \subseteq X$ is a κ^+-Borel set of diameter strictly less than $2n$. Then there are κ^+-Borel sets $(B_i)_{i \in n}$ such that every E_G-class which intersects B intersects some B_i in 1 or 2 points.

Proof of Lemma 56

Set $B_0 = B$.

Let A_{i+1} denote the domain of the tree obtained by pruning $G \restriction B_i$.

By Lemma 5, there is a κ^+-Borel set $B_{i+1} \subseteq X$ of the same diameter as A_{i+1} such that $A_{i+1} \subseteq B_{i+1}$. ⊠

Proof of Theorem 55 (continued)

The desired covering can therefore be obtained by intersecting with elements of a basis.

By Theorem 44, we can assume that there is a continuous homomorphism φ from $(G_0^n)_{n \in \omega}$ to $(G^n)_{n \in \omega}$.

Lemma 57

Suppose that $n \in \omega$. Then $d_G^{-1}(n)$ is meager.

Proof of Lemma 57

By Kuratowski-Ulam, it is enough to show that if $d_G^{-1}(n)_x$ has the Baire property, then it is meager.

Suppose that $d_G^{-1}(n)_x$ is non-meager.

Proof of Lemma 57 (continued)

Then there exists $(y, z) \in G_0^{2n} \upharpoonright d_G^{-1}(n)_x$.

Then $(\varphi(y), \varphi(z)) \notin G^{2n}$, a contradiction. ⊠

Proof of Theorem 55 (continued)

Set $D = (\varphi \times \varphi)^{-1}(\Delta(X))$ and $F = (\varphi \times \varphi)^{-1}(E)$.

Then F is a meager equivalence relation which contains E_0.

By Lemma 27, there is a continuous embedding ψ of $(\Delta(2^\omega), E_0)$ into (D, F).

Then $\varphi \circ \psi$ is a continuous embedding of E_0 into E. ☒

Part VII

The local G_0 dichotomy

Example

The *digraph on 2^ω associated with* $T \subseteq \bigcup_{n \in \omega} 2^n \times 2^n$ is given by

$$H_T = \{(t(0)^\frown 0^\frown x, t(1)^\frown 1^\frown x) \mid t \in T \text{ and } x \in 2^\omega\}.$$

In particular, if $S \subseteq 2^{<\omega}$, then $G_S = H_{\Delta(S)}$.

Definition

A set $T \subseteq \bigcup_{n \in \omega} 2^n \times 2^n$ is *dense* if

$$\forall s \in 2^{<\omega} \times 2^{<\omega} \exists t \in T \forall i \in 2\ (s(i) \sqsubseteq t(i)).$$

Lemma 58

Suppose that $T \subseteq \bigcup_{n \in \omega} 2^n \times 2^n$ is dense and $R \subseteq 2^\omega \times 2^\omega$ is a transitive set with the Baire property for which $H_T \subseteq R$. Then R is meager or comeager.

Proof of Lemma 58

Suppose, towards a contradiction, that there exist $u, v \in 2^{<\omega} \times 2^{<\omega}$ with R comeager in $\mathcal{N}_{u(1)} \times \mathcal{N}_{v(0)}$ and meager in $\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)}$.

Proof of Lemma 58 (continued)

Fix $s, t \in T$ such that $u(i) \sqsubseteq s(i)$ and $v(i) \sqsubseteq t(i)$ for all $i \in 2$.

Then $\forall^* x, y \in 2^\omega \ (s(0) \frown 0 \frown x R s(1) \frown 1 \frown x R t(0) \frown 0 \frown y R t(1) \frown 1 \frown y)$.

This contradicts the fact that R is meager in $\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)}$. ⊠

Definition

Fix sequences $s_{2n} \in 2^{2n}$ and $t_{2n+1} \in 2^{2n+1} \times 2^{2n+1}$ such that the sets $S = \{s_{2n} \mid n \in \omega\}$ and $T = \{t_{2n+1} \mid n \in \omega\}$ are dense.

Define $G_0^{\mathrm{even}} = G_S$ and $H_0^{\mathrm{odd}} = H_T$.

Lemma 59

Suppose that κ is a good aleph, X is a Hausdorff space, E is a weakly κ-Souslin equivalence relation on X, R is a weakly κ-Souslin quasi-order on X, and (A_0, A_1) is an $(E \cap R)$-independent pair of weakly κ-Souslin sets. Then there is an $(E \cap R)$-independent pair (B_0, B_1) of κ^+-Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is upward $(E \cap R)$-invariant, and B_1 is downward $(E \cap R)$-invariant.

Proof of Lemma 59

Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$.

Proof of Lemma 59 (continued)

Given an $(E \cap R)$-independent pair $(A_{0,n}, A_{1,n})$ of weakly κ-Souslin sets, fix an $(E \cap R)$-independent pair $(B_{0,n}, B_{1,n})$ of κ^+-Borel subsets of X such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$.

Set $A_{0,n+1} = [B_{0,n}]^{E \cap R}$ and $A_{1,n+1} = [B_{1,n}]_{E \cap R}$.

Define $B_0 = \bigcup_{n \in \omega} B_{0,n}$ and $B_1 = \bigcup_{n \in \omega} B_{1,n}$. ⊠

Lemma 60

Suppose that κ is a good aleph, X is a Hausdorff space, E is a weakly κ-Souslin equivalence relation on X, R is a weakly bi-κ-Souslin quasi-order on X, and (A_0, A_1) is an $(E \setminus R)$-independent pair of weakly κ-Souslin sets. Then there is an $(E \setminus R)$-independent pair (B_0, B_1) of κ^+-Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is downward $(E \cap R)$-invariant, and B_1 is upward $(E \cap R)$-invariant.

Proof of Lemma 60

Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$.

Proof of Lemma 60 (continued)

Given an $(E \setminus R)$-independent pair $(A_{0,n}, A_{1,n})$ of weakly κ-Souslin sets, fix an $(E \setminus R)$-independent pair $(B_{0,n}, B_{1,n})$ of κ^+-Borel subsets of X such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$.

Set $A_{0,n+1} = [B_{0,n}]_{E \cap R}$ and $A_{1,n+1} = [B_{1,n}]^{E \cap R}$.

Define $B_0 = \bigcup_{n \in \omega} B_{0,n}$ and $B_1 = \bigcup_{n \in \omega} B_{1,n}$. ⊠

Definition

An equivalence relation E on X is κ*-smooth* if there is a κ^+-Borel reduction of E to $\Delta(2^\kappa)$.

Theorem 61

Suppose that κ is an aleph, X is a Hausdorff space, G is a κ-Souslin digraph on X, and E is a κ-Souslin equivalence relation on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of $F \cap G$, for some κ-smooth equivalence relation F on X with $E \subseteq F$.
2. There is a continuous homomorphism from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to the pair (G, E).

Definition

A quasi-order R on X is κ-*lexicographically reducible* if for some $\alpha \in \kappa^+$ there is a κ^+-Borel reduction of R to $R_{\mathrm{lex}}(2^\alpha)$.

Theorem 62

Suppose that κ is an aleph, X is a Hausdorff space, G is a κ-Souslin digraph on X, and R is a κ-Souslin quasi-order on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of $\equiv_S \cap G$, for some κ-lexicographically reducible quasi-order S on X with $R \subseteq S$.
2. There is a continuous homomorphism from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to the pair (G, R).

Proof of Theorem 62

We will establish the special case of the theorem for good κ.

Lemma 63

It is sufficient to handle the special case that $X = \kappa^\omega$.

Proof of Lemma 63

We can clearly assume that $X \neq \emptyset$, so there is a continuous surjection $\varphi\colon \kappa^\omega \to X$.

Set $G' = (\varphi \times \varphi)^{-1}(G)$ and $R' = (\varphi \times \varphi)^{-1}(R)$.

Proof of Lemma 63 (continued)

If there is a κ^+-Borel κ-coloring of $\equiv_{S'} \cap G'$, for some κ-lexicographically reducible quasi-order S' on κ^ω with $R' \subseteq S'$, then Lemmas 5 and 59 can be used to produce the desired coloring c and quasi-order S.

If $\psi \colon 2^\omega \to \kappa^\omega$ is a continuous homomorphism from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to (G', R'), then $\varphi \circ \psi$ is a continuous homomorphism from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to (G, R). ⊠

Definition

An *approximation* is a triple of the form $a = (n^a, \varphi^a, (\psi^a_k)_{k \in n^a})$, where $n^a \in \omega$, $\varphi^a \colon 2^{n^a} \to \kappa^{n^a}$, and $\psi^a_k \colon 2^{n^a-(k+1)} \to \kappa^{n^a}$.

We say that an approximation a is *extended* by an approximation b if for all $k \in n^a$, the following conditions are satisfied:

1. $n^a \leq n^b$.
2. $\forall r \in 2^{n^a} \forall s \in 2^{n^b} \; (r \sqsubseteq s \implies \varphi^a(r) \sqsubseteq \varphi^b(s))$.
3. $\forall r \in 2^{n^a-(k+1)} \forall s \in 2^{n^b-(k+1)} \; (r \sqsubseteq s \implies \psi^a_k(r) \sqsubseteq \psi^b_k(s))$.

If $n^b = n^a + 1$, then we say that b is a *one-step extension* of a.

Proof of Theorem 62 (continued)

Fix a κ-length well-ordering of the set of all approximations.

Fix trees T_G and T_R on $\kappa \times (\kappa \times \kappa)$ such that $G = \mathrm{proj}_{\kappa^\omega \times \kappa^\omega}[T_G]$ and $R = \mathrm{proj}_{\kappa^\omega \times \kappa^\omega}[T_R]$.

Definition

A *configuration* is a triple of the form $\gamma = (n^\gamma, \varphi^\gamma, (\psi_k^\gamma)_{k \in n^\gamma})$, where $n^\gamma \in \omega$, $\varphi^\gamma \colon 2^{n^\gamma} \to \kappa^\omega$, and $\psi_k^\gamma \colon 2^{n^\gamma - (k+1)} \to \kappa^\omega$, such that

$$(\psi_k^\gamma(s), (\varphi^\gamma(s_k \frown 0 \frown s), \varphi^\gamma(s_k \frown 1 \frown s))) \in [\PHplaneTree_G]$$

for all even $k \in n^\gamma$ and $s \in 2^{n^\gamma - (k+1)}$, and

$$(\psi_k^\gamma(s), (\varphi^\gamma(t_k(0) \frown 0 \frown s), \varphi^\gamma(t_k(1) \frown 1 \frown s))) \in [\PHplaneTree_R]$$

for all odd $k \in n^\gamma$ and $s \in 2^{n^\gamma - (k+1)}$.

Definition

A configuration γ is *compatible* with an approximation a if:

1. $n^a = n^\gamma$.
2. $\forall s \in 2^{n^a} \ (\varphi^a(s) \sqsubseteq \varphi^\gamma(s))$.
3. $\forall k \in n^a \forall s \in 2^{n^a-(k+1)} \ (\psi^a_k(s) \sqsubseteq \psi^\gamma_k(s))$.

Suppose that $Y \subseteq \kappa^\omega$ is κ^+-Borel and S is a κ-lexicographically reducible quasi-order on κ^ω such that $R \subseteq S$.

We say that γ is *compatible* with S if $\varphi^\gamma[2^{n^\gamma}] \times \varphi^\gamma[2^{n^\gamma}] \subseteq S$.

We say that γ is *compatible* with Y if $\varphi^\gamma[2^{n^\gamma}] \subseteq Y$.

Definition

We use $\Gamma(a, S, Y)$ to denote the family of all configurations which are compatible with a, S, and Y.

We say that an approximation a is (S, Y)-*terminal* if $\Gamma(b, S, Y) = \emptyset$ for all one-step extensions b of a.

Definition

We say that an approximation a is *even* if n^a is even.

Let $T_{\text{even}}(S, Y)$ be the set of (S, Y)-terminal even approximations.

For each even approximation a, define $A(a, S, Y) \subseteq Y$ by

$$A(a, S, Y) = \{\varphi^\gamma(s_{n^a}) \mid \gamma \in \Gamma(a, S, Y)\}.$$

Lemma 64

Suppose that a is an even approximation for which $A(a, S, Y)$ is not $(\equiv_S \cap G)$-independent. Then a is not (S, Y)-terminal.

Proof of Lemma 64

Fix configurations $\gamma_0, \gamma_1 \in \Gamma(a, S, Y)$ with the property that

$$(\varphi^{\gamma_0}(s_{n^a}), \varphi^{\gamma_1}(s_{n^a})) \in \equiv_S \cap G.$$

Fix $x \in \kappa^\omega$ such that $(x, (\varphi^{\gamma_0}(s_{n^a}), \varphi^{\gamma_1}(s_{n^a}))) \in [\Tree_G]$.

Proof of Lemma 64 (continued)

Let γ denote the configuration given by:

1. $n^\gamma = n^a + 1$.
2. $\forall i \in 2 \forall s \in 2^{n^a} \; (\varphi^\gamma(s^\frown i) = \varphi^{\gamma_i}(s))$.
3. $\forall i \in 2 \forall k \in n^a \forall s \in 2^{n^a-(k+1)} \; (\psi_k^\gamma(s^\frown i) = \psi_k^{\gamma_i}(s))$.
4. $\psi_{n^a}^\gamma(\emptyset) = x$.

Let b denote the approximation given by:

1. $n^b = n^\gamma$.
2. $\forall s \in 2^{n^b} \; (\varphi^b(s) = \varphi^\gamma(s) \restriction n^b)$.
3. $\forall k \in n^b \forall s \in 2^{n^b-(k+1)} \; (\psi_k^b(s) = \psi_k^\gamma(s) \restriction n^b)$.

Proof of Lemma 64 (continued)

Clearly γ is compatible with b.

Clearly b is a one-step extension of a.

It follows that a is not (S, Y)-terminal. ⊠

Lemma 65

Suppose that a is an even, (S, Y)-terminal approximation. Then there is an $(\equiv_S \cap G)$-independent, κ^+-Borel set $B(a, S, Y) \subseteq \kappa^\omega$ such that $A(a, S, Y) \subseteq B(a, S, Y)$.

Proof of Lemma 65

Lemma 64 ensures that $A(a, S, Y)$ is $(\equiv_S \cap G)$-independent.

The desired result therefore follows from Lemma 5. ⊠

Definition

Set $Y' = Y \setminus \bigcup_{a \in T_{\text{even}}(S,Y)} B(a,S,Y)$.

Lemma 66

There is a κ^+-Borel κ-coloring of $(\equiv_S \cap G) \restriction (Y \setminus Y')$.

Proof of Lemma 66

Define $c(y) = \min\{a \in T(S,Y) \mid y \in B(a,S,Y)\}$ for $y \in Y \setminus Y'$.

As $c^{-1}(\{a\}) \subseteq B(a,S,Y)$ for all $a \in T(S,Y)$, it follows that c is a coloring of $(\equiv_S \cap G) \restriction (Y \setminus Y')$. $\boxtimes$

Definition

We say that an approximation a is *odd* if n^a is odd.

Let $T_{\text{odd}}(S, Y)$ be the set of (S, Y)-terminal odd approximations.

For each odd approximation a and $i \in 2$, define $A_i(a, S, Y) \subseteq Y$ by

$$A_i(a, S, Y) = \{\varphi^\gamma \circ t_{n^a}(i) \mid \gamma \in \Gamma(a, S, Y)\}.$$

Lemma 67

Suppose that a is an odd approximation for which the pair $(A_0(a, S, Y), A_1(a, S, Y))$ is not $(\equiv_S \cap R)$-independent. Then a is not (S, Y)-terminal.

Proof of Lemma 67

Fix configurations $\gamma_0, \gamma_1 \in \Gamma(a, S, Y)$ with the property that

$$(\varphi^{\gamma_0} \circ t_{n^a}(0), \varphi^{\gamma_1} \circ t_{n^a}(1)) \in \equiv_S \cap R.$$

Fix $x \in \kappa^\omega$ such that $(x, (\varphi^{\gamma_0} \circ t_{n^a}(0), \varphi^{\gamma_1} \circ t_{n^a}(1))) \in [\text{🌿}_R]$.

Proof of Lemma 67 (continued)

Let γ denote the configuration given by:

1. $n^\gamma = n^a + 1$.
2. $\forall i \in 2 \forall s \in 2^{n^a} \ (\varphi^\gamma(s^\frown i) = \varphi^{\gamma_i}(s))$.
3. $\forall i \in 2 \forall k \in n^a \forall s \in 2^{n^a-(k+1)} \ (\psi_k^\gamma(s^\frown i) = \psi_k^{\gamma_i}(s))$.
4. $\psi_{n^a}^\gamma(\emptyset) = x$.

Let b denote the approximation given by:

1. $n^b = n^\gamma$.
2. $\forall s \in 2^{n^b} \ (\varphi^b(s) = \varphi^\gamma(s) \upharpoonright n^b)$.
3. $\forall k \in n^b \forall s \in 2^{n^b-(k+1)} \ (\psi_k^b(s) = \psi_k^\gamma(s) \upharpoonright n^b)$.

Proof of Lemma 67 (continued)

Clearly γ is compatible with b.

Clearly b is a one-step extension of a.

It follows that a is not (S, Y)-terminal. ⊠

Lemma 68

Suppose that a is an odd approximation which is (S, Y)-terminal. Then there is an $(\equiv_S \cap R)$-independent, κ^+-Borel pair of sets $(B_0(a, S, Y), B_1(a, S, Y))$ such that $A_0(a, S, Y) \subseteq B_0(a, S, Y)$, $A_1(a, S, Y) \subseteq B_1(a, S, Y)$, $B_0(a, S, Y)$ is upward $(\equiv_S \cap R)$-invariant, and $B_1(a, S, Y)$ is downward $(\equiv_S \cap R)$-invariant.

Proof of Lemma 68

Lemma 67 ensures that the pair of sets $(A_0(a, S, Y), A_1(a, S, Y))$ is $(\equiv_S \cap R)$-independent.

The desired result therefore follows from Lemma 59. ⊠

Definition

Let S' denote the κ-lexicographically reducible quasi-order generated by S and the sequence $(B_0(a, S, Y))_{a \in T_{\mathrm{odd}}(S,Y)}$.

Lemma 69

The quasi-order R is contained in S'.

Proof of Lemma 69

The main point is that $B_0(a, S, Y)$ is upward $(\equiv_S \cap R)$-invariant.

As $R \subseteq S$, it follows that $R \subseteq S'$. ⊠

Definition

Recursively define a sequence $(S_\alpha, Y_\alpha)_{\alpha \in \kappa^+}$ by κ^ω by

$$(S_\alpha, Y_\alpha) = \begin{cases} (\kappa^\omega \times \kappa^\omega, \kappa^\omega) & \text{if } \alpha = 0, \\ (S'_\beta, Y'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta \in \alpha} S_\beta, \bigcap_{\beta \in \alpha} Y_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Fix $\alpha \in \kappa^+$ such that $T_{\text{even}}(S_\alpha, Y_\alpha) = T_{\text{even}}(S_{\alpha+1}, Y_{\alpha+1})$ and $T_{\text{odd}}(S_\alpha, Y_\alpha) = T_{\text{odd}}(S_{\alpha+1}, Y_{\alpha+1})$.

Lemma 70

Suppose that the trivial approximation is (S_α, Y_α)-terminal. Then there is a κ^+-Borel κ-coloring of $\equiv_S \cap G$, for some κ-lexicographically reducible quasi-order S on X with $R \subseteq S$.

Proof of Lemma 70

Note first that $Y_{\alpha+1} = \emptyset$, thus $\kappa^\omega = \bigcup_{\beta\leq\alpha} Y_\beta \setminus Y_{\beta+1}$.

As all $(\equiv_{S_\alpha} \cap G) \restriction (Y_\beta \setminus Y_{\beta+1})$ admit κ^+-Borel κ-colorings, so too does $\equiv_{S_\alpha} \cap G$. ⊠

Lemma 71

Suppose that a is an approximation which is not (S', Y')-terminal. Then there is a one-step extension which is not (S, Y)-terminal.

Proof of Lemma 71

Suppose first that a is even.

Fix a one-step extension b of a for which $\Gamma(b, S, Y') \neq \emptyset$.

Fix a configuration $\gamma \in \Gamma(b, S, Y')$.

Then $\varphi^\gamma(s_{n^b}) \in Y'$, thus b is not (S, Y)-terminal.

Proof of Lemma 71 (continued)

Suppose now that a is odd.

Fix a one-step extension b of a for which $\Gamma(b, S', Y) \neq \emptyset$.

Fix a configuration $\gamma \in \Gamma(b, S', Y)$.

Then $\varphi^\gamma \circ t_{n^b}(0) \equiv_{S'} \varphi^\gamma \circ t_{n^b}(1)$, thus b is not (S, Y)-terminal. ⊠

Lemma 72

Suppose that the trivial approximation is not (S_α, Y_α)-terminal. Then there is a continuous homomorphism from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to the pair (G, R).

Proof of Lemma 72

By Lemma 71, there are approximations $a_n = (n, \varphi_n, (\psi_{k,n})_{k \in n})$ that are not (S_α, Y_α)-terminal, each extended by the next.

Define $\varphi \colon 2^\omega \to \kappa^\omega$ and $\psi_k \colon 2^\omega \to \kappa^\omega$ by

$$\varphi(x) = \bigcup_{n \in \omega} \varphi_n(x \upharpoonright n) \text{ and } \psi_k(x) = \bigcup_{k \in n \in \omega} \psi_{k,n}(x \upharpoonright (n - (k+1))).$$

Proof of Lemma 72 (continued)

It remains to show that if $k \in \omega$ and $x \in 2^\omega$, then

$$(\psi_k(x), (\varphi(s_k {}^\frown 0 {}^\frown x), \varphi(s_k {}^\frown 1 {}^\frown x))) \in [\mathbb{T}_G]$$

if k is even, and

$$(\psi_k(x), (\varphi(t_k(0) {}^\frown 0 {}^\frown x), \varphi(t_k(1) {}^\frown 1 {}^\frown x))) \in [\mathbb{T}_R]$$

if k is odd.

We will handle the case that k is even, as the other case is identical.

Proof of Lemma 72 (continued)

It is enough to show that every open neighborhood U of the pair $(\psi_k(x), (\varphi(s_k \frown 0 \frown x), \varphi(s_k \frown 1 \frown x)))$ contains a point of $[\text{⚘}_G]$.

Towards this end, fix $n \in \omega$ sufficiently large that $k \in n$ and

$$\mathcal{N}_{\psi_{k,n}(s)} \times (\mathcal{N}_{\varphi_n(s_k \frown 0 \frown s)} \times \mathcal{N}_{\varphi_n(s_k \frown 1 \frown s)}) \subseteq U,$$

where $s = x \upharpoonright (n - (k+1))$.

Proof of Lemma 72 (continued)

Our choice of a_n ensures the existence of $\gamma \in \Gamma(a_n, S_\alpha, Y_\alpha)$.

Then $(\psi^\gamma(s), (\varphi^\gamma(s_k{}^\frown 0^\frown s), \varphi^\gamma(s_k{}^\frown 1^\frown s))) \in [\text{🌿}_G] \cap U$. ⊠ ⊠

Part VIII

Applications

Definition

We say that a quasi-order R is κ-*linearizable* if it is contained in a κ-lexicographically reducible quasi-order S for which $\equiv_R = \equiv_S$.

Theorem 73 (Harrington-Marker-Shelah)

Suppose that κ is an aleph, X is a Hausdorff space, and R is a weakly ω-universally Baire, bi-κ-Souslin quasi-order on X. Then at least one of the following holds:

1. The quasi-order R is κ-linearizable.
2. There is a continuous embedding of $\Delta(2^\omega)$ into R.

Proof of Theorem 73

We will establish the special case of the theorem for good κ.

Proof of Theorem 73 (continued)

Set $G = R^c$.

Suppose first that there is a κ^+-Borel κ-coloring c of $\equiv_S \cap G$, for some κ-lexicographically reducible quasi-order S on X with $R \subseteq S$.

Lemma 74

The quasi-order R is κ-linearizable.

Proof of Lemma 74

By Lemma 60, there are $(\equiv_S \setminus R)$-independent pairs (A_α, B_α) of κ^+-Borel sets such that $c^{-1}(\{\alpha\}) \subseteq A_\alpha \cap B_\alpha$, A_α is downward $(\equiv_S \cap R)$-invariant, and B_α is upward $(\equiv_S \cap R)$-invariant.

Let T denote the κ-lexicographically reducible quasi-order generated by S and the sequence $(B_\alpha)_{\alpha\in\kappa}$.

Then $R \subseteq T$ and $\equiv_R = \equiv_T$, thus R is κ-linearizable. ⊠

Proof of Theorem 73 (continued)

By Theorem 62, we can therefore assume that there is a continuous homomorphism φ from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to (G, R).

Set $S = (\varphi \times \varphi)^{-1}(R)$.

Essentially by Lemma 15, the equivalence relation $\equiv_S$ is meager.

By Lemma 58, the quasi-order S is meager.

By Mycielski, there is a continuous embedding ψ of $\Delta(2^\omega)$ into S.

Then $\varphi \circ \psi$ is a continuous embedding of $\Delta(2^\omega)$ into R. $\boxtimes$

Theorem 75 (Harrington-Kechris-Louveau, Ditzen, Foreman-Magidor)

Suppose that κ is an aleph, X is a Hausdorff space, and E is a weakly ω-universally Baire, bi-κ-Souslin equivalence relation on X. Then at least one of the following holds:

1. The equivalence relation E is κ-smooth.
2. There is a continuous embedding of E_0 into E.

Proof of Theorem 75

We will establish the special case of the theorem for good κ.

Set $G = E^c$.

Suppose that there is a κ^+-Borel κ-coloring c of $F \cap G$, for some κ-smooth equivalence relation F on X with $E \subseteq F$.

By Lemma 60, we can assume each $c^{-1}(\{\alpha\})$ is E-invariant.

Then E is the intersection of F with the smooth equivalence relation generated by c, and is therefore smooth.

By Theorem 61, we can therefore assume that there is a continuous homomorphism φ from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to (G, E).

Set $D = (\varphi \times \varphi)^{-1}(\Delta(X))$ and $F = (\varphi \times \varphi)^{-1}(E)$.

Essentially by Lemma 15, the equivalence relation F is meager.

Lemma 76

There is a continuous embedding ψ of $(\Delta(2^\omega), E_0)$ into (D, F).

Proof of Lemma 76

Fix a decreasing sequence of dense, open sets $U_n \subseteq D^c$ such that $F \cap \bigcap_{n\in\omega} U_n = \emptyset$.

It is enough to construct $k_n \in \omega$ and $u_{i,n} \in 2^{k_n}$ such that:

1. $\forall n \in \omega \forall s, t \in 2^n \ (\mathcal{N}_{\psi_{n+1}(s^\frown 0)} \times \mathcal{N}_{\psi_{n+1}(t^\frown 1)} \subseteq U_n)$.
2. $\forall n \in \omega \exists t \in T \forall i \in 2 \ (t(i)^\frown i = \psi_{n+1}(0^n{}^\frown i))$.

Here $\psi_n \colon 2^n \to 2^{\sum_{m\in n} k_m}$ is given by $\psi_n(s) = \bigoplus_{m\in n} u_{s(m),m}$.

Proof of Lemma 76 (continued)

Suppose that we have found k_m and $u_{i,m}$ for all $i \in 2$ and $m \in n$.

Fix an enumeration $(s_k, t_k)_{k \leq \ell}$ of $2^n \times 2^n$.

Recursively construct increasing sequences $(u_{i,k,n})_{k \leq \ell}$ such that

$$\forall k \leq \ell \; (\mathcal{N}_{\psi_n(s_k)^\frown u_{0,k,n}} \times \mathcal{N}_{\psi_n(t_k)^\frown u_{1,k,n}} \subseteq U_n).$$

Fix extensions $u_{i,n}$ of $u_{i,\ell,n}$ of the same length k_n for which there exists $t \in T$ such that $t(i)^\frown i = \psi_n(0^n)^\frown u_{i,n}$ for all $i \in 2$. ⊠

Clearly $\varphi \circ \psi$ is a continuous embedding of E_0 into E. ⊠

Definition

Let R_0 denote the partial order on 2^ω given by

$$x <_{R_0} y \iff (x E_0 y \text{ and } x \circ \delta(x,y) < y \circ \delta(x,y)),$$

where $\delta(x,y) = \max\{n \in \omega \mid x(n) \neq y(n)\}$.

Theorem 77 (Kanovei, Louveau)

Suppose that κ is an aleph, X is a Hausdorff space, and R is a weakly ω-universally Baire, bi-κ-Souslin quasi-order on X. Then at least one of the following holds:

1. The quasi-order R is κ-linearizable.
2. There is a continuous embedding of E_0 or R_0 into R.

Proof of Theorem 77

We will establish the special case of the theorem for good κ.

Proof of Theorem 77 (continued)

Set $G = R^c$.

By Theorem 62 and Lemma 74, we can assume that there is a continuous homomorphism φ from $(G_0^{\mathrm{even}}, H_0^{\mathrm{odd}})$ to (G, R).

Set $D = (\varphi \times \varphi)^{-1}(2^\omega)$ and $S = (\varphi \times \varphi)^{-1}(R)$.

Essentially by Lemma 15 and 58, the quasi-order S is meager.

Lemma 78

There is a continuous homomorphism ψ from $(\Delta(2^\omega)^c, R_0, E_0^c)$ to the triple (D^c, S, S^c).

Proof of Lemma 78

Fix a decreasing sequence of dense, open sets $U_n \subseteq D^c$ such that $S \cap \bigcap_{n\in\omega} U_n = \emptyset$.

It is enough to construct $k_n \in \omega$ and $u_{i,n} \in 2^{k_n}$ such that:

1. $\forall n \in \omega \forall s, t \in 2^n \, (\mathcal{N}_{\psi_{n+1}(s^\frown 0)} \times \mathcal{N}_{\psi_{n+1}(t^\frown 1)} \subseteq U_n)$.
2. $\forall n \in \omega \exists t \in T \forall i \in 2 \, (t(i)^\frown i = \psi_{n+1}(i^{n\frown}(1-i)))$.

Here $\psi_n \colon 2^n \to 2^{\sum_{m\in n} k_m}$ is given by $\psi_n(s) = \bigoplus_{m\in n} u_{s(m),m}$.

Proof of Lemma 78 (continued)

Suppose that we have found k_m and $u_{i,m}$ for all $i \in 2$ and $m \in n$.

Fix an enumeration $(s_k, t_k)_{k \leq \ell}$ of $2^n \times 2^n$.

Recursively construct increasing sequences $(u_{i,k,n})_{k \leq \ell}$ such that

$$\forall k \leq \ell \; (\mathcal{N}_{\psi_n(s_k) \frown u_{0,k,n}} \times \mathcal{N}_{\psi_n(t_k) \frown u_{1,k,n}} \subseteq U_n).$$

Fix extensions $u_{i,n}$ of $u_{i,\ell,n}$ of the same length k_n for which there exists $t \in T$ such that $t(i)^\frown i = \psi_n(i^n)^\frown u_{1-i,n}$ for all $i \in 2$. ⊠

Then the function $\pi = \varphi \circ \psi$ is a continuous, injective homomorphism from (R_0, E_0^c) to (R, R^c).

Proof of Theorem 77 (continued)

Suppose now there are comeagerly many $x \in 2^\omega$ such that

$$\forall y \in [x]_{E_0} \ (\pi(x) \equiv_R \pi(y)).$$

As E_0 continuously embeds into its restriction to any comeager set, such a function can be composed with π to obtain a continuous embedding of E_0 into R.

Proof of Theorem 77 (continued)

Suppose now that there are comeagerly many $x \in 2^\omega$ such that

$$\exists y \in [x]_{E_0} \ (\pi(x) \not\equiv_R \pi(y)).$$

Let σ denote the successor function for R_0.

As every Borel partial transversal of E_0 is meager, it follows that the set $C = \{x \in 2^\omega \mid \varphi(x) <_R \varphi \circ \sigma(x)\}$ is non-meager.

As R_0 continuously embeds into its restriction to any non-meager Borel set, such a function can be composed with π to obtain a continuous embedding of R_0 into R. ⊠