Dichotomy theorems and limits superior

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Part I

Introduction

I. Introduction

Sequential dichotomy theorems


Dichotomy theorems for sequences of graphs have proven particularly useful in the graph-theoretic approach to descriptive set theory.

We will discuss one family of such dichotomy theorems.

This is joint work with Clinton Conley and Dominique Lecomte.

For the sake of notational and technical simplicity, we work in $\mathtt{ZFC}$.

I. Introduction

Basic definitions and notation

Notation

Suppose that $I \subseteq \mathbb{N}$.

Notation

Let $[I]^{\mathbb{N}}$ denote the family of countably infinite subsets of I.

Definition

The limit inferior of a sequence of sets $(X_i)_{i\in I}$ is the set

$$\liminf_{i\in I} X_i = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in I\setminus i} X_j$$

of points which appear in all but finitely many of the sets X_i.

Definition

The limit superior of a sequence of sets $(X_i)_{i\in I}$ is the set

$$\limsup_{i\in I} X_i = \bigcap_{i\in\mathbb{N}} \bigcup_{j\in I\setminus i} X_j$$

of points which appear in infinitely many of the sets X_i.

I. Introduction
An example

Example

Suppose that $U_i = \{x \in 2^{\mathbb{N}} \mid x(i) = 0\}$ for all $i \in \mathbb{N}$. Then for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. The set $\liminf_{i \in I} U_i$ is meager and null.
2. The set $\limsup_{i \in I} U_i$ is comeager and conull.

Remark

Suppose that $I \subseteq \mathbb{N}$ is co-infinite and identify $2^{\mathbb{N}}$ with $\mathcal{P}(\mathbb{N})$. Then:

1. $\bigcap_{i \notin I} U_i \cap [\mathbb{N}]^{\mathbb{N}} = [I]^{\mathbb{N}}$.
2. $\liminf_{i \notin I} U_i \cap [\mathbb{N}]^{\mathbb{N}} = \bigcup_{J \subseteq \mathbb{N} \setminus I \text{ finite}} [I \cup J]^{\mathbb{N}}$.
3. $\limsup_{i \notin I} U_i \cap [\mathbb{N}]^{\mathbb{N}} = \bigcup_{J \subseteq \mathbb{N} \setminus I \text{ co-infinite}} [I \cup J]^{\mathbb{N}}$.

Remark

We are interested in notions of size for which there are no corresponding sequences of this form.

Remark

We focus on notions originating in descriptive set-theoretic dichotomy theorems.

I. Introduction
A theorem

Theorem 1 (Komjáth, Laczkovich)

Suppose that X is a Polish space and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\limsup_{i\in I} A_i$ is countable.
2. The set $\bigcap_{i\in I} A_i$ has a non-empty perfect subset.

Remark

Such results can be turned into true dichotomies by asking that either of the alternatives occurs for densely many $I \in [\mathbb{N}]^{\mathbb{N}}$.

Remark

When $X = [\mathbb{N}]^{\mathbb{N}}$, such results yield information connecting the underlying notions of size with the structure of Ramsey blocks.

Part II

Two ubiquitous examples

Proposition 2

There is a sequence $(U_i)_{i\in\mathbb{N}}$ of clopen subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with the property that for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. There exist a meager set $M \subseteq 2^{\mathbb{N}}$ and a function $f\colon M \to 2^{\mathbb{N}}$ such that $\liminf_{i\in I} U_i = \mathrm{graph}(f)$.
2. The set $\limsup_{i\in I} U_i$ is comeager.

Proof

Fix an enumeration $(s_i)_{i\in\mathbb{N}}$ of $2^{<\mathbb{N}}$ and put $\ell_i = |s_i|$.

For each $i \in \mathbb{N}$, set $U_i = \{(\tau, x) \in (\prod_{n\in\mathbb{N}} S_{2^n}) \times 2^{\mathbb{N}} \mid \tau_{\ell_i}(s_i) \sqsubseteq x\}$.

Proof of Proposition 2 (continued)

Suppose that $I \in [\mathbb{N}]^{\mathbb{N}}$.

For each $\tau \in \prod_{n\in\mathbb{N}} S_{2^n}$, set $T_\tau = \bigcup_{i\in I}\{s \in 2^{<\mathbb{N}} \mid s \sqsubseteq \tau_{\ell_i}(s_i)\}$.

Fix $\varphi\colon \prod_{n\in\mathbb{N}} S_{2^n} \to 2^{\mathbb{N}}$ such that $\forall \tau \in \prod_{n\in\mathbb{N}} S_{2^n}\ \varphi(\tau) \in [T_\tau]$.

Clearly $\liminf_{i\in I} U_i \subseteq \mathrm{graph}(\varphi)$.

Proof of Proposition 2 (continued)

For each $k \in \mathbb{N}$, let M_k denote the set of $\tau \in \prod_{n\in\mathbb{N}} S_{2^n}$ such that

$$\forall i, j \in I \setminus k \ (\tau_{\ell_i}(s_i) \sqsubseteq \tau_{\ell_j}(s_j) \text{ or } \tau_{\ell_j}(s_j) \sqsubseteq \tau_{\ell_i}(s_i)).$$

Clearly each M_k is meager and $\liminf_{i\in I} U_i \subseteq \bigcup_{k\in\mathbb{N}} M_k \times 2^{\mathbb{N}}$.

Clearly $\forall x \in 2^{\mathbb{N}} \forall^* \tau \in \prod_{n\in\mathbb{N}} S_{2^n} \ (\tau, x) \in \limsup_{i\in I} U_i$. $\boxtimes$

Definition

A partial transversal of an equivalence relation is a set which intersects every equivalence class in at most one point.

Definition

The equivalence relation E_0 on $2^{\mathbb{N}}$ is given by

$$x \; E_0 \; y \Leftrightarrow \exists m \in \mathbb{N} \forall n \geq m \; x(n) = y(n).$$

II. Two ubiquitous examples

Non-smooth equivalence relations

Proposition 3

There is a sequence $(U_i)_{i\in\mathbb{N}}$ of clopen subsets of $2^{\mathbb{N}}$ such that for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. The set $\liminf_{i\in I} U_i$ is a partial transversal of E_0.
2. The set $\limsup_{i\in I} U_i$ is comeager.

Proof

For each $i \in \mathbb{N}$, define $U_i = \bigcup_{s\in 2^i} \mathcal{N}_{s^\frown s}$.

Given $I \in [\mathbb{N}]^{\mathbb{N}}$, it is easily checked that $\limsup_{i\in I} U_i$ is comeager.

Clearly $\liminf_{i\in I} U_i$ is a partial transversal of E_0. $\boxtimes$

Part III

Families of independent sets

III. Families of independent sets

Basic definitions

Definition

A graph on X is an irreflexive symmetric set $G \subseteq X \times X$.

Definition

The restriction of G to $Y \subseteq X$ is given by $G \restriction Y = G \cap (Y \times Y)$.

Definition

A set $Y \subseteq X$ is G-independent if $G \restriction Y = \emptyset$.

Definition

Let IND_G denote the family of G-independent subsets of X.

Definition

Let $\sigma(\mathscr{F})$ denote the σ-ideal generated by the Borel sets in $\mathscr{F}$.

Definition

We say that a graph G has uncountable Borel chromatic number if X is in $\sigma(\mathrm{IND}_G)$.

Definition

The graph G_0 on $2^{\mathbb{N}}$ is given by

$$G_0 = \bigcup_{n\in\mathbb{N}}\{(s_n{}^\frown(i)^\frown x, s_n{}^\frown(1-i)^\frown x) \mid i \in 2 \text{ and } x \in 2^{\mathbb{N}}\},$$

where the sequences $s_n \in 2^n$ are dense in the sense that

$$\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq s_n.$$

Definition

For each $n \in \mathbb{N}$, let $G^{(n)}$ denote the graph given by

$$G^{(n)} = \{(x, y) \mid x \neq y \text{ and } \exists x_0\ G \cdots G\ x_n\ (x = x_0 \text{ and } y = x_n)\}.$$

Theorem 4

Suppose that X is a Hausdorff space and G is a locally countable analytic graph on X of uncountable Borel chromatic number. Then there is a sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of X such that for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. The set $\operatorname{liminf}_{i\in I} K_i$ is G-independent.
2. There is a continuous homomorphism from the graph G_0 to the graph $G \restriction \operatorname{limsup}_{i\in I} K_i$.

Proposition 5

Suppose that X is a Hausdorff space, G is a locally countable analytic graph on X, μ is a Borel probability measure on X, every G-independent Borel set is μ-null, and $(B_i)_{i\in\mathbb{N}}$ is a sequence of μ-measurable subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\limsup_{i\in I} B_i$ is μ-null.
2. The set $\bigcap_{i\in I} B_i$ is not $G^{(2)}$-independent.

Definition

The tower number is the least cardinal $\mathfrak{t}$ for which there is a sequence $(I_\alpha)_{\alpha<\mathfrak{t}}$ of infinite subsets of $\mathbb{N}$ which is maximal with the property that $\forall \alpha < \beta < \mathfrak{t}$ $(|I_\alpha \setminus I_\beta| = \aleph_0$ and $|I_\beta \setminus I_\alpha| < \aleph_0)$.

Definition

We say that G is zeed if for every sequence $(x_\alpha, y_\alpha)_{\alpha<\mathfrak{t}}$ of directed edges of G, there exist $\alpha < \beta < \mathfrak{t}$ such that $(x_\alpha, y_\beta) \in G$.

Remark

If G is the union of fewer than $\mathfrak{t}$-many rectangles, then G is zeed.

Remark

If the complement of G is transitive and $|X| \leq 2^{\kappa}$, then G is the union of κ-many rectangles.

Proposition 6

Suppose that X is a set, G is a zeed graph on X, and $(X_i)_{i\in\mathbb{N}}$ is a sequence of subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\limsup_{i\in I} X_i$ is G-independent.
2. The set $\bigcap_{i\in I} X_i$ is G-dependent.

Proof

Suppose that $(X_i)_{i\in\mathbb{N}}$ is a sequence of subsets of X.

Suppose, moreover, that $\forall I \in [\mathbb{N}]^{\mathbb{N}}\ \limsup_{i\in I} X_i \notin \mathcal{I}_G$.

Recursively construct $I_\alpha \in [\mathbb{N}]^{\mathbb{N}}$ and $x_\alpha, y_\alpha \in X$ such that:

1. $\forall \alpha < \mathfrak{t}\ (x_\alpha, y_\alpha) \in G \restriction \limsup_{i\in I_\alpha} X_i$.
2. $\forall \alpha < \mathfrak{t}\ x_\alpha \in \bigcap_{i\in I_{\alpha+1}} X_i$.
3. $\forall \alpha < \beta < \mathfrak{t}\ |I_\beta \setminus I_\alpha| < \aleph_0$.

Proof of Proposition 6 (continued)

Fix $\alpha < \beta < \mathfrak{t}$ such that $(x_\alpha, y_\beta) \in G$.

Set $I = \{i \in I_\alpha \cap I_\beta \mid y_\beta \in X_i\}$.

Clearly $(x_\alpha, y_\beta) \in G \restriction \bigcap_{i \in I} X_i$. $\boxtimes$

Part IV

From independent sets to $\aleph_0$-colorings

IV. From independent sets to $\aleph_0$-colorings

Technical strengthenings

Proposition 7

Suppose that X and Y are sets, G is a zeed graph on X, and $(R_i)_{i\in\mathbb{N}}$ is a sequence of subsets of $X \times Y$. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\mathrm{proj}_X(\limsup_{i\in I} R_i)$ is G-independent.
2. The set $\mathrm{proj}_X(\bigcap_{i\in I} R_i)$ is not G-independent.

IV. From independent sets to $\aleph_0$-colorings

Technical strengthenings

Proposition 8

Suppose that X is Hausdorff, $\mathcal{I}$ is a σ-ideal on X, $\mathscr{F} \subseteq \mathcal{P}(X)$, and for all sequences $(A_i)_{i\in\mathbb{N}}$ of analytic subsets of X, there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\operatorname{limsup}_{i\in I} A_i$ is in $\mathcal{I}$.
2. The set $\bigcap_{i\in I} A_i$ is in $\mathscr{F}$.

Suppose further that Y is Hausdorff, $R \subseteq X \times Y$ is an analytic set with countable vertical sections, and $(R_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of $X \times Y$. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\operatorname{proj}_X(R \cap \operatorname{limsup}_{i\in I} R_i)$ is in $\mathcal{I}$.
2. The set $\operatorname{proj}_X(R \cap \bigcap_{i\in I} R_i)$ is in $\mathscr{F}$.

IV. From independent sets to $\aleph_0$-colorings

The main theorem

Theorem 9

Suppose that X is a Hausdorff space, G is an analytic graph on X, $\mathcal{I}$ is a σ-ideal on X, every analytic set in $\mathcal{I}$ is contained in a Borel set in $\mathcal{I}$, and $(A_i)_{i\in\mathbb{N}}$ is an $\mathbb{N}$-sequence of analytic subsets of X such that for all $n \in \mathbb{N}$ and analytic sets $R \subseteq \bigcup_{x\in X}[x]_{E_G} \times [x]^n_{E_G}$, there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\mathrm{proj}_X(R \cap \limsup_{i\in I} A_i \times A_i^n)$ is in $\mathcal{I}$.
2. The set $\mathrm{proj}_X(R \cap \bigcap_{i\in I} A_i \times A_i^n)$ is not G-independent.

Then for some $I \in [\mathbb{N}]^{\mathbb{N}}$ at least one of the following holds:

1. The set $\mathrm{proj}_X(R \cap \limsup_{i\in I} A_i \times A_i^n)$ is in $\mathcal{I}$.
2. There is a continuous homomorphism from the graph G_0 to the graph $G \restriction \mathrm{proj}_X(R \cap \bigcap_{i\in I} A_i \times A_i^n)$.

IV. From independent sets to $\aleph_0$-colorings

The main theorem

Remark

The natural generalization to κ-Souslin graphs holds for $\kappa < \mathfrak{t}$.

Part V

Applications

Theorem 10

Suppose that X is a Hausdorff space, G is a zeed analytic graph on X, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. There is a Borel $\mathbb{N}$-coloring of $G \restriction \limsup_{i\in I} A_i$.
2. There is a continuous homomorphism from G_0 to $G \restriction \bigcap_{i\in I} A_i$.

Theorem 11

Suppose that X is a Hausdorff space, G is a locally countable analytic graph on X, μ is a Borel probability measure on X, every G-independent Borel set is μ-null, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\limsup_{i\in I} A_i$ is μ-null.
2. There is a continuous homomorphism from the graph G_0 to the graph $G^{(2)} \restriction \bigcap_{i\in I} A_i$.

Theorem 12

Suppose that X is a Hausdorff space, G is an open graph on X, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. There is a Borel $\mathbb{N}$-coloring of $G \restriction \limsup_{i\in I} A_i$.
2. The set $\bigcap_{i\in I} A_i$ contains a perfect G-clique.

V. Applications

Counting equivalence classes

Theorem 13 (Gao-Jackson-Kieftenbeld)

Suppose that X is Hausdorff, E is a co-analytic equivalence relation on X, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. There are only countably many $(E \restriction \limsup_{i\in I} A_i)$-classes.
2. There are perfectly many $(E \restriction \bigcap_{i\in I} A_i)$-classes.

Theorem 14

Suppose that X is a Hausdorff space, R is a linear co-analytic quasi-order on X, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic sets of non-empty open R-intervals. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. Some countable set intersects every interval in $\limsup_{i\in I} A_i$.
2. There is a perfect pairwise disjoint subfamily of $\bigcap_{i\in I} A_i$.

Theorem 15

Suppose that X and Y are Hausdorff spaces, $R \subseteq X \times Y$ is an analytic set with only countably many uncountable vertical sections, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of R. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. There are countably many partial functions f_n, whose graphs are Borel subsets of R, with $\limsup_{i\in I} A_i = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n)$.
2. Some vertical section of $\bigcap_{i\in I} A_i$ contains a perfect set.

Theorem 16

Suppose that X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an analytic set with uncountably many uncountable vertical sections. Then there is a sequence $(K_i)_{i \in \mathbb{N}}$ of compact subsets of X such that for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. The set $\liminf_{i \in I} K_i$ is the graph of a partial function whose domain's complement contains a perfect set.
2. There is a perfect subset of X on which the vertical sections of $\limsup_{i \in I} K_i$ have perfect subsets.

Theorem 17

Suppose that X is Hausdorff and E is a non-smooth Borel equivalence relation on X. Then there is a sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of X such that for all $I \in [\mathbb{N}]^{\mathbb{N}}$ both of the following hold:

1. The set $\liminf_{i\in I} K_i$ is a partial transversal of E.
2. There is a continuous embedding of E_0 into $E \restriction \limsup_{i\in I} K_i$.

Theorem 18

Suppose that X is a Hausdorff space, E is a non-smooth countable analytic equivalence relation on X, μ is a Borel probability measure on X, there are no Borel partial transversals of E of positive μ-measure, and $(A_i)_{i\in\mathbb{N}}$ is a sequence of analytic subsets of X. Then there exists $I \in [\mathbb{N}]^{\mathbb{N}}$ such that at least one of the following holds:

1. The set $\limsup_{i\in I} A_i$ is μ-null.
2. There is a continuous embedding of E_0 into $E \restriction \bigcap_{i\in I} A_i$.

Definition

We say that a family $\mathscr{F}$ of co-infinite subsets of $\mathbb{N}$ is dense if every co-infinite subset of $\mathbb{N}$ is contained in a set in $\mathscr{F}$.

Theorem 19

Suppose that E is a countable analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ whose Borel partial transversals are μ-null. Then there are densely many co-infinite sets $C \subseteq \mathbb{N}$ for which $E \restriction [C]^{\mathbb{N}}$ is non-smooth.