The smooth ideal

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Part I

Equivalence relations

Suppose that X and Y are Hausdorff spaces.

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Suppose that E and F are equivalence relations on X and Y.

Definition

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We say E is uncountable if its equivalence classes are uncountable.

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We say that $A \subseteq X$ is analytic if there is a closed set $F \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous surjection $\varphi \colon F \to A$.

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We say that $B \subseteq X$ is $\aleph_0$-universally Baire if for every continuous function $\pi\colon \mathbb{N}^{\mathbb{N}} \to X$, the set $\pi^{-1}(B)$ has the Baire property.

Definition

A homomorphism from E to F is a function $\pi\colon X \to Y$ such that

$$w\,E\,x \Rightarrow \pi(w)\;F\;\pi(x).$$

Definition

A homomorphism from E to F is a function $\pi\colon X \to Y$ such that

$$w \; E \; x \Rightarrow \pi(w) \; F \; \pi(x).$$

Definition

A cohomomorphism from E to F is a function $\pi\colon X \to Y$ such that

$$\neg \, w \; E \; x \Rightarrow \neg \, \pi(w) \; F \; \pi(x).$$

Definition

A reduction of E to F is a function which is both a homomorphism and a cohomomorphism.

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Definition

An embedding of E into F is an injective reduction.

We will use the natural generalizations to sequences of relations.

Definition

The diagonal on X is the equivalence relation $\Delta(X)$ on X given by

$$\Delta(X) = \{(w, x) \in X \times X \mid w = x\}.$$

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Definition

We say that a set $W \subseteq X$ is E-smooth if $E \restriction W$ is smooth.

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A partial transversal of E is a set which intersects every equivalence class of E in at most one point.

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Definition

When E is countable, the smooth ideal of E is the σ-ideal $\mathcal{I}_E$ on X generated by the family of Borel partial transversals.

Definition

The equivalence relation E_0 on $2^{\mathbb{N}}$ is given by

$$x \; E_0 \; y \Leftrightarrow \exists m \in \mathbb{N} \forall n \geq m \; x(n) = y(n).$$

Theorem 1 (Silver)

Suppose that X is an analytic Hausdorff space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:

1. The equivalence relation E has only countably many classes.
2. There is a continuous embedding of $\Delta(2^{\mathbb{N}})$ into E.

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Theorem 2 (Harrington-Kechris-Louveau)

Suppose that X is an analytic Hausdorff space and E is a Borel equivalence relation on X. Then exactly one of the following holds:

1. The equivalence relation E is smooth.
2. There is a continuous embedding of E_0 into E.

Theorem 3 (Galvin)

Suppose that X is an analytic Hausdorff space, E is a co-analytic equivalence relation on X, and F is an $\aleph_0$-universally Baire equivalence relation on X. If E has uncountably many classes, then there is a non-empty perfect set $P \subseteq X$ with the property that $E \upharpoonright P = \Delta(P)$ and $F \upharpoonright P \in \{\Delta(P), P \times P\}$.

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Theorem 4 (Kanovei-Zapletal)

Suppose that X is an analytic Hausdorff space, E is a Borel equivalence relation on X, and F is an $\aleph_0$-universally Baire equivalence relation on X. If E is non-smooth, then there is an E-non-smooth perfect set $P \subseteq X$ for which $F \restriction P \in \{\Delta(P), E \restriction P, P \times P\}$.

Theorem 5

Suppose X is an analytic Hausdorff space and E, F are co-analytic equivalence relations on X. Then exactly one of the following holds:

- The set X is the union of countably many classes of E and F.
- There is a continuous embedding of $(\Delta(2^{\mathbb{N}}), \Delta(2^{\mathbb{N}}))$ into (E, F).

Theorem 5

Suppose X is an analytic Hausdorff space and E, F are co-analytic equivalence relations on X. Then exactly one of the following holds:

- The set X is the union of countably many classes of E and F.
- There is a continuous embedding of $(\Delta(2^{\mathbb{N}}), \Delta(2^{\mathbb{N}}))$ into (E, F).

Theorem 6

Suppose X is an analytic Hausdorff space and E, F are Borel equivalence relations on X. Then exactly one of the following holds:

- There is a smooth equivalence relation E' on X such that $[x]_{E \cap F} \subseteq [x]_{E'} \subseteq [x]_E$ or $[x]_{E \cap F} \subseteq [x]_{E'} \subseteq [x]_F$.
- There is a continuous embedding of (E_0, E_0) into (E, F).

Suppose $F \subseteq E$ are countable Borel equivalence relations on X.

Definition

We say that a set $B \subseteq X/F$ is Borel if $\bigcup B$ is Borel.

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Definition

A partial quasi-transversal of E/F is a set whose intersection with each equivalence class of E/F is finite.

Definition

When E/F is countable, the quasi-smooth ideal of E/F is the σ-ideal $\mathcal{I}_{E/F}$ generated by the family of Borel partial quasi-transversals.

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Definition

We say E/F is quasi-smooth if X/F is in $\mathcal{I}_{E/F}$.

Theorem 7

Suppose that X is an analytic Hausdorff space, E is a countable Borel equivalence relation on X, and F is a Borel equivalence relation on X. Then exactly one of the following holds:

- The equivalence relation $E/(E \cap F)$ is quasi-smooth.
- There is a continuous embedding of $(E_0, \Delta(2^{\mathbb{N}}))$ into (E, F).

Theorem 7

Suppose that X is an analytic Hausdorff space, E is a countable Borel equivalence relation on X, and F is a Borel equivalence relation on X. Then exactly one of the following holds:

- The equivalence relation $E/(E \cap F)$ is quasi-smooth.
- There is a continuous embedding of $(E_0, \Delta(2^{\mathbb{N}}))$ into (E, F).

Theorem 8

Suppose that X is an analytic Hausdorff space, E is an uncountable Borel equivalence relation on X, and F is a countable Borel equivalence relation on X. Then the following are equivalent:

- There is a continuous embedding of E_0 into E.
- There is a continuous embedding of $(E_0, \Delta(2^{\mathbb{N}}))$ into (E, F).

Part II

Ideals

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y.

Definition

A homomorphism from $\mathcal{I}$ to $\mathcal{J}$ is a function $\pi\colon X \to Y$ such that

$$B \in \mathcal{I} \Rightarrow \pi(B) \in \mathcal{J}.$$

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Definition

A cohomomorphism from $\mathcal{I}$ to $\mathcal{J}$ is a function $\pi\colon X \to Y$ such that

$$B \notin \mathcal{I} \Rightarrow \pi(B) \notin \mathcal{J}.$$

Definition

A reduction of $\mathcal{I}$ to $\mathcal{J}$ is a function which is both a homomorphism and a cohomomorphism.

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A reduction of $\mathcal{I}$ to $\mathcal{J}$ is a function which is both a homomorphism and a cohomomorphism.

Definition

An embedding of $\mathcal{I}$ into $\mathcal{J}$ is an injective reduction.

Definition

We say that $\mathcal{I}$ is cohomogeneous if for every Borel set $B \notin \mathcal{I}$, there is a Borel cohomomorphism from $\mathcal{I}$ to $\mathcal{I} \restriction B$.

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We say that $\mathcal{I}$ is cohomogeneous if for every Borel set $B \notin \mathcal{I}$, there is a Borel cohomomorphism from $\mathcal{I}$ to $\mathcal{I} \upharpoonright B$.

Question (Zapletal)

Are there naturally occurring non-cohomogeneous σ-ideals?

Definition

We say that E is hyperfinite if there is an increasing sequence of finite Borel equivalence relations F_n on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

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Theorem 9 (Dougherty-Jackson-Kechris)

Suppose that X is an analytic Hausdorff space and E is a countable Borel equivalence relation on X. Then E is hyperfinite if and only if there is a Borel embedding of E into E_0.

Corollary 10

Suppose that X is an analytic Hausdorff space and E is a hyperfinite equivalence relation on X. Then for every analytic set $A \notin \mathcal{I}_E$, there is a Borel embedding of $\mathcal{I}_E$ into $\mathcal{I}_E \upharpoonright A$, so $\mathcal{I}_E$ is cohomogeneous.

Question

Can $\mathcal{I}_E$ fail to be cohomogeneous when E is not hyperfinite?

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Question

More generally, how much of E is encoded in $\mathcal{I}_E$?

Part III

Equivalence relations and ideals

Suppose again that E and F are equivalence relations on X and Y.

Definition

A quasi-homomorphism from E to F is a homomorphism from a finite-index subequivalence relation of E to F.

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Definition

We say that a quasi-homomorphism $\pi \colon X \to Y$ from E to F is smooth-to-one if for all $y \in Y$ the set $\pi^{-1}(\{y\})$ is E-smooth.

Definition

A quasi-cohomomorphism from E to F is a cohomomorphism from E to a finite-index subequivalence relation of F.

Definition

A quasi-cohomomorphism from E to F is a cohomomorphism from E to a finite-index subequivalence relation of F.

Definition

A quasi-reduction of E to F is a function which is both a quasi-homomorphism and a quasi-cohomomorphism.

Theorem 11

Suppose that X and Y are analytic Hausdorff spaces and E and F are countable Borel equivalence relations on X and Y. Then the following are equivalent:

- There is a Borel cohomomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$.
- There is a smooth-to-one Borel quasi-homomorphism from E to F.

Theorem 12

Suppose that X is an analytic Hausdorff space and E is a countable Borel equivalence relation on X. Then the following are equivalent:

- The σ-ideal $\mathcal{I}_E$ is cohomogeneous.
- The equivalence relation E is hyperfinite.

Theorem 13

Suppose that X and Y are analytic Hausdorff spaces and E and F are countable Borel equivalence relations on X and Y. Then the following are equivalent:

- There is a Borel reduction of $\mathcal{I}_E$ to $\mathcal{I}_F$.
- There is a Borel quasi-reduction of E to F.

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