An antibasis result for graphs of infinite Borel chromatic number

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1 A graph not above the shift

A *graph* on a set X is a symmetric, irreflexive subset of $X \times X$. For a graph G on X, we let $\deg_G(x) = |\{y \in X : (x, y) \in G\}|$. If $\deg_G(x)$ is countable for all $x \in X$ we say that G is *locally countable*. If, moreover, $\deg_G(x)$ is finite for all $x \in X$, we say that G is *locally finite*. The *restriction* of G to a set $A \subseteq X$, denoted by $G|A$, is simply $G \cap (A \times A)$. A set $A \subseteq X$ is (G-)*independent* if $G|A$ is empty. A κ*-coloring* of G is a function $c : X \to \kappa$ such that $c^{-1}(i)$ is independent for each $i \in \kappa$. The *chromatic number* of G, $\chi(G)$, is the least cardinal κ for which there exists a κ-coloring of G. Analogously, the *Borel chromatic number*, $\chi_B(G)$, of a graph on a standard Borel space X is the least cardinality of a Polish space Y for which there is a Borel function $c : X \to Y$ with $c^{-1}(y)$ a G-independent set for each $y \in Y$.

For a graph G, let E_G denote the equivalence relation generated by G. The classes of E_G are called the *connected components* of G, and G is *connected* if E_G has only one class. We say that G has *indecomposably infinite Borel chromatic number* if there is no way of partitioning the underlying space into countably many Borel E_G-invariant sets such that the restriction of G to each has finite Borel chromatic number.

We identify Ramsey space $[\mathbb{N}]^{\mathbb{N}}$ with the collection of increasing sequences of natural numbers. We then define the *unilateral shift*, $s : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$, by

$$s(x)(i) = x(i + 1),$$

for all $x \in [\mathbb{N}]^{\mathbb{N}}$ and $i \in \mathbb{N}$. Denote by G_s the graph on $[\mathbb{N}]^{\mathbb{N}}$ generated by s, i.e.,

$$xG_sy \Leftrightarrow x = s(y) \text{ or } y = s(x).$$

Thus, G_s is an acyclic, locally finite, Borel graph on $[\mathbb{N}]^{\mathbb{N}}$. By a Galvin-Prikry argument it is shown in [6] that $\chi_B(G_s) = \aleph_0$, and it is therein conjectured that G_s is in some sense minimal among the collection of graphs of infinite Borel chromatic number. This note investigates the possibility of such minimality.

Towards this end, we will borrow some tools from the study of measure-preserving group actions. If G is a Borel graph on a standard probability space (X, μ), the (μ-)*measurable chromatic number* of G, $\chi_\mu(G)$, is the least cardinality of a Polish space Y for which there is a (μ-)measurable function $c : X \to Y$ with $c^{-1}(y)$ a G-independent set for each $y \in Y$. We see immediately that $\chi_\mu(G) \leq \chi_B(G)$. If Γ is a countable group with generating set S (assumed not to contain the identity) and a is a free, measure-preserving action of Γ on (X, μ), we define the graph $G(S, a)$ on X by

$$xG(S,a)y \Leftrightarrow \exists s \in S \ (x = s \cdot y \text{ or } y = s \cdot x).$$

We denote by $\mathbb{F}_n$ ($n \geq 2$) the free group on n generators, and by $\mathbb{F}_\infty$ the free group on $\aleph_0$-many generators. Fixing a set of free generators $F_\infty = \{\gamma_0, \gamma_1, \ldots\}$ for $\mathbb{F}_\infty$, we may canonically identify $\mathbb{F}_n$ with the subgroup of $\mathbb{F}_\infty$ generated by $F_n = \{\gamma_1, \ldots, \gamma_n\}$ (note that γ_0 is unused). Equip $2^{\mathbb{F}_\infty}$ with the product measure μ_0, and denote by a_∞ the shift action of $\mathbb{F}_\infty$ on (X_0, μ_0), where $X_0 \subseteq 2^{\mathbb{F}_\infty}$ is the conull set on which this shift action is free. For each $n \geq 2$, let a_n denote the (free) action of $\mathbb{F}_n$ on (X_0, μ_0) obtained by restricting a_∞ to $\langle F_n \rangle$. Finally, let $G_n = G(F_n, a_n) \subseteq G(F_\infty, a_\infty)$, so G_n is an acyclic, Borel graph with $\deg_{G_n}(x) = 2n$ for each $x \in X_0$. Moreover, each E_{G_n} is ergodic.

From [6], 4.6 it follows that $\chi_B(G_n) \leq 2n + 1$, and we will next sketch the argument that $\chi_{\mu_0}(G_n) \to \infty$ as $n \to \infty$. Let b_n denote the free part of the shift of $\mathbb{F}_n$ on $2^{\mathbb{F}_n}$, equipped with the usual product measure. Each a_n is isomorphic to the product $(b_n)^{\mathbb{N}}$ acting on $(2^{\mathbb{F}_n})^{\mathbb{N}}$. In turn, $(b_n)^{\mathbb{N}}$ is isomorphic to the shift action of $\mathbb{F}_n$ on $(2^{\mathbb{N}})^{\mathbb{F}_n}$. Bowen [1] has shown that this last shift action is weakly equivalent to b_n itself (for a definition and discussion of weak equivalence, see [4], 10, 10.1). In [2] and independently [7] it is shown that the graph associated with the action b_n (and a free set of generators) has a bound on the measure of an independent set tending towards zero as $n \to \infty$. Since this bound respects isomorphism and weak equivalence, we see by the chain of actions above that $\chi_{\mu_0}(G_n) \to \infty$ as $n \to \infty$. More details of this argument may be found in [2], 4.2, 4.17, 4.18.

We amalgamate these graphs into a single graph $G_{\mathbb{F}}$ on $X_{\mathbb{F}} = X_0 \times 2^{\mathbb{N}}$ (with the product measure $\mu_{\mathbb{F}} = \mu_0 \times (1/2, 1/2)^{\mathbb{N}}$) with countably infinite Borel and measurable chromatic numbers. Partition $2^{\mathbb{N}}$ into countably many Borel parts $A_2, A_3, \ldots$, each of positive measure. Fix some aperiodic, ergodic, measure-preserving, Borel automorphism $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, and define $T : X_{\mathbb{F}} \to X_{\mathbb{F}}$ by $T(x, y) = (\gamma_0 \cdot x, \sigma(y))$. Finally, define $G_{\mathbb{F}}$ by

$$(x, y) G_{\mathbb{F}} (x', y') \Leftrightarrow (x G_n x' \text{ and } y = y' \in A_n) \text{ or } (x, y) = T^{\pm 1}(x', y').$$

Here we are pasting several copies of G_n into each A_n, and using T to tie them together. We see that $G_{\mathbb{F}}$ is an acyclic, locally finite, Borel graph with $\chi_B(G_{\mathbb{F}}) = \chi_\mu(G_{\mathbb{F}}) = \aleph_0$. Moreover, $E_{G_{\mathbb{F}}}$ is an ergodic equivalence relation, so there is no way to partition $X_{\mathbb{F}}$ into countably many $E_{G_{\mathbb{F}}}$-invariant pieces on which $G_{\mathbb{F}}$ has finite measurable (thus Borel) chromatic number.

Of course, the same idea works in general to sew countably many acyclic graphs together using an ergodic automorphism, and acyclicity is preserved provided this automorphism is sufficiently "free" from the original graphs. We examine this idea in more detail in Section 2.

Recall that a *homomorphism* from a graph G on X to a graph H on Y is a function $\varphi : X \to Y$ such that

$$x G x' \Rightarrow \varphi(x) H \varphi(x').$$

With any function $f : X \to X$ we may associate a graph G_f on X by

$$x G_f y \Leftrightarrow x \neq y \text{ and } (x = f(y) \text{ or } y = f(x)).$$

We abbreviate E_{G_f} by E_f.

Proposition 1. *Suppose that $f : X \to X$ is a Borel function. Then there is no Borel homomorphism from $G_{\mathbb{F}}$ to G_f.*

Proof. Suppose towards a contradiction that $\varphi : X_{\mathbb{F}} \to X$ is a Borel homomorphism from $G_{\mathbb{F}}$ to G_f. Denote by $(\mu_{\mathbb{F}})_*$ the push-forward of $\mu_{\mathbb{F}}$ by φ, i.e., the measure given by

$$(\mu_{\mathbb{F}})_*(A) = \mu_{\mathbb{F}}(\varphi^{-1}(A)),$$

for all Borel $A \subseteq X$. In [8] it is shown that $\chi_\mu(G_f) = 3$ with respect to any Borel probability measure, thus in particular there is a $(\mu_{\mathbb{F}})_*$-measurable 3-coloring $c : X \to 3$. But then of course $c \circ \varphi$ would be a $\mu_{\mathbb{F}}$-measurable 3-coloring of $G_{\mathbb{F}}$, the desired contradiction. □

Remark 2. The above argument in fact also rules out a Borel homomorphism from $G_{\mathbb{F}}$ to any acyclic Borel graph G whose associated equivalence relation E_G is measure hyperfinite.

Proposition 3. *Suppose that $f : X \to X$ is a Borel function. If there is a Borel homomorphism from G_f to $G_{\mathbb{F}}$, then $\chi_B(G_f)$ is finite.*

Proof. Suppose that $\varphi : X \to X_{\mathbb{F}}$ is a Borel homomorphism from G_f to $G_{\mathbb{F}}$. We may use φ to pull some of the structure of $G_{\mathbb{F}}$ back to G_f. We define subgraphs H_T, H of G_f by

$$xH_Ty \Leftrightarrow xG_fy \text{ and } \varphi(x) = T^{\pm 1}(\varphi(y))$$
$$xHy \Leftrightarrow xG_fy \text{ and } \operatorname{proj}_{2^{\mathbb{N}}}(\varphi(x)) = \operatorname{proj}_{2^{\mathbb{N}}}(\varphi(y)),$$

where $\operatorname{proj}_{2^{\mathbb{N}}}$ denotes projection onto the $2^{\mathbb{N}}$ coordinate. Observe that $G_f = H_T \sqcup H$. Moreover, since T is a Borel automorphism, H_T has Borel chromatic number at most 3. It therefore suffices to argue that $\chi_B(H)$ is finite, since that would force $\chi_B(G_f)$ to be at most $3\chi_B(H)$. We will in fact show that $\chi_B(H) \leq 3$ as well.

Lemma 4. *Suppose that G is a Borel graph on X with $G \subseteq G_f$. Then there is a Borel function $g : X \to X$ with $G = G_g$. In particular, if $\chi_B(G)$ is finite, then $\chi_B(G) \leq 3$.*

Proof. Note that each connected component of G contains at most one point x such that x and $f(x)$ are not G-related. Simply define $g : X \to X$ by

$$g(x) = \begin{cases} f(x) \text{ if } xGf(x) \\ x \text{ otherwise,} \end{cases}$$

so that $G = G_g$. If also $\chi_B(G_g)$ is finite we have $\chi_B(G_g) \leq 3$ (see [6], 5.1). □

For $n \geq 2$, put $B_n = \varphi^{-1}(X_0 \times A_n)$, where $A_n \subseteq 2^{\mathbb{N}}$ is as in the definition of $G_{\mathbb{F}}$. Each B_n is a union of connected components of H (since the only edges in $G_{\mathbb{F}}$ connecting distinct $X_0 \times A_n$, $X_0 \times A_m$ are those induced by T). Moreover, for each $n \geq 2$, the Borel chromatic number of $H|B_n$ is finite, since $\varphi(H)|(X_0 \times A_n)$ has degree bounded by $2n$ (again applying [6], 4.6). By then applying Lemma 4 to H on each B_n, we see $\chi_B(H) \leq 3$ as promised. □

Remark 5. By applying [6], 5.1 one last time, we may of course improve the conclusion of the above proposition to $\chi_B(G_f) \leq 3$.

Corollary 6. *There is no Borel homomorphism from G_s to $G_{\mathbb{F}}$ nor from $G_{\mathbb{F}}$ to G_s.*

Suppose now that we have a graph G on X and an injective Borel homomorphism $\varphi : X \to [\mathbb{N}]^{\mathbb{N}}$ of G to G_s. Put $f = \varphi^{-1} \circ s \circ \varphi$. It is clear that $G \subseteq G_f$, and so by Lemma 4 there is a Borel function $g : X \to X$ such that $G = G_g$. Suppose moreover that there is a Borel homomorphism from G to $G_{\mathbb{F}}$. Then Proposition 3 implies that $\chi_B(G)$ is finite. In particular, we have:

Corollary 7. *There is no Borel graph G of infinite Borel chromatic number which is minimal under injective Borel homomorphism for all acyclic, locally finite, Borel graphs of indecomposably infinite Borel chromatic number.*

2 A more general construction

Suppose that E and F are equivalence relations on a measure space (X, μ). We say that E and F are *independent*, in symbols $E \perp F$, if there is no sequence $x_0, x_1, \ldots, x_n = x_0$ with $n > 1$, $x_i \neq x_j$ $(0 \leq i < j < n)$, and $x_0 E x_1 F x_2 E x_3 \ldots$ (i.e., there are no nontrivial cycles whose "edges" alternate between E and F). We say that E and F are *μ-independent* if they are independent after removing a μ-null set. The following is a generalization of the central argument in [4], II, 17.2. Recall that $\mathrm{Aut}(X, \mu)$ is the set of μ-preserving automorphisms of X, which is a Polish space once equipped with the weak topology (see [4], I, 1). Recall that the *support* of $T \in \mathrm{Aut}(X, \mu)$ is the set $\{x \in X : T(x) \neq x\}$.

Proposition 8. *Suppose that E is a μ-preserving, countable, Borel equivalence relation on a standard probability space (X, μ). Then the set of automorphisms T with E and E_T μ-independent is comeager in* $\mathrm{Aut}(X, \mu)$.

Proof. Since every countable, Borel equivalence relation is contained in an aperiodic, countable, Borel equivalence relation, we may assume without loss of generality that E is aperiodic. We may then (see [3]) write $E \setminus \{(x, x) : x \in X\}$ as the union of the graphs of countably many fixed-point-free involutions $\iota_0, \iota_1, \ldots$, with of course each $\iota_n \in \mathrm{Aut}(X, \mu)$. To each formal word w in the alphabet $\{\iota_n : n \in \mathbb{N}\} \sqcup \mathbb{Z}$ and $T \in \mathrm{Aut}(X, \mu)$ we may associate an automorphism $T_w \in \mathrm{Aut}(X,\mu)$ inductively by

$$\begin{aligned} T_\emptyset &= \mathrm{id}, \\ T_{\iota_n w_0} &= \iota_n \circ T_{w_0}, \text{ for } \gamma \in \Gamma, \\ T_{z w_0} &= T^z \circ T_{w_0}, \text{ for } z \in \mathbb{Z}. \end{aligned}$$

Intuitively, we simply replace each $z \in \mathbb{Z}$ with the corresponding power T^z of T.

We abuse notation somewhat and say that a word v is a *subword* of a word w, in symbols $v \subseteq w$, if there are (possibly empty) words v_0, v_1 such that $v_0 v v_1 = w$. As usual, v is a *proper subword* of w, written $v \subsetneq w$, if $v \subseteq w$ and $v \neq w$. From now on, we restrict our attention to words which alternate symbols from $\{\iota_n : n \in \mathbb{N}\}$ and $\mathbb{Z} \setminus \{0\}$.

To establish the proposition, it therefore suffices to show that for all words w, the set

$$\{T \in \mathrm{Aut}(X,\mu) : \mu(\mathrm{supp}(T_w)) = 1\}$$

is comeager in $\mathrm{Aut}(X,\mu)$. We proceed by induction on $|w|$. If $|w| = 1$, then either $w = \iota_n$ for some $n \in \mathbb{N}$, or $w = z$ for some $z \in \mathbb{Z} \setminus \{0\}$. In both cases the conclusion is immediate, since our starting involutions and the generic automorphism have conull support. The argument is equally trivial for words of length two.

Suppose then that $n \geq 3$ and that the set

$$C_w = \{T \in \mathrm{Aut}(X,\mu) : \mu(\mathrm{supp}(T_v)) = 1 \text{ for all words } v \subsetneq w\}$$

is comeager in $\mathrm{Aut}(X,\mu)$. Fix now some word w of length n. Since the set

$$\{T \in \mathrm{Aut}(X,\mu) : \mu(\mathrm{supp}(T_w)) = 1\}$$

is certainly G_δ in the weak topology, it suffices to show that it is dense.

Towards that end, fix a nonempty basic open set

$$U = \{T \in \mathrm{Aut}(X,\mu) : \mu(T(A_i) \bigtriangleup B_i) < \varepsilon_0\},$$

determined by finite sequences (A_i), (B_i) of Borel sets, and a positive tolerance ε_0.

By assumption C_w is comeager, so we may fix some $S \in U \cap C_w$. Write $X = X^{\mathrm{supp}} \sqcup X^{\mathrm{fix}}$, where

$$\begin{aligned} X^{\mathrm{supp}} &= \{x \in X : S_w(x) \neq x\} = \mathrm{supp}(S_w) \\ X^{\mathrm{fix}} &= \{x \in X : S_w(x) = x\} = X \setminus \mathrm{supp}(S_w). \end{aligned}$$

Enumerate the terminal subwords of w as $\emptyset = w_0, w_1, w_2, \ldots, w_{n-1}, w_n = w$, so that $|w_l| = l$ for each $l \leq |w|$.

Lemma 9. *Suppose that $T \in C_w$ and Y is a Borel subset of X. Then there is a partition of Y (modulo μ-null sets) into $N = |w|^2 + 1$ Borel parts $Y_1, \ldots, Y_N$ such that for each $i \leq N$ the collection $\{T_{w_l}(Y_i) : l < |w|\}$ forms a pairwise disjoint family. If moreover $Y \subseteq \mathrm{supp}(T_w)$, we may ensure that the collection $\{T_{w_l}(Y_i) : l \leq |w|\}$ forms a pairwise disjoint family.*

Proof. Discarding a null set, we may assume each T_v, $v \subsetneq w$, has full support. Consider the graph $G_{T,w}$ on Y given by

$$xG_{T,w}y \Leftrightarrow x \neq y \text{ and } \{T_{w_l}(x) : l < |w|\} \cap \{T_{w_l}(y) : l < |w|\} \neq \emptyset.$$

The graph $G_{T,w}$ has degree bounded by $|w|^2$ and thus admits a Borel coloring by $|w|^2 + 1 = N$ colors. We choose these colors as our sets Y_i. Suppose, towards a contradiction, that $T_{w_l}(Y_i) \cap T_{w_{l+m}}(Y_i) \neq \emptyset$, and fix $x, y \in Y_i$ with $T_{w_l}(x) = T_{w_{l+m}}(y)$. Since Y_i is $G_{T,w}$-independent, we must have $x = y$. But then, writing $w_{l+m} = vw_l$, we see $T_v(T_{w_l}(x)) = T_{w_l}(x)$, which contradicts T_v having full support.

In the case that $Y \subseteq \mathrm{supp}(T_w)$, simply consider the slightly larger graph

$$xG'_{T,w}y \Leftrightarrow x \neq y \text{ and } \{T_{w_l}(x) : l \leq |w|\} \cap \{T_{w_l}(y) : l \leq |w|\} \neq \emptyset,$$

and repeat the same argument. $\square$

Apply the lemma to partition X^{supp} into $X_1^{\mathrm{supp}}, \ldots, X_N^{\mathrm{supp}}$ so that for each $i \leq N$ the sets $X_i^{\mathrm{supp}}, S_{w_1}(X_i^{\mathrm{supp}}), \ldots, S_{w_{n-1}}(X_i^{\mathrm{supp}}), S_w(X_i^{\mathrm{supp}})$ are pairwise disjoint. Analogously, partition X^{fix} into countably many Borel pieces $X_1^{\mathrm{fix}}, \ldots, X_N^{\mathrm{fix}}$ so that for each $i \leq N$ the sets $X_i^{\mathrm{fix}}, S_{w_1}(X_i^{\mathrm{fix}}), \ldots, S_{w_{n-1}}(X_i^{\mathrm{fix}})$ are pairwise disjoint.

Our goal is to perturb S to some automorphism $T \in U$ such that $\mathrm{supp}(T_w)$ is a conull subset of X. Towards that end, fix the least k so that the terminal subword w_k contains an occurrence of some element of $\mathbb{Z}$ (so $k \in \{1, 2\}$). We pause to recall some basic facts about the measure algebra.

Given a finite collection $\mathcal{B}$ of subsets of X, let $\mathcal{A}(\mathcal{B})$ denote the finite set of atoms of the Boolean algebra generated by $\mathcal{B}$. It is well known that for any Borel set $A \subseteq X$ and any finite collection $\mathcal{B}$ of Borel subsets of X, there is an involution $I \in \mathrm{Aut}(X, \mu)$ such that $\mathrm{supp}(I) = A$ and $I(B) = B$ for all $B \in \mathcal{A}(\mathcal{B})$ (where everything is viewed modulo μ-null sets); we say such an I *respects* $\mathcal{A}(\mathcal{B})$. Note that this implies $I(B) = B$ for all $B \in \mathcal{B}$. Of course, for such an involution I, we may find a Borel set $A_0 \subseteq A$ such that $\forall^\mu x \in A$ $(x \in A_0 \Leftrightarrow I(x) \notin A_0)$; we call such an A_0 an *I-half* of A.

We are now ready to proceed in N steps. Our goal in each step is to manipulate S by a well chosen involution $I_i \in \mathrm{Aut}(X, \mu)$ with support $S_{w_k}(X_i^{\mathrm{fix}})$ for each $i \leq N$, thus adding X_i^{fix} to the support of $(I \circ S)_w$. This initial goal is a bit too simplistic, but we will nevertheless get the job done by taking a bit more care. For convenience, let $\mathcal{B}$ denote the (finite) set

$$\{S(A_i) \cup \{S_{w_l}(X_i^{\mathrm{supp}}) : i \leq N, l \leq |w|\} \cup \{S_{w_l}(X_i^{\mathrm{fix}}) : i \leq N, l < |w|\}.$$

In the first step, we may find an involution I_1 with support $S_{w_k}(X_1^{\mathrm{fix}})$ which respects $\mathcal{A}(\mathcal{B})$. Let $Y_1 \subseteq X_1^{\mathrm{fix}}$ be a Borel set chosen so that $S_{w_k}(Y_1)$ is an I_1-half of $S_{w_k}(X_1^{\mathrm{fix}})$. Then we have for $x \in X_1^{\mathrm{fix}}$, $x \in Y_1 \Leftrightarrow (I_1 \circ S)_w \notin Y_1$. Note that for all $x \in X_i^*$ (where $*$ denotes either supp or fix) and $l \leq |w|$, we have $(I_1 \circ S)_{w_l}(x) \in S_{w_l}(X_i^*)$ (since I_1 respects each $S_{w_l}(X_i^*)$). Surely then for all proper subwords $v \subsetneq w$ and $x \in X$ we have $(I_1 \circ S)_v(x) \neq x$, since we may write some terminal word $w_{l+m} = vw_l$ and note that $S_{w_l}(X_i^*)$ is disjoint from $S_{w_{l+m}}(X_i^*)$ by the construction in Lemma 9 (and x is in some $S_{w_l}(X_i^*)$, as they partition X). Consequently, $I_1 \circ S \in C_w$.

In the second step, we reexamine X_2^{fix}. It may be the case that there are now some $x \in X_2^{\mathrm{fix}}$ such that $(I_1 \circ S)_w(x) \neq x$; we denote by X_2^{moved} the set of such points. By another application of Lemma 9, we may find a Borel partition $(X_{2,m}^{\mathrm{moved}})_{m \leq N}$ of X_2^{moved} such that for each m the sets $X_{2,m}^{\mathrm{moved}}$, $(I_1 \circ S)_{w_1}(X_{2,m}^{\mathrm{moved}}), \ldots, (I_1 \circ S)_{w_{n-1}}(X_{2,m}^{\mathrm{moved}}), (I_1 \circ S)_w(X_{2,m}^{\mathrm{moved}})$ are pairwise disjoint. We may now fix a Borel involution I_2 with support $S_{w_k}(X_2^{\mathrm{fix}} \setminus X_2^{\mathrm{moved}})$ respecting $\mathcal{A}(\mathcal{B} \cup \{S_{w_l}(Y_1) : l < |w|\} \cup \{(I_1 \circ S)_{w_l}(X_{2,m}^{\mathrm{moved}}) : l \leq |w|, m \leq N\}$. Let $Y_2 \subseteq X_2^{\mathrm{fix}} \setminus X_2^{\mathrm{moved}}$ be a Borel set chosen so that $S_{w_k}(Y_2)$ is an I_2-half of $S_{w_k}(X_2^{\mathrm{fix}} \setminus X_2^{\mathrm{moved}})$. By exactly the same arguments as before, for $x \in X_2^{\mathrm{fix}} \setminus X_2^{\mathrm{moved}}$, $x \in Y_2 \Leftrightarrow (I_2 \circ I_1 \circ S)_w \notin Y_2$, and also $I_2 \circ I_1 \circ S \in C_w$.

Proceeding in this fashion, in the i^{th} step, we find an involution I_i with support $S_{w_k}(X_i^{\mathrm{fix}} \setminus X_i^{\mathrm{moved}})$ which respects $\mathcal{A}(\mathcal{B} \cup \{S_{w_l}(Y_j) : l < |w|, j < i\} \cup \{(I_{j-1} \circ \cdots \circ I_1 \circ S)_{w_l}(X_{j,m}^{\mathrm{moved}}) : l \leq |w|, 2 \leq j \leq i, m \leq N\})$. After N steps we have involutions $I_1, \ldots, I_N$, and we put $T = I_N \circ \cdots \circ I_1 \circ S$. We see immediately that $T(A_i) = S(A_i)$ since each involution fixes $S(A_i)$ as a set, so $T \in U$. It remains to check that T_w has conull support.

First, suppose that $x \in X_i^{\mathrm{supp}}$ for some $i \leq N$. Then for each $l \leq |w|$ we have $T_{w_l}(x) \in S_{w_l}(X_i)$, since each involution I_i fixes (as sets) each $S_{w_l}(X_i)$. In particular, $T_w(x) \in S_w(X_i)$, which is disjoint from X_i^{supp}, and consequently $x \in \mathrm{supp}(T_w)$.

Second, suppose that $x \in X_{i,m}^{\mathrm{moved}}$ for some $i, m \leq N$. Then for each $l \leq |w|$ we have $T_{w_1}(x) \in (I_{i-1} \circ \cdots \circ I_1 \circ S)_{w_l}(X_{i,m}^{\mathrm{moved}})$, since each involution I_j $(j > i)$ fixes (as sets) each $(I_{i-1} \circ \cdots \circ I_1 \circ S)_{w_l}(X_{i,m}^{\mathrm{moved}})$. In particular, $T_w(x) \in (I_{i-1} \circ \cdots \circ I_1 \circ S)_w(X_{i,m}^{\mathrm{moved}})$, which is disjoint from $X_{i,m}^{\mathrm{moved}}$, and consequently $x \in \mathrm{supp}(T_w)$.

Third, suppose that $x \in X_i^{\mathrm{fix}} \setminus X_i^{\mathrm{moved}}$ for some i. Recall that by construction, $x \in Y_i \Leftrightarrow (I_i \circ \cdots \circ I_1 \circ S)_w(x) \notin Y_i$. Since each subsequent involution respects the images of Y_i, it is routine to verify that $x \in Y_i \Leftrightarrow T_w(x) \notin Y_i$. In particular, $x \in \mathrm{supp}(T_w)$.

Since $\bigcup_{i \leq N}(X_i^{\mathrm{supp}} \cup X_i^{\mathrm{fix}})$ has full measure, we see that T has conull support, completing the proof. □

Before generalizing the construction of $G_{\mathbb{F}}$ in the previous section, we digress a bit to discuss costs. The relevant notions are defined in [5], III.

Corollary 10. *Suppose that E is a μ-preserving, aperiodic, countable, Borel equivalence relation on a standard probability space (X, μ). Then for all $\alpha > 0$ there is an μ-preserving, countable, Borel equivalence relation $E_\alpha \supseteq E$ such that $C_\mu(E_\alpha) = C_\mu(E) + \alpha$. Moreover, if E is treeable then E_α may be taken to be treeable, and additionally there is a treeable $E_\infty \supseteq E$ with $C_\mu(E_\infty) = \infty$.*

Proof. Write $\alpha = n + \varepsilon$ for some $n \in \mathbb{N}$ and $0 \leq \varepsilon < 1$. Apply Proposition 8 in succession $n+1$ times (discarding a μ-null set if necessary) to obtain automorphisms $T_0, \ldots, T_n \in \mathrm{Aut}(X, \mu)$ with, for each $i \leq n$,

$$E_{T_i} \perp E \vee \bigvee_{j<i} E_{T_j}.$$

Fix a Borel set $A_\varepsilon \subseteq X$ with $\mu(A_\varepsilon) = \varepsilon$ and set $T_n' = T|A_\varepsilon$, i.e.,

$$T_n'(x) = \begin{cases} T_n(x) \text{ if } x \in A_\varepsilon, \\ x \text{ if } x \notin A_\varepsilon. \end{cases}$$

Then, by a fundamental result of Gaboriau's theory of costs (see, e.g., [5], 27.2), the equivalence relation

$$E_\alpha = E \vee \bigvee_{i<n} E_{T_i} \vee E_{T_n'}$$

has cost $C_\mu(E) + \alpha$.

If E is treeable, it is evident that this E_α is also treeable by simply adding to the treeing of E the (symmetrized) graphs of $T_0, \ldots, T_{n-1}, T'_n$. Also in this case we may apply Proposition 8 countably infinitely many times to obtain a treeable equivalence relation E_∞ with infinite cost. □

Suppose now that we have a countable sequence of measure-preserving, Borel graphs $(G_n)_{n\in\mathbb{N}}$ on standard probability spaces (X_n, μ_n), respectively (i.e., for each $n \in \mathbb{N}$, μ_n is E_{G_n}-invariant). We build another standard measure space (X, μ) by setting $X = \bigsqcup_{n\in\mathbb{N}} X_n$ and for Borel $A \subseteq X$,

$$\mu(A) = \sum_{n\in\mathbb{N}} 2^{-(n+1)} \mu_n(A \cap X_n).$$

We then may define a Borel graph $G_\mathbb{N}$ on X by

$$xG_\mathbb{N}y \Leftrightarrow \exists n \in \mathbb{N}\ (x, y \in X_n \text{ and } xG_ny).$$

Note then that μ is $E_{G_\mathbb{N}}$-invariant.

By Proposition 8, the generic $T \in \mathrm{Aut}(X, \mu)$ satisfies $E_T \perp E_{G_\mathbb{N}}$ (after, as usual, discarding a μ-null set). We may fix such a T which is ergodic, since of course the generic element of $\mathrm{Aut}(X, \mu)$ is ergodic. We then define the *ergodic amalgamation of* (G_n) *by* T to be the Borel graph

$$G_{(G_n),T} = G_\mathbb{N} \cup G_T$$

on X. When the particular choice of T is unimportant, we say G is an *ergodic amalgamation* of (G_n) if $G = G_{(G_n),T}$ for some T as above. Note of course that E_G is an ergodic equivalence relation. We collect here some useful facts about ergodic amalgamations.

Proposition 11. *Suppose that* $(G_n)_{n\in\mathbb{N}}$ *is a sequence of Borel, measure-preserving graphs on* (X_n, μ_n)*, respectively, and that* G *on* (X, μ) *is an ergodic amalgamation of* (G_n)*.*

(i) *If each* G_n *is acyclic, then* G *is acyclic. Moreover, the girth of* G *is equal to the least girth among the graphs* G_n*. Similarly, the odd girth and the clique number of* G *is equal to the least value of the respective parameter among the graphs* G_n*.*

(ii) *If each* G_n *is locally finite, then* G *is locally finite. Moreover, if every* G_n *has degree bounded by a fixed* d*, then* G *has degree bounded by* $d+2$*.*

(iii) *The Borel chromatic number of G, $\chi_B(G)$, is bounded below by* $\sup_n \chi_B(G_n)$ *and above by* $3 \sup_n \chi_B(G_n)$.

(iv) *The measurable chromatic number of G, $\chi_\mu(G)$, is bounded below by* $\sup_n \chi_{\mu_n}(G_n)$ *and above by* $3 \sup_n \chi_{\mu_n}(G_n)$.

In particular, if G is an ergodic amalgamation of some sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with measurable chromatic numbers tending towards infinity, and f is any Borel function on a Polish space such that G_f has infinite Borel chromatic number, the arguments in Proposition 1 and 3 ensure that there is no Borel homomorphism from G to G_f nor from G_f to G.

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