A SIMPLE MARRIAGE PROBLEM

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ABSTRACT. We produce an example of a Borel set $S \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that: (1) every horizontal and vertical section of S is countably infinite, (2) the graph associated with S is acyclic, (3) the equivalence relation associated with S is hyperfinite, and (4) no Borel partial injection whose graph is contained in S has comeager domain or range. In particular, the marriage problem associated with S has no Baire measurable solution while, under $\mathbf{add}(\text{null}) = \mathfrak{c}$, it has a universally measurable solution.

We will recursively construct sets $R_n, S_n \subseteq 2^n \times 2^n$. The sets R_n represent restrictions on the manner in which latter stages of the construction proceed, and the sets S_n represent collections of links which approximate the desired set. Along the way, we will ensure that, for all $n \in \mathbb{N}$, the following conditions hold:

(i) $\forall u \in 2^n \, \exists v \in 2^n \; ((u, v) \in R_n)$.

(ii) $\forall v \in 2^n \, \exists u \in 2^n \; ((u, v) \in R_n)$.

(iii) $\exists u \in 2^n \, \forall v \in 2^n \; ((u, v) \in R_n \setminus S_n)$.

(iv) $\exists v \in 2^n \, \forall u \in 2^n \; ((u, v) \in R_n \setminus S_n)$.

Following common practice, we use $\emptyset$ to denote both the empty string and the empty set. On the set $2^0 = \{\emptyset\}$, we define $R_0 := 2^0 \times 2^0 = \{(\emptyset, \emptyset)\}$ and $S_0 := \emptyset$.

Fix an enumeration $((i_n, j_n), s_n)$ of $(2 \times 2) \times 2^{<\mathbb{N}}$ such that $\forall n \in \mathbb{N} \; (|s_n| \leq n)$. We will construct (R_{n+1}, S_{n+1}) from (R_n, S_n) and $((i_n, j_n), s_n)$, by either adding a single new link to the set

$$S'_{n+1} := \{(si, ti) : (s, t) \in S_n \text{ and } i \in \{0, 1\}\},$$

or by removing all but one point of a horizontal or vertical section from the set

$$R'_{n+1} := \{(si, tj) : (s, t) \in R_n \text{ and } i, j \in \{0, 1\}\}.$$

It is easy to see that this will ensure that conditions (i) – (iv) hold of (R_{n+1}, S_{n+1}). Here are the specifics as to how we proceed:

1. If $i_n = 0$, then we add a new link without augmenting the restrictions:
   (a) If $j_n = 0$, then fix $u_n \in 2^n$ such that $s_n \subseteq u_n$. By condition (i),

   $$\exists v_n \in 2^n \; ((u_n, v_n) \in R_n).$$

   Set $R_{n+1} := R'_{n+1}$ and $S_{n+1} := S'_{n+1} \cup \{(u_n 0, v_n 1)\}$.

(b) If $j_n = 1$, then fix $v_n \in 2^n$ such that $s_n \subseteq v_n$. By condition (ii),

$$\exists u_n \in 2^n \ ((u_n, v_n) \in R_n).$$

Set $R_{n+1} := R'_{n+1}$ and $S_{n+1} := S'_{n+1} \cup \{(u_n 0, v_n 1)\}$.

2. If $i_n = 1$, then we add a new restriction without augmenting the links:

(a) If $j_n = 0$, then condition (iii) ensures that

$$\exists u_n \in 2^n \, \forall v \in 2^n \ ((u_n, v) \in R_n \setminus S_n).$$

Fix $v_n \in 2^n$ such that $s_n \subseteq v_n$, and set $S_{n+1} := S'_{n+1}$ and

$$R_{n+1} := \{(u, v) \in R'_{n+1} : u = u_n 0 \Rightarrow v = v_n 0\}.$$

(b) If $j_n = 1$, then condition (iv) ensures that

$$\exists v_n \in 2^n \, \forall u \in 2^n \ ((u, v_n) \in R_n \setminus S_n).$$

Fix $u_n \in 2^n$ such that $s_n \subseteq u_n$, and set $S_{n+1} := S'_{n+1}$ and

$$R_{n+1} := \{(u, v) \in R'_{n+1} : v = v_n 0 \Rightarrow u = u_n 0\}.$$

This completes the description of (R_n, S_n). Finally, define $S \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$S = \bigcup_{n \in \mathbb{N}} \{(sx, tx) : (s, t) \in S_n \text{ and } x \in 2^{\mathbb{N}}\}.$$

Lemma 1. *The graph associated with S is acyclic and the equivalence relation associated with S is hyperfinite.*

Proof. It will be convenient to have a copy of S sitting in a product of disjoint spaces. Set $X := \mathcal{N}_0$, $Y := \mathcal{N}_1$, and $Z := X \sqcup Y = 2^{\mathbb{N}}$, and define $S_* \subseteq X \times Y$ by

$$S_* := \{(0x, 1y) : (x, y) \in S\}.$$

The graph associated with S_* (equivalently, S) is given by $\mathcal{G} := S_* \cup S_*^{-1}$. To see that this is acyclic, simply observe that any $\mathcal{G}$-cycle is necessarily of the form $0s_0x, 1s_1x, \ldots, 0s_kx = 0s_0x$, where $\forall i < k \ (|s_i| = n)$, for some $n \in \mathbb{N}$. Then $s_0, s_1, \ldots, s_k = s_0$ is a cycle with respect to the graph $\mathcal{G}_n := S_n \cup S_n^{-1}$. But a straightforward induction shows that the graphs of the latter form are acyclic, and we obtain a contradiction.

The equivalence relation E associated with S_* (equivalently, S) is that generated by $\mathcal{G}$. As $\mathcal{G}$ is contained in E_0, so too is E, and it follows that E is hyperfinite. □

While S does not literally satisfy the requirement that every horizontal and vertical section is countably infinite, it is true generically:

Lemma 2. $\forall^* x \in 2^{\mathbb{N}} \ (|S_x| = |S^x| = \aleph_0)$.

Proof. We will just prove that $\forall^* x \in 2^{\mathbb{N}}$ $(|S_x| = \aleph_0)$. It is enough to show that

$$\forall s \in 2^{<\mathbb{N}}\, \exists t \in 2^{<\mathbb{N}}\ (s \subseteq t \text{ and } |[S_{|s|}]_s| < |[S_{|t|}]_t|).$$

Towards this end, fix $n > |s|$ such that $(i_n, j_n) = (0, 0)$ and $s \subseteq u_n$, and observe that $t = u_n 0$ is as desired. $\square$

As $S \subseteq E_0$, there is an E_0-invariant comeager Borel set $C \subseteq 2^{\mathbb{N}}$ such that $\forall x \in C$ $(|S_x| = |S^x| = \aleph_0)$. Fix any Borel set $S' \subseteq (2^{\mathbb{N}} \setminus C) \times (2^{\mathbb{N}} \setminus C)$ whose sections are countably infinite, whose associated graph is acyclic, and whose associated equivalence relation is hyperfinite, and define $S'' \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$S'' = (S \cap (C \times C)) \cup S'.$$

It is clear that every section of S'' is countably infinite, the graph associated with S'' is acyclic, and the equivalence relation associated with S'' is hyperfinite. Moreover, if S'' contains the graph of a Borel partial injection whose domain or range is comeager, then so too does S. The following fact therefore rules this out:

Lemma 3. *Suppose that $f \vdots 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel partial injection whose graph is contained in S. Then neither the domain nor range of f is comeager.*

Proof. For each finite set $S \subseteq \mathbb{N}$ and $x \in 2^{\leq \mathbb{N}}$, let $i_S(x)$ be the sequence obtained by flipping the value of the n^{th} digit of x, for each $n \in S$, i.e.,

$$[i_S(x)](n) = \begin{cases} x(n) & \text{if } n < |x| \text{ and } n \notin S, \\ 1 - x(n) & \text{if } n < |x| \text{ and } n \in S, \text{ and} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now suppose, towards a contradiction, that $\operatorname{dom}(f)$ is comeager. Then there exists $s \in 2^{<\mathbb{N}}$ and a finite set $S \subseteq \mathbb{N}$ such that

$$\forall^* x \in \mathcal{N}_s\ (f(x) = i_S(x)).$$

Without loss of generality, we can assume that $|s| > \max_{n \in S} n$. Put $t = i_S(s)$, noting that $f(\mathcal{N}_s)$ is comeager in $\mathcal{N}_t$. Fix $n > |s|$ such that $(i_n, j_n) = (1, 0)$ and $t \subseteq s_n$. Then $t \subseteq v_n$, and it follows that $f(\mathcal{N}_{u_n 0}) \subseteq \mathcal{N}_{v_n 0} \subseteq \mathcal{N}_t$. As $\operatorname{graph}(f) \subseteq E_0$, however, it follows that $f(\mathcal{N}_{u_n 0})$ is non-meager, thus $f(\mathcal{N}_s) \cap f(\mathcal{N}_{u_n 0}) \neq \emptyset$. As $\mathcal{N}_s \cap \mathcal{N}_{u_n 0} = \emptyset$, this contradicts the injectivity of f.

An essentially identical proof shows that $\operatorname{rng}(f)$ is not comeager. $\square$

Remark 4. A similar argument shows that S'' does not even contain the graph of a finite-to-one Borel partial function with comeager domain.

In contrast, we have the following result in the measure-theoretic context:

Theorem 5 (add(null) = $\mathfrak{c}$). *Suppose that X and Y are Polish spaces, $S \subseteq X \times Y$ is Borel, the graph associated with S is acyclic, the equivalence relation associated with S is hyperfinite (or just measure amenable), and every horizontal and vertical section of S is of cardinality at least 3. Then there is a universally measurable injection $f : X \to Y$ such that* $\operatorname{graph}(f) \subseteq S$.

Proof (Sketch). Let $\mathcal{G}$ be the graph associated with S, and E the equivalence relation associated with S. By Kechris's generalization of Adams's result on end selection, after throwing out an E-invariant, μ-null Borel set, we can assume that either: (1) $\mathcal{G}$ is induced by a Borel function, or (2) there is a Borel way of choosing a line out of each $\mathcal{G}$-component. In each of these cases, we can easily build a Borel injection whose graph is contained in S. Using $\mathbf{add}(\text{null}) = \mathfrak{c}$, we can simultaneously do this for all measures. $\square$