Math 223D: Borel Dynamics

Contained herein are the lecture notes from a course in Borel dynamics that I gave during the spring term of 2007 at UCLA. They should be accessible to anyone with a reasonable understanding of algebra, analysis, set theory, and topology at the undergraduate level. The main topics include algebraic properties of full groups and the existence of invariant measures. We also introduce various basic facts of classical descriptive set theory as they become necessary.

1 Separable automorphisms

A *σ-algebra* on a set X is a non-empty set $\mathcal{B} \subseteq \mathcal{P}(X)$ which is closed under complements, countable unions, and countable intersections. A *Borel space* is a pair $(X, \mathcal{B})$, consisting of a set X and a σ-algebra $\mathcal{B}$ on X. An *automorphism* of $(X, \mathcal{B})$ is a permutation $T \in S_X$ such that

$$\forall B \subseteq X \ (B \in \mathcal{B} \Leftrightarrow T(B) \in \mathcal{B}).$$

We use $\mathrm{Aut}(X, \mathcal{B})$ to denote the set of all automorphisms of $(X, \mathcal{B})$.

The *orbit* of a point x with respect to a permutation $T \in S_X$ is given by

$$[x]_T = \{T^n(x) : n \in \mathbb{Z}\}.$$

The *orbit equivalence relation* induced by T is given by

$$x E_T^X y \Leftrightarrow [x]_T = [y]_T.$$

A *separating family* for a set $B \subseteq X$ is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

$$\forall x, y \in B \ (x \neq y \Rightarrow \exists A \in \mathcal{A} \ (x \in A \text{ and } y \notin A)).$$

A *separating family* for a permutation $T \in S_X$ is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

$$\forall x \in X \ (\mathcal{A} \text{ is a separating family for } [x]_T).$$

We say that an automorphism T of $(X, \mathcal{B})$ is *separable* if $\mathcal{B}$ contains a countable separating family for T.

Example 1.1. Suppose that (X, τ) is a second countable Hausdorff space, and $\tau \subseteq \mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra. Then every automorphism of $(X, \mathcal{B})$ is separable.

Example 1.2. Suppose that X is an uncountable set and $\mathcal{B} \subseteq \mathcal{P}(X)$ is the σ-algebra consisting of all countable and co-countable subsets of X. Then $S_X = \mathrm{Aut}(X, \mathcal{B})$, but the only separable automorphisms of $(X, \mathcal{B})$ are those which fix all but countably many points of X.

Exercise 1. Give examples of separable and non-separable automorphisms of Borel spaces which are different from those given above.

For each $k \in \mathbb{Z}^+$ and $T \in S_X$, we say that a set $B \subseteq X$ is (T,k)-*discrete* if the sets $B, T(B), \ldots, T^k(B)$ are pairwise disjoint. When $k = 1$, we say also that B is T-*discrete.* A (T,k)-discrete set $B \subseteq X$ is *maximal* (T,k)-*discrete* if

$$\forall C \subseteq X \ (B \subset C \Rightarrow C \text{ is not } (T,k)\text{-discrete}).$$

While `AC` ensures the existence of maximal (T,k)-discrete sets, it says little about whether any such sets are in $\mathcal{B}$.

Proposition 1.3. *Suppose that* $(X,\mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X,\mathcal{B})$. *Then the following are equivalent:*

1. *T is separable;*
2. *$\forall k \in \mathbb{Z}^+$ ($\mathcal{B}$ contains a maximal (T,k)-discrete set).*

Proof. To see (1) $\Rightarrow$ (2), suppose that T is separable and $k \in \mathbb{Z}^+$. Fix a countable separating family $\mathcal{A} \subseteq \mathcal{B}$ for T, as well as an enumeration $\langle A_n \rangle_{n \in \mathbb{N}}$ of the closure of $\mathcal{A}$ under finite intersections. Set $B_0 = \emptyset$. Given B_n, define $C_n = A_n \setminus (T^{-1}(A_n) \cup \cdots \cup T^{-k}(A_n))$, $D_n = T^{-k}(B_n) \cup \cdots \cup T^k(B_n)$, and

$$B_{n+1} = B_n \cup (C_n \setminus D_n).$$

As C_n is (T,k)-discrete, it follows that if B_n is (T,k)-discrete, then so too is B_{n+1}. As B_0 is trivially (T,k)-discrete, it follows that each B_n is (T,k)-discrete, thus so too is the set $B_\infty = \bigcup_{n \in \mathbb{N}} B_n$. Now suppose, towards a contradiction, that B_∞ is not maximal (T,k)-discrete. Then there exists $x \in X \setminus B_\infty$ such that $\{x\} \cup B_\infty$ is (T,k)-discrete. Fix $n \in \mathbb{N}$ such that $x \in A_n$ and $T(x), \ldots, T^k(x) \notin A_n$, and observe that $x \in C_n$, thus $x \in B_{n+1} \subseteq B_\infty$, the desired contradiction.

To see (2) $\Rightarrow$ (1), simply observe that if B_k is a maximal (T,k)-discrete subset of B, for each $k \in \mathbb{Z}^+$, then the family of sets of the form $T^i(B_k)$, for $i \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, is a countable separating family for T, thus T is separable. □

The *aperiodic part* of T is given by $\mathrm{Aper}(T) = \{x \in X : [x]_T \text{ is infinite}\}$. We say that T is *aperiodic* if $X = \mathrm{Aper}(T)$.

Exercise 2. A topological space is *zero-dimensional* if it has a basis consisting of clopen sets. Suppose that (X,d) is a compact zero-dimensional metric space. Show that if T is an aperiodic homeomorphism of (X,d), then there is a maximal T-discrete clopen subset of (X,d).

The *period k part* of T is given by $\mathrm{Per}_k(T) = \{x \in X : |[x]_T| = k\}$, and the *periodic part* of T is given by $\mathrm{Per}(T) = \bigcup_{k \in \mathbb{Z}^+} \mathrm{Per}_k(T)$. We say that T is *periodic* if $X = \mathrm{Per}(T)$.

Proposition 1.4. *Suppose that* $(X,\mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X,\mathcal{B})$ *is separable. Then* $\forall k \in \mathbb{Z}^+$ $(\mathrm{Per}_k(T) \in \mathcal{B})$, *thus* $\mathrm{Per}(T), \mathrm{Aper}(T) \in \mathcal{B}$.

Proof. Proposition 1.3 ensures that for each $k \in \mathbb{Z}^+$, there is maximal (T, k)-discrete set $B_k \in \mathcal{B}$. The T-*saturation* of a set $B \subseteq X$ is given by

$$[B]_T = \bigcup_{n \in \mathbb{Z}} T^n(B).$$

It is clear that each $[B_k]_T$ is in B, so it is enough to show that

$$X \setminus [B_k]_T = \bigcup_{l \leq k} \mathrm{Per}_l(T),$$

for all $k \in \mathbb{Z}^+$. Towards this end, note first that if $\ell \leq k$ and $x \in \mathrm{Per}_l(T)$, then $T^l(x) = x$, so $x \notin B_k$, thus $\mathrm{Per}_l(T) \cap [B_k]_T = \emptyset$. On the other hand, if $|[x]_T| > k$ and $[x]_T \cap B_k = \emptyset$, then $\{x\} \cup B_k$ is (T, k)-discrete, which contradicts our assumption that B_k is maximal (T, k)-discrete. □

A *partial transversal* of T is a set $B \subseteq X$ which intersects every orbit of T in at most one point. A set $B \subseteq X$ is T-*complete* if it intersects every orbit of T in at least one point. A *transversal of* T is a complete partial transversal of T. We say that T is *smooth* if $\mathcal{B}$ contains a transversal of T.

Proposition 1.5. *Every smooth automorphism of a Borel space is separable.*

Proof. Simply observe that if $(X, \mathcal{B})$ is a Borel space, $T \in \mathrm{Aut}(X, \mathcal{B})$, and $B \in \mathcal{B}$ is a transversal of T, then the family of sets of the form $T^n(B)$, for $n \in \mathbb{Z}$, is a countable separating family for T. □

Smoothness should be viewed as a notion of triviality, as the questions which interest us here will always be simple to answer for smooth automorphisms. The following observation therefore suggests that we will really only be interested in aperiodic separable automorphisms:

Proposition 1.6. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is periodic. Then the following are equivalent:*

1. *T is separable;*
2. *T is smooth.*

Proof. In light of Proposition 1.5, it is enough to show (1) ⇒ (2). By Proposition 1.3, there are maximal (T, k)-discrete sets $B_k \in \mathcal{B}$, for each $k \in \mathbb{Z}^+$. It then follows from Proposition 1.4 that the set

$$B = \mathrm{Per}_1(T) \cup \bigcup_{k \in \mathbb{Z}^+} B_k \cap \mathrm{Per}_{k+1}(T)$$

is in $\mathcal{B}$, and since it is a transversal of T, it follows that T is smooth. □

A *measure on* $(X, \mathcal{B})$ is a function $\mu : \mathcal{B} \to [0, \infty]$ which is *countably additive*, in the sense that if $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ is pairwise disjoint, then

$$\mu \left(\bigcup_{n \in \mathbb{N}} B_n \right) = \sum_{n \in \mathbb{N}} \mu(B_n).$$

We say that μ is a *probability measure* if $\mu(X) = 1$, and we say that μ is T-*invariant* if $\forall B \in \mathcal{B}\ (\mu(B) = \mu(T(B)))$. The following simple observation gives a wealth of examples of non-smooth separable automorphisms:

Proposition 1.7. *Suppose that* $(X, \mathcal{B})$ *is a Borel space,* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is aperiodic, and* T *admits an invariant probability measure. Then* T *is not smooth.*

Proof. Simply observe that if μ is a T-invariant probability measure on $(X, \mathcal{B})$ and $B \in \mathcal{B}$ is a transversal of T, then the family of sets of the form $T^n(B)$, for $n \in \mathbb{Z}$, is a partition of X into infinitely many sets of the same measure, which contradicts our assumption that $\mu(X) = 1$. □

As we have already begun to see, much of the usefulness behind our assumption of separability will come from the existence of maximal discrete sets, which serves a role similar to that of Rokhlin's Lemma in ergodic theory. In fact, as we shall see presently, maximal discrete sets can be used to prove a generalized version of Rokhlin's Lemma.

A *finitely additive measure on* $(X, \mathcal{B})$ is a function $\mu : \mathcal{B} \to [0, \infty]$ which is *finitely additive*, in the sense that if $\langle B_i \rangle \in \mathcal{B}^n$ is pairwise disjoint, then

$$\mu \left(\bigcup_{i < n} B_n \right) = \sum_{i < n} \mu(B_n).$$

We say that μ is a *finitely additive probability measure* if $\mu(X) = 1$.

Proposition 1.8. *Suppose that* $(X, \mathcal{B})$ *is a Borel space,* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is aperiodic, and* μ *is a finitely additive probability measure on* $(X, \mathcal{B})$*. Then the following are equivalent:*

1. *T is separable;*
2. *For all $k \in \mathbb{Z}^+$ and $\epsilon > 0$, there is a T-complete set $D \in \mathcal{B}$ such that the sets $D, T(D), \ldots, T^k(D)$ partition a set of measure at least $1 - \epsilon$.*

Proof. To see (1) $\Rightarrow$ (2), suppose that $k \in \mathbb{Z}^+$ and $\epsilon > 0$, and fix a positive integer $n > 1/\epsilon$. By Proposition 1.3, there is a maximal (T, kn)-discrete set $B \in \mathcal{B}$. Put $C = T^{-k}(B) \cup \cdots \cup T^{-1}(B)$, and observe that the sets of the form $T^{ik}(C)$, for $i < n$, are pairwise disjoint. In particular, there exists $i < n$ such that $\mu(T^{ik}(C)) < \epsilon$. For each $j \in \mathbb{Z}^+$, set

$$\begin{aligned} B_j &= \{x \in T^{ik}(B) : j \text{ is least positive integer with } T^j(x) \in T^{ik}(B)\} \\ &= T^{ik}(B) \cap T^{ik-j}(B) \setminus \bigcap_{0<l<j} T^{ik-l}(B), \end{aligned}$$

and define $D \subseteq X$ by

$$\begin{aligned} D &= \bigcup_{j \in \mathbb{Z}^+} \{T^{l(k+1)}(x) : x \in B_j \text{ and } l(k+1) < j - k\} \\ &= \bigcup_{l(k+1) < j-k} T^{l(k+1)}(B_j). \end{aligned}$$

It is clear that D is (T, k)-discrete. Suppose that $x \in X \setminus (D \cup T(D) \cup \cdots \cup T^k(D))$. Fix $j \in \mathbb{Z}^+$ and $l < j$ such that $x \in T^l(B_j)$. As $x \notin D \cup T(D) \cup \cdots \cup T^k(D)$, the definition of D ensures that $l \geq j - k$, and it follows that

$$x \in T^{-k}(T^{ik}(B)) \cup \cdots \cup T^{-1}(T^{ik}(B)) = T^{ik}(C).$$

As $x \in X \setminus (D \cup T(D) \cup \cdots \cup T^k(D))$ was arbitrary, this implies that

$$\mu(X \setminus (D \cup T(D) \cup \cdots \cup T^k(D))) \leq \mu(T^{ik}(C)) < \epsilon,$$

thus the sets $D, T(D), \ldots, T^k(D)$ partition a set of measure at least $1 - \epsilon$.

To see (2) $\Rightarrow$ (1), simply observe that if $B_k \in \mathcal{B}$ is a complete (T, k)-discrete set, for each $k \in \mathbb{N}$, then the family of sets of the form $T^i(B_k)$, for $i \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, is a separating family for T. □

The *very full group* of $T \in \text{Aut}(X, \mathcal{B})$ is given by

$$[T]^\star = \{U \in \text{Aut}(X, \mathcal{B}) : \forall x \in X \, \exists n \in \mathbb{Z} \, (U(x) = T^n(x))\}.$$

Note that $U \in [T]^\star$ if and only if there is a partition $\langle B_n \rangle \in \mathcal{P}(X)^{\mathbb{Z}}$ such that

$$U = \bigcup_{n \in \mathbb{Z}} T^n | B_n.$$

Proposition 1.9. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \text{Aut}(X, \mathcal{B})$ is separable. Then every element of $[T]^\star$ is separable.*

Proof. If $U \in [T]$, then every orbit of U is contained in an orbit of T, thus any countable separating family for T is a countable separating family for U. □

The *full group* (or *closure under countable decomposition*) of T is the group $[T]$ of all automorphisms which are of the form $\bigcup_{n \in \mathbb{Z}} T^n | B_n$, for some partition $\langle B_n \rangle \in \mathcal{B}^{\mathbb{Z}}$ of X. It is clear that $[T]$ is a subgroup of $[T]^\star$.

Proposition 1.10. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \text{Aut}(X, \mathcal{B})$ is separable. Then $[T] = [T]^\star$.*

Proof. Suppose that $U \in [T]^\star$. Then for each $n \in \mathbb{Z}$, the automorphism $T^{-n} \circ U$ is in $[T]^\star$, so Proposition 1.9 implies that $T^{-n} \circ U$ is separable, thus Proposition 1.4 implies that the set $A_n = \{x \in X : T^n(x) = U(x)\} = \text{Per}_1(T^{-n} \circ U)$ is in $\mathcal{B}$. Fix an enumeration $\langle k_n \rangle_{n \in \mathbb{N}}$ of $\mathbb{Z}$, and recursively define

$$B_{k_n} = A_{k_n} \setminus \bigcup_{m < n} A_{k_m}.$$

Then $\langle B_n \rangle \in \mathcal{B}^{\mathbb{Z}}$ is a partition of X and $U = \bigcup_{n \in \mathbb{Z}} T^n | B_n$, thus $U \in [T]$. □

2 Baire category

Here we review some basic facts concerning the Baire category theorem. Suppose that (X, τ) is a topological space. We say that a set $M \subseteq X$ is meager if there is a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of dense open sets such that $M \cap \bigcap_{n \in \mathbb{N}} U_n = \emptyset$. We say that a set $C \subseteq X$ is *comeager* if its complement is meager. We say that (X, τ) is a *Baire space* if every comeager subset of X is dense.

Theorem 2.1 (Baire)**.** *Every complete metric space is a Baire space.*

Proof. Suppose that (X, d) is a complete metric space and $C \subseteq X$ is comeager. Fix a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of dense open subsets of X such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. We must show that for all open sets $U \subseteq X$, the set $U \cap \bigcap_{n \in \mathbb{N}} U_n$ is non-empty. Towards this end, fix $x_0 \in U$, and given $x_n \in U \cap \bigcap_{i<n} U_i$, fix $\epsilon_n > 0$ sufficiently small that $\overline{B_d(x_n, \epsilon_n)} \subseteq U \cap \bigcap_{i<n} U_i$, and fix $x_{n+1} \in B_d(x_n, \epsilon_n) \cap U_n$. It is clear that $\langle x_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence, and it follows that it has a limit point which is necessarily in $U \cap \bigcap_{n \in \mathbb{N}} U_n$. □

We say that a set $B \subseteq X$ has the *Baire property* if there is an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager, and we use $\mathrm{BP}(X, \tau)$ to denote the family of subsets of X with the Baire property. Note that $\tau \subseteq \mathrm{BP}(X, \tau)$.

Proposition 2.2. *Suppose that* (X, τ) *is a topological space. Then* $\mathrm{BP}(X, \tau)$ *is a* σ*-algebra.*

Proof. Suppose first that $B \in \mathrm{BP}(X, \tau)$, and fix an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager. As $B \bigtriangleup U = (X \setminus B) \bigtriangleup (X \setminus U)$, it follows that the latter is meager. As $U \cup (X \setminus \overline{U})$ is a dense open set, it follows that $\overline{U} \setminus U$ is meager, thus so too is $(X \setminus B) \bigtriangleup (X \setminus \overline{U})$, hence $X \setminus B \in \mathrm{BP}(X, \tau)$.

Suppose now that $\langle B_n \rangle \in \mathrm{BP}(X, \tau)^{\mathbb{N}}$, and for each $n \in \mathbb{N}$, fix an open set $U_n \subseteq X$ such that $B_n \bigtriangleup U_n$ is meager. Set $B = \bigcup_{n \in \mathbb{N}} B_n$ and $U = \bigcup_{n \in \mathbb{N}} U_n$, and observe that

$$B \bigtriangleup U \subseteq \bigcup_{n \in \mathbb{N}} B_n \bigtriangleup U_n,$$

so $B \bigtriangleup U$ is meager, thus $B \in \mathrm{BP}(X, \tau)$. □

We say that B is *comeager in* U if $U \setminus B$ is meager.

Proposition 2.3. *Suppose that* (X, τ) *is a topological space and* $B \in \mathrm{BP}(X, \tau)$ *is non-meager. Then there is a non-empty open set* $U \subseteq X$ *such that* B *is comeager in* U*.*

Proof. Fix an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager. Then $U \setminus B \subseteq B \bigtriangleup U$, so B is comeager in U, and since B is non-meager, the set U is non-empty. □

Finally, we note that homeomorphisms interact well with meager sets:

Proposition 2.4. *Suppose that* (X, τ) *is a topological space and* $T : X \to X$ *is a homeomorphism. Then* T *sends meager sets to meager sets.*

Proof. Simply observe that homeomorphisms send open sets to open sets and dense sets to dense sets. □

Theorem 2.5 (Montgomery, Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then for each open set $U \subseteq X$, the set $\{x \in X : R_x$ is meager in $U\}$ is Borel.*

Proof. Fix a countable basis $\langle U_n \rangle$ for Y. We will show, by induction on the construction of R, that the sets of the form

$$R_{U_n} = \{x \in X : R_x \cap U_n \text{ is non-meager}\}$$

are Borel. If R is of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are Borel, then R_{U_n} is either $\emptyset$ or A, and therefore Borel. If R is of the form $\bigcup_{k \in \mathbb{N}} R_k$, and we have already established the theorem for each R_k, then $R_{U_n} = \bigcup_{k \in \mathbb{N}} (R_k)_{U_n}$ is Borel. Finally, if $R = (X \times Y) \setminus S$ and we have already establish the theorem for S, then $R_{U_n} = \bigcup_{U_m \subseteq U_n} X \setminus S_{U_m}$ is Borel. □

We write $\forall^* x \in U\ \varphi(x)$ to indicate that $\{x \in X : \varphi(x)\}$ is comeager in U.

Exercise 3 (Kuratowski-Ulam)**.** Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then for all non-empty open sets $U \subseteq X$ and $V \subseteq Y$,

$$R \text{ is comeager in } U \times V \Leftrightarrow \forall^* x \in U\, \forall^* y \in V\ ((x, y) \in R).$$

3 Marker sequences

Warning: The notion of balanced marker sequences given below is not correct.

We say that an equivalence relation is *finite* if all of its equivalence classes are finite. We say that a set $B \subseteq X$ is T-*birecurrent* if

$$\forall x \in B\, \exists m < 0 < n\ (T^m(x), T^n(x) \in B).$$

Associated with any T-complete birecurrent set $B \subseteq X$ is the finite equivalence relation $E \subseteq E_T^X$ on $X \setminus B$ given by

$$x E T^n(x) \Leftrightarrow \forall i \leq n\ (T^i(x) \notin B),$$

for all $n \in \mathbb{N}$ and $x \in X$.

Suppose now that T is aperiodic. A *marker sequence* for T is a decreasing, vanishing sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ of T-birecurrent sets. Note that if $\langle B_n \rangle$ is a marker sequence, then the corresponding increasing sequence $\langle E_n \rangle$ of finite equivalence relations has union E_T^X.

Proposition 3.1. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic. Then the following are equivalent:*

1. *T admits a marker sequence;*

2. T is separable.

Proof. To see (1) ⇒ (2), simply observe that if $\langle B_n \rangle$ is a marker sequence for T, then the family of sets of the form $T^i(B_n)$, for $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, is a separating family for T. To see (2) ⇒ (1), we need first some definitions. The *return time function* associated with a T-birecurrent set $B \in \mathcal{B}$ is the $\mathcal{B}$-measurable map $r_B : B \to \mathbb{Z}^+$ given by

$$r_B(x) = \min\{k \in \mathbb{Z}^+ : T^k(x) \in B\},$$

and the *induced transformation* associated with B is the automorphism of $(B, \mathcal{B}|B)$ given by $T_B(x) = T^{r_B(x)}(x)$. Suppose now that T is separable. By repeated application of Proposition 1.3, we can build a sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ such that $B_0 = X$ and B_{n+1} is maximal T_{B_n}-discrete. Let $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$ and $B = [B_\infty]_T$, noting that B_∞ is a transversal of $T|B$. Clearly $\langle B_n \setminus B \rangle$ is a marker sequence for $T|(X \setminus B)$, so it only remains to produce a marker sequence $\langle A_n \rangle$ for $T|B$. For each $n \in \mathbb{N}$, define $I_n \subseteq \mathbb{Z}$ by

$$I_n = \bigcup_{m \geq n} \{i \in \mathbb{Z} : i \equiv 2^m \pmod{2^{m+1}}\},$$

and observe that the sets $A_n = \bigcup_{i \in I_n} T^i(B_\infty)$ are as desired. □

Exercise 4. Suppose that (X, d) is a compact zero-dimensional metric space and $T : X \to X$ is an aperiodic homeomorphism of (X, d). Must T have a clopen marker sequence?

The *T-boundary* of a set $B \subseteq X$ is given by

$$\begin{aligned}\partial_T(B) &= \{x \in B : T^{-1}(x) \notin B \text{ or } T(x) \notin B\} \\ &= (B \setminus T(B)) \cup (B \setminus T^{-1}(B)).\end{aligned}$$

Suppose that $\langle B_n \rangle$ is a marker sequence with corresponding sequence $\langle E_n \rangle$ of finite equivalence relations. We say that $\langle B_n \rangle$ is balanced if

$$\forall n \in \mathbb{N}\, \forall x \in B_n \setminus B_{n+1}\ (\partial_T([x]_{E_{n+1}}) \subseteq B_n).$$

As we shall see, balanced marker sequences are useful because they can be used to reduce various algebraic questions about $[T]$ to straightforward combinatorial questions about finite sets. We will now show that every aperiodic separable automorphism admits a balanced marker sequence.

We begin by introducing another class of automorphisms of Borel spaces. We say that a set $B \subseteq X$ has *arbitrarily large gaps* with respect to T if for all $n \in \mathbb{N}$ and $x \in X$, there exists $y \in [x]_T$ such that $T(x), \ldots, T^n(x) \notin B$. We say that T is *gapped* if there is a T-complete set $B \in \mathcal{B}$ with arbitrarily large gaps. Refining Proposition 1.5, we have the following:

Proposition 3.2. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic. Then each of the following implies the next:*

1. *T is smooth;*

2. *T is gappable;*

3. *T is separable.*

Proof. To see (1) ⇒ (2), note that transversals are themselves T-complete sets with arbitrarily large gaps. To see (2) ⇒ (3), suppose that T is gappable, fix a T-complete set $B \in \mathcal{B}$ with arbitrarily large gaps, and observe that $\langle T^n(B)\rangle_{n\in\mathbb{Z}}$ is a countable separating family for T, thus T is separable. □

In the measure-theoretic context, every automorphism is gapped:

Proposition 3.3. *Suppose that $(X, \mathcal{B})$ is a Borel space, $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic separable, and μ is a probability measure on $(X, \mathcal{B})$. Then there is a T-invariant set $C \in \mathcal{B}$ such that $\mu(C) = 1$ and $T|C$ is gapped.*

Proof. By Proposition 1.3, there are maximal $(T, k\cdot 3^k)$-discrete sets $B_k \in \mathcal{B}$, for each $k \in \mathbb{Z}^+$. By replacing each B_k with its image under an appropriate iterate of T, we can assume that $\mu(\bigcup_{i<k} T^i(B_k)) \leq 1/3^k$. Set $B = \bigcup_{i<k<\omega} T^i(B_k)$, and define $A = [X \setminus B]_T$. Then A is a T-invariant set of positive μ-measure such that $T|A$ is gapped (as witnessed by $A \cap B$). By applying this argument countably many times, we obtain the desired T-invariant set $C \in \mathcal{B}$ such that $\mu(C) = 1$ and $T|C$ is gapped. □

We can now connect gapped automorphisms to the task at hand:

Proposition 3.4. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is gapped. Then T admits a balanced marker sequence.*

Proof. We need first the following observation:

Lemma 3.5. *Suppose that $B \in \mathcal{B}$ is T-complete. Then there is a T-invariant set $A \in \mathcal{B}$ such that $T|A$ is smooth and $B \setminus A$ is birecurrent.*

Proof. Define $C \subseteq B$ by

$$\begin{aligned} C &= \{x \in B : \forall n \in \mathbb{Z}^+ \ (T^n(x) \notin B)\} \\ &= B \setminus \bigcup_{n\in\mathbb{Z}^+} T^{-n}(B), \end{aligned}$$

and define $D \subseteq B$ by

$$\begin{aligned} D &= \{x \in B : \forall n \in \mathbb{Z}^+ \ (T^{-n}(x) \notin B)\} \\ &= B \setminus \bigcup_{n\in\mathbb{Z}^+} T^{n}(B). \end{aligned}$$

Then $C \cup (D \setminus [C]_T)$ is a transversal for the restriction of T to the set $A = [C \cup D]_T$, and $B \setminus A$ is clearly birecurrent. □

Suppose now that $B \in \mathcal{B}$ is T-complete and has arbitrarily large gaps. By Proposition 3.2 and Lemma 3.5, we can assume that for each $n \in \mathbb{N}$, the set

$$B_n = \{x \in X : T^{-n}(x), \ldots, T^n(x) \notin B\} = X \setminus \bigcup_{|i| \leq n} T^i(B)$$

is T-birecurrent. Then $\langle B_n \rangle$ is a balanced marker sequence. □

Propositions 3.3 and 3.4 already imply that every aperiodic separable automorphism admits a balanced marker sequence on a set of full measure. Unfortunately, there is more to do, as there are separable automorphisms which are not gapped. To see this, recall that a metric d on X is an *ultrametric* if

$$\forall x, y, z \in X \ (d(x, z) = \max(d(x, y), d(y, z))),$$

Note that a metric d on X is an ultrametric if and only if

$$\forall x, y \in X \, \forall 0 \leq r \leq s \ (B_d(x, r) \subseteq B_d(y, s) \text{ or } B_d(x, r) \cap B_d(y, s) = \emptyset).$$

Recall that a point x in a topological space is *isolated* if $\{x\}$ is open, and a topological space is *perfect* if it has no isolated points.

Proposition 3.6. *Suppose that (X, d) is a complete perfect ultrametric space, $\tau \subseteq \mathcal{B} \subseteq \mathrm{BP}(X, d)$ is a σ-algebra on X, and $T \in \mathrm{Aut}(X, \mathcal{B})$ is an isometry of (X, d) with dense orbits. Then T is not gapped.*

Proof. Suppose that $B \in \mathcal{B}$ is T-complete. Then $X = \bigcup_{n \in \mathbb{Z}} T^n(B)$, so Theorem 2.1 and Proposition 2.4 ensure that B is non-meager, thus Proposition 2.3 implies that there is a non-empty open set $U \subseteq X$ such that B is comeager in U. Fix $x \in U$. As (X, d) is perfect, there is a non-empty open set $V \subseteq U$ such that $x \notin V$. As the orbits of T are dense, there exists $k \in \mathbb{Z}$ such that $T^k(x) \in V$. As (X, d) is an ultrametric space and T is an isometry, it follows that $U = T^k(U)$. As homeomorphisms preserve meager sets, it follows that $\forall i \in \mathbb{Z}$ ($T^{ik}(B)$ is comeager in U), thus the set

$$C = \bigcap_{i \in \mathbb{Z}} T^{ik}(B)$$

is comeager in U. Fix $y \in C$, and observe that $\forall i \in \mathbb{Z}$ ($T^{ik}(y) \in B$), thus there is no gap in $B \cap [y]_T$ of length strictly greater than k. □

Example 3.7. We use $2^{\mathbb{N}}$ to denote the *Cantor space* of all infinite sequences $x = x_0 x_1 \ldots$ of 0's and 1's, equipped with the topology generated by the sets

$$\mathcal{N}_s = \{x \in 2^{\mathbb{N}} : s \subseteq x\},$$

where s is in the set $2^{<\mathbb{N}}$ of finite strings of 0's and 1's. It is clear that Cantor space is perfect, and it has a compatible complete ultrametric given by $d(x, y) =$

$1/2^n$, where n is least such that $x_n \neq y_n$. The *odometer* is the isometry of Cantor space given by

$$\sigma(x) = \begin{cases} 0^n 1y & \text{if } x = 1^n 0y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

It is easy to see that $\sigma \in \mathrm{Aut}(2^{\mathbb{N}}, \mathrm{BP})$ is separable, and a straightforward induction shows that the orbits of σ are dense, thus Proposition 3.6 implies that σ is not gapped.

We are now ready to return to the general case:

Proposition 3.8. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic separable. Then T admits a balanced marker sequence.*

Proof. We will recursively construct a decreasing sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$. We begin by fixing a T-complete set $B_0 \in \mathcal{B}$ such that both $T^{-1}(B_0) \cap B_0$ and $X \setminus B_0$ are T-birecurrent. This can be achieved, for example, by appealing to Proposition 1.3 to find a maximal $(T, 2)$-discrete set $A \in \mathcal{B}$, and then setting $B_0 = X \setminus A$. Suppose now that we have a T-complete set $B_n \subseteq B_0$ such that $T^{-1}(B_n) \cap B_n$ is T-birecurrent. Then the T-complete set

$$C_n = T^{-1}(B_n) \cap B_n \setminus T(B_n)$$

is also T-birecurrent. Fix a maximal $(T_{C_n}, n+2)$-discrete set $D_n \in \mathcal{B}$, and define

$$B_{n+1} = B_n \setminus \left\{ T^i(x) : x \in D_n \text{ and } 1 \leq i \leq \sum_{j \leq n} r_{T, C_n}(T^j_{C_n}(x)) \right\}.$$

It is clear that $B_{n+1} \subseteq B_n$ is a T-complete set in $\mathcal{B}$, $T^{-1}(B_n) \cap B_n$ is T-birecurrent, and B_{n+1} has gaps of size at least $n+1$ on every orbit of T.

Set $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$ and $B = [B_\infty]_T$. Then $\langle B_n \setminus B \rangle$ is a balanced marker sequence for $T|(X \setminus B)$. As the set B_∞ is $(T|B)$-complete and has arbitrarily large gaps, it follows that $T|B$ is gapped, and therefore has a balanced marker sequence by Proposition 3.4. □

Exercise 5. Suppose that (X, d) is a compact zero-dimensional metric space and $T : X \to X$ is an aperiodic homeomorphism of (X, d). Is there a T-invariant Borel set $B \subseteq X$ and a sequence $\langle U_n \rangle$ of clopen sets such that $T|(X \setminus B)$ is smooth and $\langle B \cap U_n \rangle$ is a balanced marker sequence for $T|B$?

4 Compositions of involutions

We say that $T \in S_X$ is an *involution* if $T^2 = \mathrm{id}$. In this section, we will explore the circumstances under which an automorphism can be written as the composition of involutions from its full group.

We say that I *anticommutes* with T if $I \circ T = T^{-1} \circ I$. Note that if I anticommutes with T, then $T \circ I = I \circ I \circ T \circ I = I \circ T^{-1} \circ I \circ I = I \circ T^{-1}$, thus I anticommutes with T^{-1}. The following observation will simplify the task of finding two involutions whose composition is T:

Proposition 4.1. *Suppose that* $(X, \mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X, \mathcal{B})$*. Then the following are equivalent:*

1. *There are involutions* $I_1, I_2 \in [T]$ *such that* $T = I_1 \circ I_2$*;*

2. *There is an involution* $I \in [T]$ *which anticommutes with* T*.*

Proof. To see (1) ⇒ (2), we actually prove something a bit stronger:

Lemma 4.2. *Suppose that* $I_1, I_2 \in [T]$ *and* $T = I_1 \circ I_2$*. Then* I_1, I_2 *both anticommute with* T*.*

Proof. Simply observe that $T^{-1} = I_2 \circ I_1$, so

$$I_1 \circ T = I_2 = T^{-1} \circ I_1 \text{ and } T \circ I_2 = I_1 = I_2 \circ T^{-1},$$

thus I_1 anticommutes with T and I_2 anticommutes with T^{-1}. □

To see (2) ⇒ (1), note that if $I \in [T]$ is an involution that anticommutes with T, then $(I \circ T) \circ (I \circ T) = I \circ T \circ T^{-1} \circ I = \mathrm{id}$, thus $I_1 = I$ and $I_2 = I \circ T$ are involutions in $[T]$ such that $T = I \circ I \circ T = I_1 \circ I_2$. □

We are now ready to make a first dent in the involutions problem:

Proposition 4.3. *Suppose that* $(X, \mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is smooth. Then there are involutions* $I_1, I_2 \in [T]$ *such that* $T = I_1 \circ I_2$*.*

Proof. By Proposition 4.1, it is enough to find an involution $I \in [T]$ which anticommutes with T. Towards this end, fix a transversal $B \in \mathcal{B}$ of T, and note that if $x \in T^m(B) \cap T^n(B)$, then $T^{-2m}(x) = T^{-2n}(x)$, thus we obtain an element of $[T]$ by setting

$$I = \bigcup_{n \in \mathbb{Z}} T^{-2n} | T^n(B).$$

To see that I is an involution, simply observe that if $x \in T^n(B)$, then $I(x) = T^{-2n}(x)$ is in $T^{-n}(B)$, thus $I(I(x)) = T^{2n}(I(x)) = T^{2n}(T^{-2n}(x)) = x$. To see that I anticommutes with T, note that if $x \in T^n(B)$, then $T(x) \in T^{n+1}(B)$, so $I \circ T(x) = T^{-2(n+1)} \circ T(x) = T^{-2n-1}(x) = T^{-1} \circ T^{-2n}(x) = T^{-1} \circ I(x)$. □

We can say even more for aperiodic automorphisms:

Proposition 4.4. *Suppose that* $(X, \mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is aperiodic. Then the following are equivalent:*

1. *There are involutions* $I_1, I_2 \in [T]$ *such that* $T = I_1 \circ I_2$*;*

2. T *is smooth.*

Proof. To see (1) ⇒ (2), suppose that $I_1, I_2 \in [T]$ are involutions and $T = I_1 \circ I_2$. By Lemma 4.2, both I_1 and I_2 anticommute with T.

Lemma 4.5. *Each T-orbit contains a unique point which is fixed by I_1 or I_2.*

Proof. The *displacement function* associated with an automorphism $U \in [T]$ is given by $d_U(x) = k$, where k is the unique integer such that $U(x) = T^k(x)$ (here we use the fact that T is aperiodic). For each $l \in \{1, 2\}$, observe that

$$\begin{aligned} I_l(T(x)) &= T^{-1}(I_l(x)) \\ &= T^{d_{I_l}(x)-1}(x) \\ &= T^{d_{I_l}(x)-2}(T(x)), \end{aligned}$$

thus $d_{I_l}(T(x)) = d_{I_l}(x) - 2$. In particular, it follows the restriction of d_{I_l} to any orbit of T is an injection whose image is either the set of all even integers or the set of all odd integers. Noting that

$$\begin{aligned} T(x) &= I_1 \circ I_2(x) \\ &= T^{d_{I_1}(I_2(x))} \circ T^{d_{I_2}(x)}(x) \\ &= T^{d_{I_1}(I_2(x))+d_{I_2}(x)}(x), \end{aligned}$$

it follows that $d_{I_1}(x)$ is even if and only if $d_{I_2}(x)$ is odd, which then implies that for each $x \in X$, there is a unique $y \in [x]_T$ such that either $d_{I_1}(y) = 0$ or $d_{I_2}(y) = 0$. □

Lemma 4.5 implies that $\mathrm{Per}_1(I_1) \cup \mathrm{Per}_1(I_2)$ is a transversal of T, so to see that T is smooth, it is enough to show the following:

Lemma 4.6. $\mathrm{Per}_1(I_1), \mathrm{Per}_1(I_2) \in \mathcal{B}$.

Proof. Fix a partition $\langle B_n \rangle \in \mathcal{B}^{\mathbb{Z}}$ such that $I_1 = \bigcup_{n \in \mathbb{Z}} T^n | B_n$, and observe that $\mathrm{Per}_1(I_1) = B_0 \in \mathcal{B}$. The proof that $\mathrm{Per}_1(I_2) \in \mathcal{B}$ is identical. □

As Proposition 4.3 gives (2) $\Rightarrow$ (1), this completes the proof. □

Along similar lines, we have the following:

Proposition 4.7. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is separable. Then the following are equivalent:*

1. *There are involutions $I_1, I_2 \in [T]$ such that $T = I_1 \circ I_2$;*
2. *T is smooth.*

Proof. To see (1) $\Rightarrow$ (2), suppose that there are involutions $I_1, I_2 \in [T]$ such that $T = I_1 \circ I_2$. By Proposition 1.4, the periodic part of T is in $\mathcal{B}$. Proposition 4.4 then implies that $T|\mathrm{Aper}(T)$ is smooth, and Proposition 1.6 implies that $T|\mathrm{Per}(T)$ is smooth, thus T is smooth. As (2) $\Rightarrow$ (1) follows from Proposition 4.3, this completes the proof of the proposition. □

Exercise 6. Give an example of a non-smooth automorphism T of a Borel space which is the composition of two involutions in $[T]$. Give an example of a non-smooth aperiodic automorphism T of a Borel space which is the composition of two involutions in $[T]^\star$.

While Proposition 4.3 allows us to write periodic permutations as a product of two involutions from their full groups, Proposition 4.7 shows that such involutions do not always exist, even for separable automorphisms. So we must content ourselves with writing permutations as the product of three involutions. The following observation suggests a route towards this goal:

Proposition 4.8. *Suppose that $(X, \mathcal{B})$ is a Borel space, $T \in \mathrm{Aut}(X, \mathcal{B})$ is separable, and $I \in [T]$ is an involution such that $I \circ T$ is periodic. Then there are involutions $I_1, I_2, I_3 \in [T]$ such that $T = I_1 \circ I_2 \circ I_3$.*

Proof. Set $I_1 = I$. Proposition 1.9 implies that $I \circ T$ is separable, so Proposition 1.6 ensures that $I \circ T$ is smooth. By Proposition 4.3, there are involutions $I_2, I_3 \in [I \circ T]$ such that $I_1 \circ T = I_2 \circ I_3$, thus $T = I_1 \circ I_2 \circ I_3$. As $[I \circ T] \subseteq [T]$, all of these involutions are in $[T]$, and the proposition follows. □

We will now focus on the problem of finding an involution $I \in [T]$ such that $I \circ T$ is periodic. As our focus will be on the separable case, Propositions 1.4, 1.6, and 4.3 allow us to assume that T is aperiodic. We can then visualize each orbit of T as a copy of $\mathbb{Z}$, and think of I as a collection of arcs that sit above the copy of $\mathbb{Z}$ associated with each orbit. We say that $x \in X$ is *covered* by an arc of I if there are integers $m \leq 0 < n$ such that $I(T^m(x)) = T^n(x)$, and we say that $I \in [T]$ is *covering* if every point of X is covered by an arc of I. We say that a set $B \subseteq X$ is *I-invariant* if $B = [B]_I$, and we say that $I \in [T]$ is *non-crossing* if its arcs do not cross, i.e., if

$$\forall x \in X \, \forall n \in \mathbb{Z}^+ \ (I(x) = T^n(x) \Rightarrow \{T^i(x)\}_{0<i<n} \text{ is } I\text{-invariant}).$$

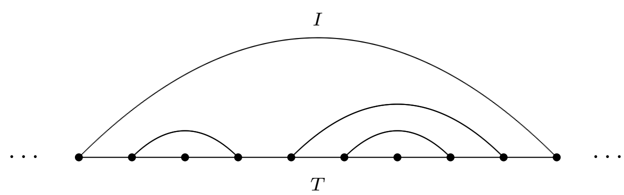

Figure 1: The arcs associated with a covering, non-crossing involution.

Proposition 4.9. *Suppose that $(X, \mathcal{B})$ is a Borel space, $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic, and $I \in [T]$ is a covering, non-crossing involution. Then $I \circ T$ is periodic.*

Proof. Given $x \in X$, fix integers $m \leq 0 < n$ such that $I(T^m(x)) = T^n(x)$, and observe that the set $\{T^i(x)\}_{m \leq i < n}$ is invariant under $I \circ T$, thus $[x]_{I \circ T}$ is finite. As $x \in X$ was arbitrary, it follows that $I \circ T$ is periodic. □

It remains to establish the existence of covering, non-crossing involutions:

Proposition 4.10. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic separable. Then T admits a covering, non-crossing involution.*

Proof. By Proposition 3.8, there is a balanced marker sequence $\langle B_n \rangle$ for T. Let $\langle E_n \rangle$ be the corresponding sequence of equivalence relations. For each $n \in \mathbb{N}$, let $L_n = T(B_n) \setminus B_n$ and $R_n = T^{-1}(B_n) \setminus B_n$ denote the set of points which are leftmost and rightmost within their E_n-classes, respectively. Let I_n be the involution of $L_n \cup R_n$ which sends each point of L_n to the unique E-related point of R_n, and observe that $I = \bigcup_{n \in \mathbb{N}} I_n$ is the desired involution. □

We can now prove the main result of this section:

Theorem 4.11. *Suppose that $(X, \mathcal{B})$ is a Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ is separable. Then there are involutions $I_1, I_2, I_3 \in [T]$ such that $T = I_1 \circ I_2 \circ I_3$.*

Proof. By Proposition 1.4, the periodic part of T is in $\mathcal{B}$. By Propositions 1.6 and 4.3, there are two involutions in $[T|\mathrm{Per}(T)]$ whose composition is $T|\mathrm{Per}(T)$. By Proposition 4.10, there is a covering, non-crossing involution for $T|\mathrm{Aper}(T)$, and Propositions 4.8 and 4.9 then imply that there are three involutions in $[T|\mathrm{Aper}(T)]$ whose composition is $T|\mathrm{Aper}(T)$. □

5 Compositions of periodic automorphisms, I

We still have yet to explore the question of whether separability is required for an automorphism to be written as the composition of three involutions from its full group. In this section, we will take care of this and much more. We begin with the following observation:

Proposition 5.1. *Suppose that $(X, \mathcal{B})$ is a Borel space, $T \in \mathrm{Aut}(X, \mathcal{B})$, and $U \in [T]$ is periodic. Then there is a T-invariant set $B \in \mathcal{B}$ such that $T|B$ is separable and $\forall x \in X \setminus B \, \exists n \in \mathbb{N} \, \forall y \in [x]_T \; ([y]_U \subseteq \{T^i(y)\}_{|i|<n})$.*

Proof. Fix $\langle B_n \rangle \in \mathcal{B}^{\mathbb{Z}}$ such that $U = \bigcup_{n \in \mathbb{Z}} T^n | B_n$. Then we obtain a function $\pi : X \to \mathbb{Z}^{\mathbb{Z}}$ by setting

$$[\pi(x)](i) = \text{ the unique } n \in \mathbb{Z} \text{ such that } T^i(x) \in B_n.$$

As $\pi^{-1}(\{\alpha \in \mathbb{Z}^{\mathbb{Z}} : \alpha(i) = n\}) = T^{-i}(B_n)$, it follows that π is $\mathcal{B}$-measurable.

Let $\mathrm{Per}(\mathbb{Z}^{\mathbb{Z}})$ denote the set of periodic sequences in $\mathbb{Z}^{\mathbb{Z}}$, and let $\mathrm{Aper}(\mathbb{Z}^{\mathbb{Z}})$ denote the complement of $\mathrm{Per}(\mathbb{Z}^{\mathbb{Z}})$. Then the T-invariant set $B = \pi^{-1}(\mathrm{Aper}(\mathbb{Z}^{\mathbb{Z}}))$ is in $\mathcal{B}$, and it is easy to see that $T|B$ is separable. Suppose now that $x \notin B$, and fix $k \in \mathbb{N}$ such that $\pi(x)$ is of period k. For each $i < k$, fix $n_i \in \mathbb{Z}$ such that $T^i(x) \in B_{n_i}$, and set $n = \sum_{i<k} |n_i|$. As the periodicity of U easily implies that

$$\forall y \in [x]_T \; ([y]_U \subseteq \{T^i(y)\}_{|i|<n}),$$

the proposition follows. □

We are now ready for the main result of this section:

Theorem 5.2. *Suppose that $(X, \mathcal{B})$ is a Borel space, T is an aperiodic automorphism of $(X, \mathcal{B})$, and there are periodic automorphisms $T_1, \ldots, T_k \in [T]$ such that $T = T_1 \circ \cdots \circ T_k$. Then T is separable.*

Proof. Suppose, towards a contradiction, that T is not separable. Proposition 5.1 ensures that by throwing out a T-invariant set in $\mathcal{B}$ on which T is separable and then restricting our attention to an appropriate T-invariant subset of what remains, we can assume the existence of $n \in \mathbb{N}$ such that

$$\forall x \in X \ ([x]_{T_1} \cup \cdots \cup [x]_{T_k} \subseteq \{T^i(x)\}_{|i|<n}).$$

The *average displacement* of an automorphism $U \in [T]$ at x is given by

$$\overline{d_U}(x) = \lim_{m \to \infty} \frac{1}{m} \sum_{i<m} d_U(T^i(x)).$$

Clearly $\overline{d_T}(x) = 1$. We will obtain a contradiction by showing that

$$\overline{d_{T_1 \circ \cdots \circ T_k}}(x) = 0.$$

Given $\epsilon > 0$, suppose that $m \geq 4k^2n^2/\epsilon$ and set $B = \{T^i(x)\}_{i<m}$. Then

$$\begin{aligned} \sum_{y \in B} d_{T_1 \circ \cdots \circ T_k}(y) &= \sum_{y \in B} \sum_{1 \leq i \leq k} d_{T_i}(T_{i+1} \circ \cdots \circ T_k(y)) \\ &= \sum_{1 \leq i \leq k} \sum_{y \in B} d_{T_i}(T_{i+1} \circ \cdots \circ T_k(y)). \end{aligned}$$

Set $C = \{T^i(x)\}_{kn \leq i < m-kn}$, and observe that if $1 \leq i \leq k$, then the set

$$C_i = [T_{i+1} \circ \cdots \circ T_k(C)]_{T_i}$$

is contained in B. As C_i is T_i-invariant, it follows that $\sum_{y \in C_i} d_{T_i}(y) = 0$, so

$$\begin{aligned} \left| \sum_{y \in B} d_{T_i}(T_{i+1} \circ \cdots \circ T_k(y)) \right| &\leq 2kn^2 + \left| \sum_{y \in C} d_{T_i}(T_{i+1} \circ \cdots \circ T_k(y)) \right| \\ &\leq 4kn^2 + \left| \sum_{y \in C_i} d_{T_i}(y) \right| = 4kn^2, \end{aligned}$$

thus

$$\left| \frac{1}{m} \sum_{1 \leq i \leq k} \sum_{y \in B} d_{T_i}(T_{i+1} \circ \cdots \circ T_k(y)) \right| \leq 4k^2n^2/m \leq \epsilon,$$

and since $\epsilon > 0$ was arbitrary, it follows that $\overline{d_{T_1 \circ \cdots \circ T_k}}(x) = 0$. □

6 Compositions of periodic automorphisms, II

Warning: This section is quite poorly written, and should be redone using covering, non-crossing involutions in place of balanced marker sequences.

We say that an automorphism is of *strict period* k if its orbits are all of cardinality 1 or k. In this section, we show that if $m \geq 2$ and $n \geq 3$, then every aperiodic separable automorphism of a Borel space is the composition of an automorphism $U \in [T]$ of strict period m and an automorphism $V \in [T]$ of strict period n. In the special case that $m = n$, we show that U and V^{-1} can be taken to be conjugate via an element of $[T]$, thus T is a commutator in $[T]$.

Suppose that $T \in \mathrm{Aut}(X, \mathcal{B})$ is aperiodic. Given x, y in the same orbit of T, we say that x is to the *left* of y, or y is to the *right* of x, if there exists $n \in \mathbb{N}$ such that $T^n(x) = y$. We use the notation $[x, y]$ to denote the set of z which are to the right of x and to the left of y. We use the notation (x, y), $(x, y]$, and $[x, y)$ similarly. We say that a periodic automorphism $U \in [T]$ is *oriented* if for each orbit $\mathcal{O}$ of U, the restriction of U to $\mathcal{O}$ is given by $(x_1^{\mathcal{O}}\ x_2^{\mathcal{O}}\ \cdots\ x_n^{\mathcal{O}})$, where $\langle x_1^{\mathcal{O}}, x_2^{\mathcal{O}}, \ldots, x_n^{\mathcal{O}} \rangle$ is the left-to-right enumeration of $\mathcal{O}$. We use $l^{\mathcal{O}}$ and $r^{\mathcal{O}}$ to denote the leftmost and rightmost points of $\mathcal{O}$, respectively. Associated with each orbit $\mathcal{O}$ of U is the *outer arc* connecting $r^{\mathcal{O}}$ to $l^{\mathcal{O}}$, as well as the *inner arcs* which connect $x_i^{\mathcal{O}}$ to $x_{i+1}^{\mathcal{O}}$, for $0 < i < n$. We also use the notation $l^{\mathcal{A}}, r^{\mathcal{A}}$ to denote the leftmost and rightmost points of an (outer or inner) arc $\mathcal{A}$ of U. We say that U is *non-crossing* if none of its arcs cross, i.e., if for every inner arc $\mathcal{A}$ of U, the interval $(l^{\mathcal{A}}, r^{\mathcal{A}})$ is U-invariant. We say that U *covers* x if there is an inner arc $\mathcal{A}$ of U such that $x \in [l^{\mathcal{A}}, r^{\mathcal{A}})$. We say that an inner arc $\mathcal{A}$ of U is n-*dromedary* if the interval $(l^{\mathcal{A}}, r^{\mathcal{A}})$ is either empty or the disjoint union of intervals $[l^{\mathcal{A}_1}, r^{\mathcal{A}_1}], \ldots, [l^{\mathcal{A}_{n-1}}, r^{\mathcal{A}_{n-1}}]$, where $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ are outer arcs of U. We say that an automorphism is (m, n)-*good* if it is periodic, oriented, non-crossing, of strict period m, and every inner arc of U is n-dromedary. We say that such an automorphism is (m, n)-*great* if every point of X is covered by an inner arc of U.

Proposition 6.1. *Suppose that $(X, \mathcal{B})$ is a Borel space, T is an aperiodic automorphism of $(X, \mathcal{B})$, $m \geq 2$, $n \geq 2$, and T admits an (m, n)-great automorphism. Then there exist $T_1, T_2 \in [T]$ of strict periods m, n such that $T = T_1 \circ T_2$.*

Proof. It is clearly enough to show that if $T_1 \in [T]$ is (m, n)-great, then $T_2 = T_1^{-1} \circ T$ is of strict period n. Towards this end, suppose that $x \in X$, and let $\mathcal{A}$ be the inner arc of T_1 of minimal width which covers x. If the interval $(l^{\mathcal{A}}, r^{\mathcal{A}})$ is empty, then $T_2(x) = x$, thus $|[x]_{T_2}| = 1$. Otherwise, let $\mathcal{O}$ denote the orbit of T_1 associated with $\mathcal{A}$, let $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ be the outer arcs of T_1 associated with $\mathcal{A}$, and note that $[x]_{T_2} = \{l^{\mathcal{O}}\} \cup \{r^{\mathcal{A}_1}, \ldots, r^{\mathcal{A}_{n-1}}\}$, thus $|[x]_{T_2}| = n$. □

We will now embark upon the proof of the existence of (m, n)-great automorphisms, for $m \geq 2$ and $n \geq 3$. We begin with some elementary observations about permutations of finite subintervals of $\mathbb{Z}$. We say that a permutation U of an interval $[i, j] \subseteq \mathbb{Z}$ is (m, n)-*great* if it is (m, n)-good (with respect to the successor S on $\mathbb{Z}$) and $U(j) = i$ (so that every point of $[i, j)$ is covered by U).

Proposition 6.2. *There is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all natural numbers $m \geq 2, n \geq 3$ and all integers $i < 0 < j$ with $i < -1$ or $j > 1$, there is an integer $2 \leq k \leq f(m, n)$ with $i < -(k + 1)$ or $j > k + 1$, and an (m, n)-great permutation of $[-k, k]$ which fixes every point of $[-k, k] \cap \{i, 0, j\}$.*

Proof. We leave this to the reader. □

We will continue to use f to denote the function of Proposition 6.2. We say that a set $A \subseteq \mathbb{Z}$ is k-*discrete* if it is (S, k)-discrete.

Proposition 6.3. *Suppose that i, $m \geq 2$, and $n \geq 3$ are natural numbers, and $A \subseteq \mathbb{Z}$ is $f(m,n)$-discrete. Then there is an integer $1 < k < f(m,n)$ and an (m,n)-great permutation of $[-(i+k), i+k]$ which fixes every point of $[-(i+k),(i+k)] \cap (S \cup \{0\})$.*

Proof. As $f(m,n) > 1$, Proposition 6.2 ensures that we can find $1 \leq k_0 \leq f(m,n)$ with $-(k_0+1) \notin S$ and an (m,n)-great permutation U_0 of $[-k_0, k_0]$ which fixes every point of $[-k,k] \cap (S \cup \{0\})$. If $k_0 \geq i$, then we are done. Otherwise, we can identify the interval $[-k_0, k_0]$ with the point 0, and appeal to Proposition 6.2 to find $1 \leq k_1 \leq f(m,n)$ with $-(k_0+k_1+1) \notin S$ and an (m,n)-great extension U_1 of U_0 to $[-(k_0+k_1), k_0+k_1]$ which fixes every point of $[-(k_0+k_1),(k_0+k_1)] \cap S$. If $k_0 + k_1 \geq i$, then we are done. Otherwise, we continue in this fashion, building natural numbers $k_2 < k_3 < \cdots$ and extensions $U_2, U_3, \ldots$. Fix n least such that $k_0 + k_1 + \cdots + k_n \geq i$, and observe that $k = k_0 + k_1 + \cdots + k_n$ and U_n are as desired. □

We are now ready for our main result:

Proposition 6.4. *Suppose that $(X, \mathcal{B})$ is a Borel space and T is an aperiodic, separable automorphism of $(X, \mathcal{B})$. Then, for every $m \geq 2$ and $n \geq 3$, there is an (m,n)-great automorphism for T.*

Proof. By Proposition 1.3, there is a maximal $(T, 3f(m,n))$-discrete set $B \in \mathcal{B}$. By Proposition 4.10, there is a balanced marker sequence $\langle A_i \rangle$ for T_B with $A_0 = X$. Set $B_i = \bigcup_{j \leq 3f(m,n)} T^j(A_i)$. Then $\langle B_i \rangle$ is a balanced marker sequence for T. Let $\langle E_i \rangle$ denote the corresponding sequence of equivalence relations.

We will recursively define a decreasing sequence $\langle C_i \rangle \in \mathcal{B}^{\mathbb{N}}$ such that

$$\forall i \in \mathbb{N} \left(C_i \subseteq B_i \subseteq \bigcup_{|j| \leq f(m,n)} T^j(C_i) \right).$$

We will simultaneously construct (m,n)-good automorphisms $T_i \in [T]$ such that $T_{i+1}|(X \setminus C_i) = T_i|(X \setminus C_i)$. We begin by setting $T_0 = \mathrm{id}$. Given T_i and C_i, suppose that $x \in C_i \setminus C_{i+1}$ and let l and r be the leftmost and rightmost points of $[x]_{E_{n+1}}$, respectively. By treating each E_i-class as if it were a single point and adding a single point to the right of $[x]_{E_{i+1}}$ if necessary, we can treat $[x]_{E_{i+1}}$ as if it were an interval of the form $[-i, i]$. Proposition 6.3 then ensures that there exists $2 \leq k \leq f(m,n)$ and an (m,n)-great extension to $[-(i+k), i+k]$. From this, it is easy to define a set $C_i \subseteq B_{i+1} \cap C_i$ and an (m,n)-great automorphism $T_{i+1} \in [T]$ which agrees with T_i on $X \setminus C_i$ and sends the rightmost point of $[x]_{F_{i+1}}$ to the leftmost point of $[x]_{F_{i+1}}$, for each $x \in C_i \setminus C_{i+1}$, where F_{i+1} denotes the equivalence relation associated with C_{i+1}.

This completes the recursive construction. It is clear that $\bigcup_{i \in \mathbb{N}} T_i|(X \setminus C_i)$ is the desired (m,n)-great automorphism for T. □

As a corollary, we now have the following:

Theorem 6.5. *Suppose that* $(X, \mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is separable aperiodic. Then, for all integers* $m \geq 2$ *and* $n \geq 3$*, there exist* $T_1, T_2 \in [T]$ *of strict periods* m, n *such that* $T = T_1 \circ T_2$*.*

Proof. This follows from Propositions 6.1 and 6.4. □

Remark 6.6. Of course, by applying Theorem 6.5 to T^{-1}, we obtain the corresponding fact for $m \geq 3$ and $n \geq 2$.

When $m = n$, it is not difficult to alter the above construction so as to simultaneously produce $U \in [T]$ such that $T_2 = U \circ T_1 \circ U^{-1}$. The idea is to simply replace Proposition 6.2 with the strengthening in which the number of fixed points of the extension on $[-k, k]$ is one greater than the number of fixed points of the composition of the inverse of the extension with the successor. This allows us to recursively build up the conjugating automorphism U as we build up T_1 in the proof of Proposition 6.4. As every smooth automorphism is conjugate to its inverse via an element of $[T]$, we therefore obtain the following:

Theorem 6.7. *Suppose that* $(X, \mathcal{B})$ *is a Borel space and* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is aperiodic separable. Then, for all integers* $k \geq 3$*, there are automorphisms* $U, V \in [T]$ *such that* U *is of strict period* k *and* $T = U \circ V \circ U^{-1} \circ V^{-1}$*. In particular, it follows that* T *is a commutator of* $[T]$*.*

7 Separable actions

We use the notation $\Gamma \curvearrowright X$ to denote an action of a group Γ on X. An *action* of a group Γ on $(X, \mathcal{B})$ is an action of Γ on X such that each map of the form $x \mapsto \gamma \cdot x$ is an automorphism of $(X, \mathcal{B})$. We use the notation $\Gamma \curvearrowright (X, \mathcal{B})$ to denote such an action. The *orbit* of a point x with respect to $\Gamma \curvearrowright X$ is

$$[x]_\Gamma = \{\gamma \cdot x : \gamma \in \Gamma\},$$

and the *orbit equivalence relation* associated with $\Gamma \curvearrowright X$ is given by

$$x E_\Gamma^X y \Leftrightarrow [x]_\Gamma = [y]_\Gamma.$$

Given a set $\mathcal{A} \subseteq \mathcal{P}(X)$, we use the notation $\Gamma(\mathcal{A})$ to denote the family of sets of the form $\gamma(A)$, where $\gamma \in \Gamma$ and $A \in \mathcal{A}$. A *separating family* for $\Gamma \curvearrowright X$ is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

$$\forall x \in X \ (\Gamma(\mathcal{A}) \text{ is a separating family for } [x]_\Gamma).$$

We say that $\Gamma \curvearrowright (X, \mathcal{B})$ is *separable* if $\mathcal{B}$ contains a countable separating family.

It turns out that many facts about separable automorphisms generalize to separable actions of countable groups. We will now discuss briefly some of these. As the proofs are nearly identical to those we have already given for separable automorphisms, we leave them to the reader.

Given a finite set $\Delta \subseteq \Gamma$, we say that a set $B \subseteq X$ is Δ-*discrete* if the sets of the form $\delta(B)$, for $\delta \in \Delta$, are pairwise disjoint.

Proposition 7.1. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$. Then the following are equivalent:*

1. *$\Gamma \curvearrowright (X, \mathcal{B})$ is separable;*
2. *$\forall \Delta \subseteq \Gamma$ finite ($\mathcal{B}$ contains a maximal Δ-discrete set).*

The *aperiodic part* of $\Gamma \curvearrowright (X, \mathcal{B})$ is given by

$$\mathrm{Aper}(\Gamma) = \{x \in X : |[x]_\Gamma| = \aleph_0\}.$$

We say that $\Gamma \curvearrowright (X, \mathcal{B})$ is *aperiodic* if $X = \mathrm{Aper}(\Gamma)$. The *period k part* of $\Gamma \curvearrowright (X, \mathcal{B})$ is given by

$$\mathrm{Per}_k(\Gamma) = \{x \in X : |[x]_\Gamma| = k\},$$

and the *periodic part* of $\Gamma \curvearrowright (X, \mathcal{B})$ is given by $\mathrm{Per}(\Gamma) = \bigcup_{k \in \mathbb{Z}^+} \mathrm{Per}_k(\Gamma)$. We say that $\Gamma \curvearrowright (X, \mathcal{B})$ is *periodic* if $X = \mathrm{Per}(\Gamma)$.

Proposition 7.2. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is separable. Then $\forall k \in \mathbb{Z}^+$ $(\mathrm{Per}_k(\Gamma) \in \mathcal{B})$, thus both of the sets $\mathrm{Per}(\Gamma), \mathrm{Aper}(\Gamma)$ are in $\mathcal{B}$.*

A *partial transversal* of $\Gamma \curvearrowright (X, \mathcal{B})$ is a set $B \subseteq X$ which intersects every orbit of Γ in at most one point. A set $B \subseteq X$ is Γ-*complete* if it intersects every orbit of Γ in at least one point. A *transversal* of $\Gamma \curvearrowright (X, \mathcal{B})$ is a complete partial transversal. We say that $\Gamma \curvearrowright (X, \mathcal{B})$ is *smooth* if $\mathcal{B}$ contains a transversal.

Proposition 7.3. *Every smooth action of a countable group on a Borel space is separable.*

The *very full group* of $\Gamma \curvearrowright (X, \mathcal{B})$ is given by

$$[\Gamma]^\star = \{U \in \mathrm{Aut}(X, \mathcal{B}) : \forall x \in X \ \ (x E_\Gamma^X U(x))\}.$$

Note that $U \in [\Gamma]^\star$ if and only if there is a partition $\langle B_\gamma \rangle \in \mathcal{P}(X)^\Gamma$ such that

$$U = \bigcup_{\gamma \in \Gamma} \gamma | B_\gamma.$$

Proposition 7.4. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is separable. Then every element of $[\Gamma]^\star$ is separable.*

The *σ-full group* (or *closure under countable decomposition*) of $\Gamma \curvearrowright (X, \mathcal{B})$ is the group $[\Gamma]$ of all automorphisms which are of the form $\bigcup_{\delta \in \Delta} \delta | B_\delta$, for some countable set $\Delta \subseteq \Gamma$ and partition $\langle B_\delta \rangle \in \mathcal{B}^\Delta$ of X. When Γ is countable, we call this the *full group* of Γ. It is clear that $[\Gamma]$ is a subgroup of $[\Gamma]^\star$.

Proposition 7.5. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is separable. Then $[\Gamma] = [\Gamma]^\star$.*

We will now prove several facts about full groups which we did not see earlier.

Proposition 7.6. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is separable. Then there are involutions $I_n \in [\Gamma]$ such that $E_\Gamma^X = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(I_n)$.*

Proof. For each $\gamma \in \Gamma$, fix a maximal γ-discrete set $B_\gamma \in \mathcal{B}$. For each $n \in \{0, 1, 2\}$, set $D_n^\gamma = \gamma^n(B_\gamma)$, $R_n^\gamma = \gamma^{n+1}(B_\gamma)$, $X_n^\gamma = X \setminus (D_n^\gamma \cup R_n^\gamma)$, and

$$I_n^\gamma = \gamma | D_n^\gamma \cup \gamma^{-1} | R_n^\gamma \cup \operatorname{id} | X_n^\gamma.$$

To see that these involutions are as desired, suppose that $x E_\Gamma^X y$, fix $\gamma \in \Gamma$ such that $\gamma \cdot x = y$, fix $n \in \{0, 1, 2\}$ such that $x \in D_n^\gamma$, and note that $I_n^\gamma(x) = y$. □

The *support* of a permutation $T \in S_X$ is given by $\operatorname{supp}(T) = X \setminus \operatorname{Per}_1(T)$.

Proposition 7.7. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, $\Gamma \curvearrowright (X, \mathcal{B})$ is separable, and $B \in \mathcal{B}$. Then there is an involution $I \in [\Gamma]$ such that $\operatorname{supp}(I) \subseteq B$ and $B \setminus \operatorname{supp}(I)$ is a partial transversal of $\Gamma \curvearrowright (X, \mathcal{B})$.*

Proof. By Proposition 7.6, there are involutions $I_n \in [\Gamma]$ such that $E_\Gamma^X = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(I_n)$. Recursively define $A_n, B_n \in \mathcal{B}$ by setting

$$A_n = X \setminus \bigcup_{i<n} B_i \text{ and } B_n = \operatorname{supp}(I_n) \cap (A_n \cap B) \cap I_n(A_n \cap B).$$

It is clear that the intersection of B with the set $A = X \setminus \bigcup_{n \in \mathbb{N}} B_n$ is a partial transversal of $\Gamma \curvearrowright (X, \mathcal{B})$, and it follows that the involution $I = \bigcup_{n \in \mathbb{N}} I_n | B_n \cup \operatorname{id} | A$ is as desired. □

We say that a set $B \in \mathcal{B}$ is Γ-*aperiodic* if $B \cap [x]_\Gamma$ is infinite, for all $x \in B$.

Proposition 7.8. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, $\Gamma \curvearrowright (X, \mathcal{B})$ is separable, and $B \in \mathcal{B}$ is Γ-aperiodic. Then there is an involution $I \in [\Gamma]$ such that $B = \operatorname{supp}(I)$.*

Proof. The previous proposition shows that the proposition holds off of a Γ-invariant set $B \in \mathcal{B}$ on which the action of Γ is smooth, so it is enough to prove the special case of the proposition in which $\Gamma \curvearrowright (X, \mathcal{B})$ is smooth. Towards this end, fix a partition $\langle B_n \rangle \in \mathcal{B}^\mathbb{N}$ of X into transversals, and let $I \in [\Gamma]$ be the involution which sends each point x of B_{2n} to the unique point of $[x]_\Gamma \cap B_{2n+1}$. □

Proposition 7.9. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is aperiodic separable. Then there is an aperiodic automorphism in $[\Gamma]$.*

Proof. The *full semigroup* of $\Gamma \curvearrowright (X, \mathcal{B})$ is the semigroup $[\![\Gamma]\!]$ of partial injections of the form $\bigcup_{\gamma \in \Gamma} \gamma | B_\gamma$, where $\langle B_\gamma \rangle \in \mathcal{B}^\Gamma$.

Lemma 7.10. *Suppose that $U \in [\![\Gamma]\!]$ and both $X \setminus \mathrm{dom}(U), X \setminus \mathrm{range}(U)$ are Γ-complete. Then there is an extension $V \in [\![\Gamma]\!]$ of U such that both $X \setminus \mathrm{dom}(V), X \setminus \mathrm{range}(V)$ are Γ-complete and $\forall x \in X \ ([x]_U \subset [x]_V)$.*

Proof. As $X \setminus \mathrm{range}(U)$ is clearly Γ-aperiodic, there is an involution $I \in [\Gamma]$ such that $\mathrm{supp}(I) = X \setminus \mathrm{range}(U)$. Let T be the bijection from $X \setminus \mathrm{range}(U)$ to $X \setminus \mathrm{dom}(U)$ in $[\![\Gamma]\!]$, fix a transversal $B \in \mathcal{B}$ of $I|\mathrm{supp}(I)$, and let V be the extension of U with domain $T(B) \cup \mathrm{dom}(U)$ which agrees with $I \circ T^{-1}$ on $T(B)$. □

Repeatedly apply Lemma 7.10 to obtain an increasing sequence $\langle U_n \rangle \in [\![\Gamma]\!]^{\mathbb{N}}$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ is in $[\![\Gamma]\!]$ and every orbit of U is infinite. Of course, the problem is that there can still be orbits of U which miss either $\mathrm{dom}(U)$ or $\mathrm{range}(U)$. It is easy to alter U on the Γ-saturations of the sets $X \setminus \mathrm{dom}(U), X \setminus \mathrm{range}(U)$ so as to ensure that $\mathrm{dom}(U) = \mathrm{range}(U) = X$, and this completes the proof of the proposition. □

Theorem 7.11. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is a countable group, and $\Gamma \curvearrowright (X, \mathcal{B})$ is aperiodic separable. Then every element of $[\Gamma]$ is a commutator.*

Proof. Suppose that $T \in [\Gamma]$. For each $n \in \mathbb{Z}^+$, let $A_n = \mathrm{Per}_n(T)$, and fix a transversal $B_n \in \mathcal{B}$ of $T|A_n$, as well as an involution $I_n \in [\Gamma]$ such that $\mathrm{supp}(I_n) \subseteq B_n$ and $B_n \setminus \mathrm{supp}(I_n)$ is a partial transversal of $\Gamma \curvearrowright (X, \mathcal{B})$. As the action of Γ on $[B_n \setminus \mathrm{supp}(I_n)]_T$ is smooth, the well known fact that every element of S_∞ is a commutator allows us to assume that $B_n = \mathrm{supp}(I_n)$.

Theorem 6.5 ensures that $T|\mathrm{Aper}(T)$ is a commutator in $[T|\mathrm{Aper}(T)] \subseteq [\Gamma|\mathrm{Aper}(T)]$, so it is enough to show that $T|A_n$ is a commutator in $[T|A_n] \subseteq [\Gamma|A_n]$, for each $n \in \mathbb{Z}^+$. Towards this end, fix a transversal $C_n \in \mathcal{B}$ of $I_n|\mathrm{supp}(I_n)$, set $D_n = \mathrm{supp}(I_n) \setminus C_n$, and define $U = T|[C_n]_T$ and $V = T|[D_n]_T$. Then $T|A_n = U \circ V$, and the map $W = \bigcup_{k<n} T^{-k} \circ I_n \circ T^{-k}|T^k(C_n)$ witnesses that U is conjugate to V^{-1}, thus $T|A_n$ is a commutator. □

8 The positive theory of full groups, I

In this section and the next, we will prove some general theorems about formulae in the language of groups. We should first be very clear about what we mean by a formula. The *atomic formulae* are those of the form $g_1^{\pm 1} \dots g_n^{\pm 1} = 1$, where $g_1, \dots, g_n$ are variables. The *formulae* are the elements of the smallest class Form which contains the atomic formulae and which satisfies the following closure properties:

1. $\forall \varphi \in \mathrm{Form} \ (\neg \varphi \in \mathrm{Form})$;
2. $\forall \kappa \, \forall \langle \varphi_\alpha \rangle \in \mathrm{Form}^\kappa \ (\bigvee_{\alpha < \kappa} \varphi_\alpha \in \mathrm{Form})$;
3. $\forall \kappa \, \forall \langle \varphi_\alpha \rangle \in \mathrm{Form}^\kappa \ (\bigwedge_{\alpha < \kappa} \varphi_\alpha \in \mathrm{Form})$;

4. $\forall \varphi \in \mathrm{Form}\, \forall \kappa\ (\exists \langle g_\alpha \rangle_{\alpha<\kappa}\ \varphi(\langle g_\alpha \rangle) \in \mathrm{Form})$;

5. $\forall \varphi \in \mathrm{Form}\, \forall \kappa\ (\forall \langle g_\alpha \rangle_{\alpha<\kappa}\ \varphi(\langle g_\alpha \rangle) \in \mathrm{Form})$.

A *sentence* in the language of groups is a formula in which no variable occurs freely. A formula φ is *positive* if $\neg$ does not occur in φ. A $(\forall^\kappa \exists)$*-formula* is one of the form $\forall \langle g_\alpha \rangle_{\alpha<\kappa}\, \exists \langle h_\beta \rangle_{\beta<\lambda}\ \psi(\langle g_\alpha \rangle, \langle h_\beta \rangle)$. The *theory* of a group G is the class $\mathrm{Th}(G)$ of sentences which hold in G. Similarly, the *positive theory* of G is the class $\mathrm{Th}_+(G)$ of positive sentences which hold in G, and the *positive* $(\forall^\kappa \exists)$*-theory* of G is the class $\mathrm{Th}_{(\forall^\kappa \exists)^+}(G)$ of positive $(\forall^\kappa \exists)$-sentences which hold in G.

Suppose that Γ is a group and I is a set. We endow $(2^I)^\Gamma$ with the product topology, as well as the Borel structure generated by its basic open sets. Breaking with our tradition thus far, whenever we mention $(2^I)^\Gamma$ we shall actually mean the corresponding Borel space. The (right) *shift action* of a group Γ on $(2^I)^\Gamma$ is given by

$$[(\gamma \cdot x)(\delta)](i) = [x(\delta\gamma)](i).$$

A *homomorphism* from $\Gamma \curvearrowright X$ to $\Gamma \curvearrowright Y$ is a function $\pi : X \to Y$ such that $\forall x \in X\, \forall \gamma \in \Gamma\ (\gamma \cdot \pi(x) = \pi(\gamma \cdot x))$. Such a map is *locally injective* if $\forall x \in X\ (\pi | [x]_\Gamma$ is injective). A *homomorphism* from $\Gamma \curvearrowright (X, \mathcal{B})$ to $\Gamma \curvearrowright (Y, \mathcal{C})$ is a homomorphism π from $\Gamma \curvearrowright X$ to $\Gamma \curvearrowright Y$ such that $\forall C \in \mathcal{C}\ (\pi^{-1}(C) \in \mathcal{B})$.

Exercise 7. Show that $\Gamma \curvearrowright (X, \mathcal{B})$ is separable if and only if there is a locally injective homomorphism from $\Gamma \curvearrowright (X, \mathcal{B})$ to $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$.

We will now prove the main result of this section:

Theorem 8.1. *Suppose that Γ is a group, κ is an infinite cardinal, $(X, \mathcal{B})$ is a Borel space, and $\Gamma \curvearrowright (X, \mathcal{B})$ is separable. Then*

$$\mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright (2^\kappa)^\Gamma]) \subseteq \mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright (X, \mathcal{B})]).$$

Proof. Suppose that $\varphi \in \mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright (2^\kappa)^\Gamma])$, and fix a quantifier-free positive formula $\psi(\langle g_\alpha \rangle_{\alpha<\kappa}, \langle h_\beta \rangle_{\beta<\lambda})$ in the language of groups such that φ is $\forall \langle g_\alpha \rangle\, \exists \langle h_\beta \rangle\ \psi(\langle g_\alpha \rangle, \langle h_\beta \rangle)$. Fix a separating family $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ for $\Gamma \curvearrowright (X, \mathcal{B})$.

Given $\langle \gamma_\alpha \rangle \in [\Gamma \curvearrowright (X, \mathcal{B})]^\kappa$, we must find $\langle \delta_\beta \rangle \in [\Gamma \curvearrowright (X, \mathcal{B})]^\lambda$ such that $\psi(\langle \gamma_\alpha \rangle, \langle \delta_\beta \rangle)$. Towards this end, fix countable sets $\Gamma_\alpha \subseteq \Gamma$ and partitions $\langle B_{(\alpha,\gamma)} \rangle \in \mathcal{B}^{\Gamma_\alpha}$ of X with

$$\gamma_\alpha = \bigcup_{\gamma \in \Gamma_\alpha} \gamma | B_{(\alpha,\gamma)}.$$

Let $K = \mathbb{N} \cup \{(\alpha, \gamma) : \alpha < \kappa \text{ and } \gamma \in \Gamma_\alpha\}$, fix a bijection $\varphi : \kappa \to K$, and define $\pi : X \to (2^\kappa)^\Gamma$ by

$$[\pi(x)(\gamma)](\alpha) = \chi_{B_{\varphi(\alpha)}}(\gamma \cdot x).$$

It is clear that π is a locally injective homomorphism from $(X, \mathcal{B})$ to $(2^\kappa)^\Gamma$.

Associated with each $\alpha < \kappa$ is a natural attempt at building an element γ'_α of $[\Gamma \curvearrowright (2^\kappa)^\Gamma]$. Namely, for each $y \in (2^\kappa)^\Gamma$, we would like γ'_α to send y to

$\gamma \cdot y$, where γ is the unique element of Γ_α such that $[y(1_\Gamma)](\varphi^{-1}(\alpha, \gamma)) = 1$. Of course, there might not be such a γ, and even if there is, the corresponding map might not define a bijection. We will deal with this by simply ignoring the piece of the space where this is problematic.

For each $(\alpha, \gamma) \in K$, define $D_{(\alpha,\gamma)}, R_{(\alpha,\gamma)} \subseteq (2^\kappa)^\Gamma$ by

$$D_{(\alpha,\gamma)} = \{y \in (2^\kappa)^\Gamma : \forall \delta \in \Gamma_\alpha \ ([y(1_\Gamma)](\alpha, \delta) = 1 \Leftrightarrow \gamma = \delta)\}$$

and

$$R_{(\alpha,\gamma)} = \{y \in (2^\kappa)^\Gamma : \forall \delta \in \Gamma_\alpha \ ([y(\delta^{-1})](\alpha, \delta) = 1 \Leftrightarrow \gamma = \delta)\}.$$

For each $\alpha < \kappa$, define $D_\alpha, R_\alpha \subseteq (2^\kappa)^\Gamma$ by

$$D_\alpha = \bigcap_{n \in \mathbb{N}} \bigcup_{\langle \gamma_i \rangle \in (\Gamma_\alpha)^n} \bigcap_{i<n} (\gamma_0 \cdots \gamma_{i-1})^{-1}(D_{(\alpha,\gamma_i)})$$

and

$$R_\alpha = \bigcap_{n \in \mathbb{N}} \bigcup_{\langle \gamma_i \rangle \in (\Gamma_\alpha)^n} \bigcap_{i<n} \gamma_0 \cdots \gamma_{i-1}(R_{(\alpha,\gamma_i)}),$$

and set $Y_\alpha = D_\alpha \cap R_\alpha$ and $Z_\alpha = (2^\kappa)^\Gamma \setminus Y_\alpha$. For each $\alpha < \kappa$, set

$$\gamma'_\alpha = \bigcup_{\gamma \in \Gamma_\alpha} \gamma | (D_{(\alpha,\gamma)} \cap Y_\alpha) \cup \mathrm{id} | Z_\alpha.$$

It is easy to see that $\gamma'_\alpha \in [\Gamma \curvearrowright (2^\kappa)^\Gamma]$ and π is a homomorphism from γ_α to γ'_α.

Now fix a sequence $\langle \delta'_\beta \rangle \in [\Gamma \curvearrowright (2^\kappa)^\Gamma]^\lambda$ such that $\psi(\langle \gamma'_\alpha \rangle, \langle \delta'_\beta \rangle)$. Then there are countable sets $\Delta_\beta \subseteq \Gamma$ and measurable partitions $\langle C'_{(\beta,\delta)} \rangle_{\delta \in \Delta_\beta}$ of $(2^\kappa)^\Gamma$ with

$$\delta'_\beta = \bigcup_{\delta \in \Delta_\beta} \delta | C'_{(\beta,\delta)}.$$

Let $C_{(\beta,\gamma)} = \pi^{-1}(C'_{(\beta,\gamma)})$, and define $\delta_\beta = \bigcup_{\delta \in \Delta_\beta} \delta | C_{(\beta,\delta)}$. The fact that π is a homomorphism easily implies that each δ_β is in $[\Gamma]$, and since π is a homomorphism of δ_β with δ'_β, it follows that $\psi(\langle \gamma_\alpha \rangle, \langle \delta_\beta \rangle)$. $\square$

It is not hard to see that if $\Gamma \curvearrowright (X, \mathcal{B})$ is aperiodic separable, then the image of the homomorphism π constructed in the proof of Theorem 8.1 is contained in the aperiodic part of $\Gamma \curvearrowright (2^\kappa)^\Gamma$. As a consequence, we obtain the following:

Theorem 8.2. *Suppose that Γ is a group, κ is an infinite cardinal, $(X, \mathcal{B})$ is a Borel space, and $\Gamma \curvearrowright (X, \mathcal{B})$ is aperiodic separable. Then*

$$\mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright \mathrm{Aper}((2^\kappa)^\Gamma)]) \subseteq \mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright (X, \mathcal{B})]).$$

9 The positive theory of full groups, II

The sequence space associated with a σ-complete Boolean algebra $\mathfrak{B}$ is

$$\mathrm{Seq}(\mathfrak{B}) = \{\langle b_n \rangle \in (\mathfrak{B}^+)^{\mathbb{N}} : \forall n \in \mathbb{N}\ (b_{n+1} \subseteq b_n)\},$$

equipped with the topology generated by the sets of the form

$$\mathcal{N}_b = \{\langle b_n \rangle \in \mathrm{Seq}(\mathfrak{B}) : \exists n \in \mathbb{N}\ (b_n \subseteq b)\},$$

for $b \in \mathfrak{B}$.

Proposition 9.1. $\mathrm{Seq}(\mathfrak{B})$ *is a Baire space.*

Proof. We must show that if $b \in \mathfrak{B}^+$ and $\langle U_n \rangle_{n \in \mathbb{N}}$ is a sequence of dense open subsets of $\mathrm{Seq}(\mathfrak{B})$, then $\mathcal{N}_b \cap \bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$. Towards this end, set $b_0 = b$ and recursively find a decreasing sequence $\langle b_n \rangle \in \mathrm{Seq}(\mathfrak{B})$ such that

$$\forall n \in \mathbb{N}\ (\mathcal{N}_{b_{n+1}} \subseteq U_n).$$

It is clear that any such sequence is in $\mathcal{N}_b \cap \bigcap_{n \in \mathbb{N}} U_n$. $\square$

A $\mathfrak{B}$-*name* is a way of specifying an element of $\mathfrak{B}$ via complements, countable unions, and countable intersections. More precisely, let $\mathrm{Name}(\mathfrak{B})$ denote the smallest class of formulae such that:

1. $\forall b \in \mathfrak{B}\ (b \in \mathrm{Name}(\mathfrak{B}))$;
2. $\forall \sigma \in \mathrm{Name}(\mathfrak{B})\ (\neg\sigma \in \mathrm{Name}(\mathfrak{B}))$;
3. $\forall I$ countable $\forall \langle \sigma_i \rangle \in \mathrm{Name}(\mathfrak{B})^I\ (\bigcup_{i \in I} \sigma_i \in \mathrm{Name}(\mathfrak{B}))$;
4. $\forall I$ countable $\forall \langle \sigma_i \rangle \in \mathrm{Name}(\mathfrak{B})^I\ (\bigcap_{i \in I} \sigma_i \in \mathrm{Name}(\mathfrak{B}))$.

We use $\sigma^{\mathfrak{B}}$ to denote the natural *interpretation* of σ in $\mathfrak{B}$, whose straightforward recursive definition we leave to the reader. We similarly use $\sigma^{\mathrm{Seq}(\mathfrak{B})}$ to denote the *interpretation* of σ in $\mathrm{Seq}(\mathfrak{B})$, obtained by replacing each occurrence of an element b of $\mathfrak{B}$ in σ with $\mathcal{N}_b$. We write $B =^* C$ to indicate that the set $B \triangle C$ is meager.

Proposition 9.2. *Suppose that* $\mathfrak{B}$ *is a* σ*-complete Boolean algebra and* $\sigma \in \mathrm{Name}(\mathfrak{B})$. *Then* $\sigma^{\mathrm{Seq}(\mathfrak{B})} =^* \mathcal{N}_{\sigma^{\mathfrak{B}}}$.

Proof. By induction on the construction of σ. The case that σ is of the form $\ulcorner b \urcorner$, for some $b \in \mathfrak{B}$, is a trivial consequence of the definition of interpretation. Granting that we have proven the proposition for σ, observe that

$$\begin{aligned} (\neg\sigma)^{\mathrm{Seq}(\mathfrak{B})} &= \mathrm{Seq}(\mathfrak{B}) \setminus \sigma^{\mathrm{Seq}(\mathfrak{B})} \\ &=^* \mathrm{Seq}(\mathfrak{B}) \setminus \mathcal{N}_{\sigma^{\mathfrak{B}}} \\ &=^* \mathcal{N}_{\neg(\sigma^{\mathfrak{B}})} \\ &= \mathcal{N}_{(\neg\sigma)^{\mathfrak{B}}}. \end{aligned}$$

Similarly, if I is countable and we have established the proposition for σ_i, then

$$\begin{aligned}\left(\bigcup_{i\in I}\sigma_i\right)^{\mathrm{Seq}(\mathfrak{B})} &= \bigcup_{i\in I}\sigma_i^{\mathrm{Seq}(\mathfrak{B})}\\ &=^* \bigcup_{i\in I}\mathcal{N}_{\sigma_i^{\mathfrak{B}}}\\ &=^* \mathcal{N}_{\bigcup_{i\in I}\sigma_i^{\mathfrak{B}}}\\ &= \mathcal{N}_{\left(\bigcup_{i\in I}\sigma_i\right)^{\mathfrak{B}}},\end{aligned}$$

and an identical argument takes care of countable intersections. □

In particular, we obtain the following corollary:

Proposition 9.3. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, $C \subseteq \mathrm{Seq}(\mathfrak{B})$ is comeager, σ and τ are $\mathfrak{B}$-names, and $\langle\sigma_n\rangle \in \mathrm{Name}(\mathfrak{B})^{\mathbb{N}}$.*

1. *If $\sigma^{\mathrm{Seq}(\mathfrak{B})} \cap C = \emptyset$, then $\sigma^{\mathfrak{B}} = \mathbb{O}$;*
2. *If $(\sigma^{\mathrm{Seq}(\mathfrak{B})} \cap \tau^{\mathrm{Seq}(\mathfrak{B})}) \cap C = \emptyset$, then $\sigma^{\mathfrak{B}} \cap \tau^{\mathfrak{B}} = \mathbb{O}$;*
3. *If $C \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n^{\mathrm{Seq}(\mathfrak{B})}$, then $\bigcup_{n\in\mathbb{N}} \sigma_n^{\mathfrak{B}} = \mathbb{1}$.*
4. *If $\langle\sigma_n^{\mathrm{Seq}(\mathfrak{B})}\rangle$ partitions C, then $\langle\sigma_n^{\mathfrak{B}}\rangle$ partitions unity.*

Proof. To see (1), simple observe that if $\sigma^{\mathfrak{B}} \neq \mathbb{O}$, then $\mathcal{N}_{\sigma^{\mathfrak{B}}}$ is a non-empty open set, thus Proposition 9.2 implies that $\sigma^{\mathrm{Seq}(\mathfrak{B})} \cap C$ is non-meager, thus non-empty. It is clear that (2) and (3) follow from (1), and (4) trivially follows from (2) and (3). □

Given a comeager set $C \subseteq \mathrm{Seq}(\mathfrak{B})$, let $\mathcal{C}$ denote the σ-algebra generated by the sets of the form $\mathcal{N}_b \cap C$, for $b \in \mathfrak{B}$. Note that $\mathcal{C}$ can also be described as the family of sets of the form $\sigma^{\mathrm{Seq}(\mathfrak{B})} \cap C$, for $\sigma \in \mathrm{Name}(\mathfrak{B})$. Given $T \in \mathrm{Aut}(\mathfrak{B})$, we use $\hat{T}$ to denote the corresponding automorphism of $(C, \mathcal{C})$.

Proposition 9.4. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, $C \subseteq \mathrm{Seq}(\mathfrak{B})$ is comeager, σ is a $\mathfrak{B}$-name, $T \in \mathrm{Aut}(\mathfrak{B})$, and $\mathrm{supp}(\hat{T}) \cap \sigma^{\mathrm{Seq}(\mathfrak{B})} \cap C = \emptyset$. Then $\forall b \leq \sigma^{\mathfrak{B}}\ (b = T(b))$.*

Proof. Suppose, towards a contradiction, that there exists $b \leq \sigma^{\mathfrak{B}}$ such that $b \neq T(b)$. Then there exists $a \leq b$ such that $a \cap T(a) = \mathbb{O}$. As $a^{\mathrm{Seq}(\mathfrak{B})} \cap \hat{T}(a^{\mathrm{Seq}(\mathfrak{B})}) = \emptyset$ and $a^{\mathrm{Seq}(\mathfrak{B})} \setminus \sigma^{\mathrm{Seq}(\mathfrak{B})}$ is meager, it follows that there exists $x \in \sigma^{\mathrm{Seq}(\mathfrak{B})} \cap C$ such that $\hat{T}(x) \neq x$, the desired contradiction. □

Suppose that Γ is a group. We use the notation $\Gamma \curvearrowright \mathfrak{B}$ to denote an action of Γ on $\mathfrak{B}$ such that each map of the form $b \mapsto \gamma \cdot b$ is an automorphism of $\mathfrak{B}$. Observe that every action $\Gamma \curvearrowright \mathfrak{B}$ induces an action $\Gamma \curvearrowright (C, \mathcal{C})$.

A *sequential name* is a sequence $\Sigma = \langle \sigma_\delta \rangle_{\delta \in \Delta}$ of $\mathfrak{B}$-names, where $\Delta \subseteq \Gamma$ is countable. We say that Σ *is a name for* $T \in [\Gamma \curvearrowright \mathfrak{B}]$ if both $\langle \sigma_\delta^{\mathfrak{B}} \rangle_{\delta \in \Delta}$ and $\langle \delta(\sigma_\delta)^{\mathfrak{B}} \rangle_{\delta \in \Delta}$ are partitions of unity, and

$$\forall \delta \in \Delta \, \forall b \leq \sigma_\delta^{\mathfrak{B}} \ (T(b) = \gamma \cdot b).$$

We say that Σ *is a name for* $T \in [\Gamma \curvearrowright (C, \mathcal{C})]$ if both $\langle \sigma_\delta^{\mathrm{Seq}(\mathfrak{B})} \cap C \rangle_{\delta \in \Delta}$ and $\langle \delta(\sigma_\delta^{\mathrm{Seq}(\mathfrak{B})}) \cap C \rangle_{\delta \in \Delta}$ are partitions of unity, and

$$\forall \delta \in \Delta \, \forall x \in \sigma_\delta^{\mathrm{Seq}(\mathfrak{B})} \cap C \ (T(x) = \delta \cdot x).$$

We say that Σ is *trivial* if it is a name for the trivial automorphism of $\mathfrak{B}$, and we say that Σ is C-*trivial* if it is a name for the trivial automorphism of $(C, \mathcal{C})$.

Proposition 9.5. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a group, $\Gamma \curvearrowright \mathfrak{B}$, and $C \subseteq \mathrm{Seq}(\mathfrak{B})$ is comeager. Then every C-trivial sequential $\mathfrak{B}$-name is a trivial sequential $\mathfrak{B}$-name.*

Proof. This follows easily from Propositions 9.3 and 9.4. □

The *σ-full group* (or *closure under countable decomposition*) of $\Gamma \curvearrowright \mathfrak{B}$ is the group $[\Gamma]$ of all automorphisms which are of the form $\bigcup_{\delta \in \Delta} \delta | B_\delta$, for some countable set $\Delta \subseteq \Gamma$ and partition $\langle B_\delta \rangle \in \mathfrak{B}^\Delta$ of unity. When Γ is countable, we call this the *full group* of Γ.

Suppose now that $T \in [\Gamma \curvearrowright (C, \mathcal{C})]$. Then there is a sequential name $\Sigma = \langle \sigma_\delta \rangle_{\delta \in \Delta}$ for T. Proposition 9.3 implies that Σ is also a name for some automorphism in $[\Gamma \curvearrowright \mathfrak{B}]$. Proposition 9.5 easily implies that this automorphism is independent of the choice of Σ. Define $\pi_C : [\Gamma \curvearrowright (C, \mathcal{C})] \to [\Gamma \curvearrowright \mathfrak{B}]$ by letting $\pi_C(T)$ be the automorphism named by $\langle \sigma_\delta \rangle_{\delta \in \Delta}$. Proposition 9.5 ensures that π_C is a homomorphism, and it is easy to see that if $T \in [\Gamma \curvearrowright \mathfrak{B}]$, then $\hat{T} \in [\Gamma \curvearrowright (C, \mathcal{C})]$ and $T = \pi_C(\hat{T})$, thus π is surjective. As a consequence, we have:

Proposition 9.6. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a group, $\Gamma \curvearrowright \mathfrak{B}$, and $C \in \mathcal{B}$ is comeager. Then $\mathrm{Th}_+([\Gamma \curvearrowright (C, \mathcal{C})]) \subseteq \mathrm{Th}_+([\Gamma \curvearrowright \mathfrak{B}])$.*

A *separating family* for $\Gamma \curvearrowright \mathfrak{B}$ is a set $\mathfrak{A} \subseteq \mathfrak{B}$ with the property that for all $b \in \mathfrak{B}^+$ and $\gamma \in \Gamma$ with $b \cap (\gamma \cdot b) = \mathbb{O}$, there exist $a \in \mathfrak{A}$ and a non-zero $c \subseteq b$ such that $c \subseteq a$ and $\gamma \cdot c \subseteq \mathbb{1} - a$. We say that $\Gamma \curvearrowright \mathfrak{B}$ is *separable* if $\mathfrak{B}$ contains a countable separating family.

Proposition 9.7. *Every action of a countable group on a complete Boolean algebra is separable.*

Proof. Simply observe that the completeness of $\mathfrak{B}$ ensures the existence of maximal discrete sets, and this implies separability as before. □

Proposition 9.8. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group, and $\Gamma \curvearrowright \mathfrak{B}$ is separable. Then there is a Γ-invariant dense G_δ set $C \subseteq \mathrm{Seq}(\mathfrak{B})$ such that $\Gamma \curvearrowright (C, \mathcal{C})$ is separable.*

Proof. Fix a countable separating family $\mathfrak{A} \subseteq \mathfrak{B}$ for $\Gamma \curvearrowright \mathfrak{B}$. It is clear that for each $\gamma \in \Gamma$, the union of the sets of the form

$$\{\langle b_n \rangle \in \mathrm{Seq}(\mathfrak{B}) : (b_i \subseteq a \text{ and } \gamma \cdot b_i \subseteq \mathbb{1} - a) \text{ or } \forall b \leq b_i \ (b = \gamma \cdot b)\},$$

where $i \in \mathbb{N}$ and $a \in \mathfrak{A}$, contains an open dense subset of $\mathrm{Seq}(\mathfrak{A})$, and the proposition easily follows. □

We say that $\Gamma \curvearrowright \mathfrak{B}$ is *aperiodic* if for all $b \in \mathfrak{B}^+$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, there is a non-zero $c \subseteq b$ and $\gamma \in \Gamma$ such that $\gamma \cdot c \cap \bigcup_{1 \leq i \leq n} \gamma_i \cdot c = \mathbb{0}$.

Proposition 9.9. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group, and $\Gamma \curvearrowright \mathfrak{B}$ is aperiodic. Then there is a Γ-invariant dense G_δ set $C \subseteq \mathrm{Seq}(\mathfrak{B})$ such that $\Gamma \curvearrowright (C, \mathcal{C})$ is aperiodic.*

Proof. It is clear that for each $n \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, the union of the sets

$$\{\langle b_n \rangle \in \mathrm{Seq}(\mathfrak{B}) : \gamma \cdot b_i \cap \bigcup_{1 \leq j \leq n} \gamma_j \cdot b_i = \mathbb{0}\},$$

where $i \in \mathbb{N}$ and $\gamma \in \Gamma$, contains an open dense subset of $\mathrm{Seq}(\mathfrak{A})$, and the proposition easily follows. □

As a corollary of the results of the last two sections, we obtain:

Proposition 9.10. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and $\Gamma \curvearrowright \mathfrak{B}$ is separable. Then*

$$\mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright (2^\kappa)^\Gamma]) \subseteq \mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright \mathfrak{B}]).$$

Moreover, if $\Gamma \curvearrowright \mathfrak{B}$ is also aperiodic, then

$$\mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright \mathrm{Aper}((2^\kappa)^\Gamma)]) \subseteq \mathrm{Th}_{(\forall^\kappa \exists)^+}([\Gamma \curvearrowright \mathfrak{B}]).$$

In particular, it follows that if $\mathfrak{B}$ is a complete Boolean algebra, then every element of $\mathrm{Aut}(\mathfrak{B})$ is the composition of three involutions (this is due originally to Ryzhikov), and if there is an aperiodic automorphism of $\mathfrak{B}$, then every element of $\mathrm{Aut}(\mathfrak{B})$ is a commutator.

Exercise 8. Show that if T is an aperiodic automorphism of $\mathfrak{B}$ which is the composition of periodic automorphisms in $[T]$, then T is separable.

10 Change of topology

A metric space (X, d) is *Polish* if it is second countable and complete.

Proposition 10.1. *Suppose that (X, d) is a Polish metric space and $C \subseteq X$ is closed. Then $(C, d|C)$ is Polish.*

Proof. It is clear that $(C, d|C)$ is second countable. To see that $d|C$ is complete, suppose that $\langle x_n \rangle \in C^{\mathbb{N}}$ is $(d|C)$-Cauchy. Then $\langle x_n \rangle$ is d-Cauchy, and therefore has a limit point x. As C is closed, it follows that $x \in C$. □

A topological space (X, τ) is *Polish* if it has a compatible Polish metric.

Proposition 10.2. *Suppose that (X, τ) is a Polish space and $U \subseteq X$ is open. Then $(U, \tau|U)$ is Polish.*

Proof. Fix a compatible Polish metric d for (X, τ). Set $C = X \setminus U$, and for each $x \in U$, let $d(x, C) = \inf_{z \in C} d(x, z)$. Now define d_U on U by

$$d_U(x, y) = d(x, y) + \left| \frac{1}{d(x, C)} - \frac{1}{d(y, C)} \right|.$$

To see that d_U satisfies the triangle inequality, observe that if $x, y, z \in U$, then

$$\begin{aligned} d_U(x, y) + d_U(y, z) &= d(x, y) + \left| \frac{1}{d(x, C)} - \frac{1}{d(y, C)} \right| + \\ &\quad\; d(y, z) + \left| \frac{1}{d(y, C)} - \frac{1}{d(z, C)} \right| \\ &\geq d(x, z) + \left| \frac{1}{d(x, C)} - \frac{1}{d(y, C)} + \frac{1}{d(y, C)} - \frac{1}{d(z, C)} \right| \\ &= d(x, z) + \left| \frac{1}{d(x, C)} - \frac{1}{d(z, C)} \right| \\ &= d_U(x, z). \end{aligned}$$

To see that d_U is complete, note that if $\langle x_n \rangle \in U^{\mathbb{N}}$ is a d_U-Cauchy sequence, then it certainly must be a d-Cauchy sequence as well, since $d|U \leq d_U$. Let x be the d-limit of $\langle x_n \rangle$, and observe that $x \in U$, since otherwise we would have that $\lim_{n \to \infty} d(x_n, C) = 0$, which easily implies that $\langle x_n \rangle$ is not d_U-Cauchy. This, in turn, easily implies that $\langle x_n \rangle$ converges to x in (U, d_U).

It only remains to verify that d_U is compatible with $(U, \tau|U)$. As $d|U \leq d_U$, it follows that $B_{d_U}(x, \epsilon) \subseteq B_d(x, \epsilon)$, for every $x \in U$ and $\epsilon < d(x, C)$, so the topology of d_U is finer than that of $d|U$. Conversely, given $x \in U$ and $\epsilon < d(x, C)$, set $\delta = \min(\epsilon/2, d(x, C)/2, \epsilon d(x, C)^2/4)$. If $d(x, y) < \delta$, then

$|d(x,C) - d(y,C)| < \delta$, and it follows that

$$\begin{aligned} d_U(x,y) &= d(x,y) + \left| \frac{1}{d(x,C)} - \frac{1}{d(y,C)} \right| \\ &= d(x,y) + \frac{|d(x,C) - d(y,C)|}{d(x,C)d(y,C)} \\ &< \delta + \frac{\delta}{d(x,C)(d(x,C) - \delta)} \\ &\leq \delta + \frac{\delta}{d(x,C)(d(x,C) - d(x,C)/2)} \\ &= \delta + \frac{2\delta}{d(x,C)^2} \\ &\leq \delta + \frac{2\epsilon d(x,C)^2/4}{d(x,C)^2} \\ &= \delta + \frac{\epsilon}{2} \\ &\leq \epsilon, \end{aligned}$$

so $B_d(x,\delta) \subseteq B_{d_U}(x,\epsilon)$, thus the topology of $d|U$ is finer than that of d_U. □

We next give two closure properties of Polish spaces under countable unions:

Proposition 10.3. *Suppose that $\langle (X_n, \tau_n) \rangle$ is a countable sequence of Polish spaces and τ is the topology generated by $\bigsqcup_n \tau_n$. Then $(\bigsqcup_n X_n, \tau)$ is Polish.*

Proof. For each n, let d_n be a compatible Polish metric on (X_n, τ_n). Then $d'_n(x,y) = \max(d(x,y), 1)$ is also a compatible Polish metric on (X_n, τ_n). Define d on $X = \bigsqcup_n X_n$ by

$$d(x,y) = \begin{cases} d'_n(x,y) & \text{if } x, y \in X_n, \\ 2 & \text{if no such } n \text{ exists.} \end{cases}$$

It is clear that d is a Polish metric compatible with (X, τ). □

Proposition 10.4. *Suppose that $\langle \tau_n \rangle_{n \in \mathbb{N}}$ is an increasing sequence of Polish topologies on a set X. Then the topology τ generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is Polish.*

Proof. Let $(Y, \tau_Y) = \prod_{n \in \mathbb{N}} (X, \tau_n)$ and define $\pi : X \to Y$ by $\pi(x) = (x, x, \ldots)$. Then $\pi(X)$ is the diagonal, and therefore closed. As products of countably many Polish spaces are Polish, it follows from Proposition 10.1 that $(\pi(X), \tau_Y|\pi(X))$ is Polish. For each $U \in \tau_n$, note that $\pi(U) = \Delta(Y) \cap \{\langle x_k \rangle : x_n \in U\}$, so π is a homeomorphism of (X, τ) with $(\pi(X), \tau_Y|\pi(X))$, thus (X, τ) is Polish. □

A *topology on* $(X, \mathcal{B})$ is a topology on X which generates $\mathcal{B}$.

Proposition 10.5. *Suppose that $(X, \mathcal{B})$ is a Borel space, τ is a Polish topology on $(X, \mathcal{B})$, and $B \in \mathcal{B}$. Then there is a Polish topology $\tau' \supseteq \tau$ on $(X, \mathcal{B})$ in which B is clopen.*

Proof. By induction on the construction of B from τ (as τ ranges over all Polish topologies on $(X, \mathcal{B})$). If B is closed, then Proposition 10.1 ensures that $(B, \tau|B)$ is Polish, Proposition 10.2 ensures that $(X \setminus B, \tau|(X \setminus B))$ is Polish, Proposition 10.3 ensures that the topology τ' generated by these topologies is Polish, and it is clear that B is clopen in (X, τ').

Suppose now that $B = \bigcup_{n \in \mathbb{N}} B_n$, where $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ is a sequence of sets for which we have already verified the proposition. Then we can recursively find an increasing sequence $\langle \tau_n \rangle$ of Polish topologies on $(X, \mathcal{B})$ such that B_n is clopen in (X, τ_n). Proposition 10.4 then ensures that the topology generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is Polish. As B is open in this topology, the base case implies that there is a Polish topology τ' on $(X, \mathcal{B})$ in which B is clopen. □

A Borel space $(X, \mathcal{B})$ is *standard* if there is a Polish topology on $(X, \mathcal{B})$.

Proposition 10.6. *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $B \in \mathcal{B}$. Then $(B, \mathcal{B}|B)$ is a standard Borel space.*

Proof. By Proposition 10.5, there is a Polish topology τ on $(X, \mathcal{B})$ in which B is clopen, and Proposition 10.1 implies that $\tau|B$ is Polish. □

Suppose that $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are standard Borel spaces. We say that $B \subseteq X$ is *Borel* if it is in $\mathcal{B}$, and we say that $\pi : X \to Y$ is *Borel* if $\pi^{-1}(\mathcal{C}) \subseteq \mathcal{B}$.

Proposition 10.7. *Suppose that $(X, \mathcal{B})$ is a standard Borel space, τ is a Polish topology on $(X, \mathcal{B})$, and $\langle B_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel subsets of X. Then there is a Polish topology $\tau' \supseteq \tau$ on $(X, \mathcal{B})$ in which each B_n is clopen.*

Proof. By Proposition 10.5, there is an increasing sequence $\langle \tau_n \rangle$ of Polish topologies on $(X, \mathcal{B})$ containing τ such that B_n is clopen in (X, τ_n). By Proposition 10.4, the topology generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is as desired. □

Proposition 10.8. *Suppose that $(X, \mathcal{B})$ is a standard Borel space and τ is a Polish topology on $(X, \mathcal{B})$. Then there is a zero-dimensional Polish topology $\tau' \supseteq \tau$ on $(X, \mathcal{B})$.*

Proof. By Proposition 10.7, there is an increasing sequence $\langle \tau_n \rangle$ of Polish topologies on $(X, \mathcal{B})$ containing τ such that every set in some countable basis for (X, τ_n) is clopen in (X, τ_{n+1}). By Proposition 10.4, the topology generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is as desired. □

Proposition 10.9. *Suppose that $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are standard Borel spaces, τ_X and τ_Y are Polish topologies on $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, and $\pi : X \to Y$ is Borel. Then there are zero-dimensional Polish topologies $\tau'_X \supseteq \tau_X$ and $\tau'_Y \supseteq \tau_Y$ on $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ with respect to which π is continuous.*

Proof. By Proposition 10.8, there is a zero-dimensional Polish topology $\tau'_Y \supseteq \tau_Y$ on $(Y, \mathcal{C})$. By Proposition 10.7, there is a Polish topology $\tau \supseteq \tau_X$ on $(X, \mathcal{B})$ in which the preimages under π of a clopen basis for (Y, τ'_Y) are all clopen. By Proposition 10.8, there is a zero-dimensional Polish topology $\tau'_X \supseteq \tau$ on $(X, \mathcal{B})$. It is clear that τ'_X and τ'_Y are as desired. □

Exercise 9. Show that if $(X, \mathcal{B})$ is a standard Borel space, τ is a Polish topology on $(X, \mathcal{B})$, and $T \in \text{Aut}(X, \mathcal{B})$, then there is a zero-dimensional Polish topology $\tau' \supseteq \tau$ on $(X, \mathcal{B})$ with respect to which T is a homeomorphism.

11 Analytic sets

Suppose that $(X, \mathcal{B})$ is a standard Borel space. A set $A \subseteq X$ is *analytic* (or $\boldsymbol{\Sigma}^1_1$) if there is a standard Borel space $(Y, \mathcal{C})$, a Borel function $\pi : Y \to X$, and a Borel set $B \subseteq X \times Y$ such that $A = \pi(B)$. A set is *coanalytic* (or $\boldsymbol{\Pi}^1_1$) if it is the complement of an analytic set.

Suppose that $A, B, C \subseteq X$. We say that C *separates* A from B if $A \subseteq C$ and $B \cap C = \emptyset$. We say that A, B are *Borel separable* if there is a Borel set $C \subseteq X$ which separates A from B.

Theorem 11.1 (Souslin)**.** *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $A, B \subseteq X$ are disjoint analytic sets. Then A, B are Borel separable.*

Proof. We note first the following:

Lemma 11.2. *Suppose that $A = \bigcup_{m \in \mathbb{N}} A_m$, $B = \bigcup_{n \in \mathbb{N}} B_n$, and $\forall m, n \in \mathbb{N}$ (A_m, B_n are Borel separable). Then A, B are Borel separable.*

Proof. For each $m, n \in \mathbb{N}$, fix $C_{mn} \in \mathcal{B}$ which separates A_m from B_n. Then the set $C_m = \bigcap_{n \in \mathbb{N}} C_{mn}$ separates A_m from B, thus the set $C = \bigcup_{m \in \mathbb{N}} C_m$ separates A from B. □

Suppose now, towards a contradiction, that A and B are not Borel separable. Fix Polish spaces V and W, Borel functions $\varphi : V \to X$ and $\psi : W \to Y$, and Borel sets $C \subseteq V$ and $D \subseteq W$ such that $\varphi(C) = A$ and $\psi(D) = B$. By Proposition 10.9, we can assume that V and W are zero-dimensional, C and D are clopen, and φ and ψ are continuous. By repeated application of Lemma 11.2, we can find decreasing sequences $\langle C_n \rangle_{n \in \mathbb{N}}$ and $\langle D_n \rangle_{n \in \mathbb{N}}$ of clopen subsets of C and D with vanishing diameter such that

$$\forall n \in \mathbb{N}\ (\varphi(C_n), \psi(D_n) \text{ are not Borel separable}).$$

Let x and y denote the unique points of $\bigcap_{n \in \mathbb{N}} C_n$ and $\bigcap_{n \in \mathbb{N}} D_n$, respectively, and set $\delta = d(\varphi(x), \psi(y))/2$. The continuity of φ and ψ ensures the existence of $n \in \mathbb{N}$ such that $\varphi(C_n) \subseteq B(\varphi(x), \delta)$ and $\psi(D_n) \subseteq B(\psi(y), \delta)$, and it follows that $B(\varphi(x), \delta)$ separates $\varphi(C_n)$ from $\psi(D_n)$, the desired contradiction. □

As a corollary, we obtain a characterization of Borel sets:

Theorem 11.3 (Souslin)**.** *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $A \subseteq X$ is analytic and coanalytic. Then A is Borel.*

Proof. By Theorem 11.1, there is a Borel set $B \subseteq X$ which separates A from $X \setminus A$, thus $A = B$ is Borel. □

We also obtain a characterization of Borel functions:

Theorem 11.4 (Souslin)**.** *Suppose that* $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$ *are standard Borel spaces and* $f : X \to Y$*. Then the following are equivalent:*

1. f *is Borel;*
2. $\mathrm{graph}(f)$ *is Borel;*
3. $\mathrm{graph}(f)$ *is analytic.*

Proof. To see (1) ⇒ (2), suppose that f is Borel, fix a countable separating family $\mathcal{A} \subseteq \mathcal{C}$ for Y, and observe that

$$\mathrm{graph}(f) = \bigcap_{A \in \mathcal{A}} (f^{-1}(A) \times A) \cup (f^{-1}(Y \setminus A) \times (Y \setminus A)).$$

To see (3) ⇒ (1), observe that if $\mathrm{graph}(f) \subseteq X \times Y$ and $B \subseteq X$ are Borel, then

$$\begin{aligned} f^{-1}(B) &= \mathrm{proj}_X(\mathrm{graph}(f) \cap (X \times B)) \\ &= X \setminus \mathrm{proj}_X(\mathrm{graph}(f) \cap (X \times (Y \setminus B))), \end{aligned}$$

so $f^{-1}(B)$ is analytic and coanalytic, thus Borel, by Theorem 11.3. □

We have also the following generalization of Theorem 11.1:

Theorem 11.5. *Suppose that* $(X, \mathcal{B})$ *is a standard Borel space and* $\langle A_n \rangle_{n \in \mathbb{N}}$ *is a pairwise disjoint sequence of analytic subsets of* X*. Then there is a pairwise disjoint sequence* $\langle B_n \rangle_{n \in \mathbb{N}}$ *of Borel subsets of* X *such that* $\forall n \in \mathbb{N}\ (A_n \subseteq B_n)$*.*

Proof. By repeated application of Theorem 11.1, there is a sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ such that $B_n \subseteq X \setminus \bigcup_{m<n} B_m$ and B_n separates A_n from $\bigcup_{m>n} A_m$, for all $n \in \mathbb{N}$. It is clear that any such sequence is as desired. □

As a corollary, we obtain the following:

Theorem 11.6. *Suppose that* $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$ *are standard Borel spaces,* $B \subseteq X$ *is Borel, and* $\pi : X \to Y$ *is a Borel injection. Then* $\pi(B)$ *is Borel.*

Proof. By Proposition 10.9, we can assume that X and Y are zero-dimensional Polish spaces, B is clopen, and π is continuous. A *tree* on $\mathbb{N}$ is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $\forall s \subseteq t\ (t \in T \Rightarrow s \in T)$. A *branch* of T is a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ (\alpha|n \in T)$. We use $[T]$ to denote the set of all branches of T. We say that T is *pruned* if $\forall t \in T\, \exists n \in \mathbb{N}\ (tn \in T)$.

Fix a pruned tree T on $\mathbb{N}$ and a sequence $\langle B_t \rangle_{t \in T}$ of non-empty clopen subsets of B such that:

1. $B_\emptyset = B$;
2. $\forall t \in T\ (\{B_{tn}\}_{tn \in T}$ partitions $B_t)$;
3. $\forall t \in T\ (\mathrm{diam}(B_t) < 1/|t|)$.

For each $t \in T$, let $T_t = \{s \in \mathbb{N}^{<\mathbb{N}} : ts \in T\}$. Proposition 11.5 ensures that for each $t \in T$, there is a pairwise disjoint family $\langle C_{tn} \rangle_{n \in T_t}$ of Borel sets such that $\pi(B_{tn}) \subseteq C_{tn}$. For each $t \in T$, set

$$D_t = \bigcap_{s \subseteq t} \overline{\pi(B_s)} \cap C_s.$$

To see that $\pi(B)$ is Borel, it is enough to show that $\pi(B) = \bigcap_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^n} D_s$. Towards this end, note first that if $x \in B$, then there exists $\alpha \in [T]$ such that $x \in \bigcap_{n \in \mathbb{N}} B_{\alpha|n}$, thus $\pi(x) \in \bigcap_{n \in \mathbb{N}} D_{\alpha|n}$, and it follows that $\pi(B) \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^n} D_s$.

Conversely, suppose that $\forall n \in \mathbb{N}\, \exists t_n \in T \cap \mathbb{N}^n\ (y \in D_{t_n})$. Then the construction of $\langle D_t \rangle$ ensures that $\forall n \in \mathbb{N}\ (t_n \subseteq t_{n+1})$. Set $\alpha = \bigcup_{n \in \mathbb{N}} t_n$, and let x be the unique point of $\bigcap_{n \in \mathbb{N}} B_{\alpha|n}$. The continuity of π ensures that $\lim_{n \to \infty} \operatorname{diam}(D_{\alpha|n}) = 0$, so $y = \pi(x)$, thus $\bigcap_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^n} D_s \subseteq \pi(B)$. □

12 Isomorphism theorems

We write $X \sqsubseteq_c Y$ to indicate the existence of a continuous injection.

Proposition 12.1. *Suppose that X is a Polish space. Then exactly one of the following holds:*

1. *X is countable;*
2. *$2^{\mathbb{N}} \sqsubseteq_c X$.*

Proof. It is clear that (1) and (2) are mutually exclusive, so it is enough to prove $\neg(1) \Rightarrow (2)$. Towards this end, we will first show that if (1) is false, then there is a sequence $\langle U_s \rangle_{s \in 2^{<\mathbb{N}}}$ of open sets such that:

1. $\forall s \in 2^{<\mathbb{N}}$ (U_s is uncountable);
2. $\forall s \in 2^{<\mathbb{N}}$ ($U_{s0}, U_{s1} \subseteq U_s$ and $\overline{U_{s0}} \cap \overline{U_{s1}} = \emptyset$);
3. $\forall n \in \mathbb{N}\, \forall s \in 2^n$ ($\operatorname{diam}(U_s) \leq 1/n$).

We begin by setting $U_\emptyset = X$. Suppose now that we have found $\langle U_s \rangle_{s \in 2^{\leq n}}$. For each $s \in 2^n$, we must describe how to build U_{s0}, U_{s1} from U_s. Let U denote the set of points which are contained in a countable open set. As X is second countable, this is a countable set. In particular, there are distinct points $x_{s0}, x_{s1} \in U_s \setminus U$. By fixing $\epsilon_n > 0$ sufficiently small, we can clearly ensure that the sets $U_{s0} = B(x_{s0}, \epsilon_n)$ and $U_{s1} = B(x_{s1}, \epsilon_n)$ are as desired.

For each $\alpha \in 2^{\mathbb{N}}$, the sequence $\langle \overline{U_{\alpha|n}} \rangle_{n \in \mathbb{N}}$ is decreasing and of vanishing diameter, thus the completeness of X ensures that we can define $\pi : 2^{\mathbb{N}} \to X$ by

$$\pi(\alpha) = \text{the unique point of } \bigcap_{n \in \mathbb{N}} \overline{U_{\alpha|n}}.$$

Conditions (2) and (3) ensure that π is injective and continuous. □

We write $X \sqsubseteq_B Y$ to indicate the existence of a Borel injection.

Proposition 12.2. *Suppose that X is a second countable Hausdorff space. Then $X \sqsubseteq_B 2^{\mathbb{N}}$.*

Proof. Fix a basis $\langle U_n \rangle_{n \in \mathbb{N}}$ for X, and define an injection $\pi : X \to 2^{\mathbb{N}}$ by

$$[\pi(x)](n) = \chi_{U_n}(x).$$

Set $V_n = X \setminus U_n$, and observe that if $s \in 2^{<\mathbb{N}}$, then

$$\pi^{-1}(\mathcal{N}_s) = \bigcap_{s(n)=0} V_n \cap \bigcap_{s(n)=1} U_n,$$

thus π is Borel. □

We write $X \cong_B Y$ to indicate the existence of a Borel isomorphism.

Proposition 12.3. *Suppose that X and Y are Polish spaces such that $X \sqsubseteq_B Y$ and $Y \sqsubseteq_B X$. Then $X \cong_B Y$.*

Proof. Fix Borel injections $\varphi : X \to Y$ and $\psi : Y \to X$. Let Z be the disjoint union of X and Y, and let $T : Z \to Z$ be the Borel injection given by $T = (\varphi|X) \cup (\psi|Y)$. Define $A = X \cap [X \setminus \psi(Y)]_T$ and $B = X \setminus A$. Theorem 11.6 ensures that the map $(\varphi|A) \cup (\psi^{-1}|B)$ is the desired isomorphism. □

We now have the main result of this section:

Theorem 12.4. *All uncountable standard Borel spaces are isomorphic.*

Proof. This follows from Propositions 12.1, 12.2, and 12.3. □

A *standard probability space* is a triple of the form $(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B})$ is a standard Borel space and μ is a probability measure on $(X, \mathcal{B})$. Such a space is *continuous* if $\forall x \in X \ (\mu(\{x\}) = 0)$.

Theorem 12.5. *All continuous standard probability spaces are isomorphic.*

Proof. We note first the following fact:

Lemma 12.6. *Suppose that $(X, \mathcal{B}, \mu)$ is a continuous standard probability space and $C \in \mathcal{B}$ has full measure. Then $(X, \mathcal{B}, \mu) \cong (C, \mathcal{B}|C, \mu|C)$.*

Proof. Fix a pairwise disjoint sequence $\langle N_k \rangle_{k \in \mathbb{N}}$ of null Borel sets of the same cardinality with $N_0 = X \setminus C$. (To see that there is such a sequence, fix a Borel injection $\pi : 2^{\mathbb{N}} \to X$, and observe that for all but countably many $\alpha \in 2^{\mathbb{N}}$, the Borel set $\pi(\{\alpha_0 \beta_0 \alpha_1 \beta_1 \ldots : \beta \in 2^{\mathbb{N}}\})$ is null.) By Theorem 12.4, there is a Borel isomorphism $\pi : X \to C$ such that $\mathrm{supp}(\pi) = \bigcup_{k \in \mathbb{N}} N_k$ and $\pi(N_k) = N_{k+1}$, and it is clear that any such map is the desired isomorphism. □

By Theorem 12.4, it is enough to show that if μ is a probability measure $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the usual Borel structure on $[0,1]$, then there is a Borel automorphism of $[0,1]$ which sends μ to the restriction m of Lebesgue measure to $\mathcal{B}$. Towards this end, let $C = \{x \in [0,1] : \forall y < x \ (\mu([0,y]) < \mu([0,x]))\}$, and observe that the map $\pi : C \to [0,1]$ given by $\pi(x) = \mu([0,x])$ is an isomorphism of $(C, \mathcal{B}|C, \mu|C)$ with $([0,1], \mathcal{B}, m)$. As Lemma 12.6 ensures that $([0,1], \mathcal{B}, \mu) \cong (C, \mathcal{B}|C, \mu|C)$, this completes the proof of the theorem. □

13 Normal subgroups

In this section, we study the normal subgroup lattice of the automorphism groups of standard Borel spaces and standard probability spaces.

Proposition 13.1. *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $T \in \mathrm{Aut}(X, \mathcal{B})$ has uncountable support. Then the normal subgroup $N \trianglelefteq \mathrm{Aut}(X, \mathcal{B})$ generated by T is $\mathrm{Aut}(X, \mathcal{B})$.*

Proof. We begin with the following observation:

Lemma 13.2. *There is an involution $I \in N$ such that $|\mathrm{supp}(I)| = |\mathrm{fix}(I)| = \mathfrak{c}$.*

Proof. Fix a maximal T-discrete Borel set $B \subseteq X$. It is clear that B is uncountable. Fix a Borel set $C \subseteq B$ such that both C and $B \setminus C$ are uncountable. By Theorem 12.4, there is a Borel involution $I : X \to X$ such that $C = \mathrm{supp}(I)$, and it easily follows that $[T, I] = T \circ I \circ T^{-1} \circ I^{-1}$ is as desired. □

As a consequence, we obtain the following:

Lemma 13.3. *N contains every Borel involution of $(X, \mathcal{B})$.*

Proof. By Theorem 12.4 and the fact that Borel involutions are smooth, any two Borel involutions with uncountable, co-uncountable support are conjugate. Lemma 13.2 therefore implies that N contains every Borel involution with uncountable, co-uncountable support. As every Borel involution is easily seen to be a composition of two involutions of this form, the lemma follows. □

As Theorem 4.11 implies that every Borel isomorphism of $(X, \mathcal{B})$ is a composition of three Borel involutions, it follows that $N = \mathrm{Aut}(X, \mathcal{B})$. □

Exercise 10. Show that under the hypotheses of Proposition 13.1, every automorphism of $(X, \mathcal{B})$ is the composition of 4 conjugates of T.

We now have the following:

Theorem 13.4 (Shortt)**.** *Suppose that $(X, \mathcal{B})$ is an uncountable standard Borel space. Then $\mathrm{Aut}(X, \mathcal{B})$ has exactly five normal subgroups:*

1. $\{\mathrm{id}\}$;
2. $\{T \in \mathrm{Aut}(X, \mathcal{B}) : T$ *has finite support and even cycle type*$\}$;

3. *$\{T \in \mathrm{Aut}(X, \mathcal{B}) : T$ has finite support$\}$;*

4. *$\{T \in \mathrm{Aut}(X, \mathcal{B}) : T$ has countable support$\}$;*

5. $\mathrm{Aut}(X, \mathcal{B})$.

Proof. By Proposition 13.1, it is enough to show that if $N \trianglelefteq \mathrm{Aut}(X, \mathcal{B})$ has no automorphisms of uncountable support, then it is given by (1), (2), (3), or (4). Towards this end, fix a countably infinite set $Z \subseteq X$, and let

$$N' = \{T \in N : \mathrm{supp}(T) \subseteq Z\}.$$

Then $N' \trianglelefteq S_Z$, thus N' is one of the following groups:

1′. $\{\mathrm{id}\}$;

2′. $\{T \in S_Z : T$ has finite support and even cycle type$\}$;

3′. $\{T \in S_Z : T$ has finite support$\}$;

4′. $\{T \in S_Z : T$ has countable support$\}$.

It is easy to see that if N' is given by (i'), then N is given by (i). □

Let $\mathrm{Aut}(X, \mathcal{B}, \mu)$ denote the group of measure-preserving automorphisms of $(X, \mathcal{B})$, modulo almost everywhere equality.

Theorem 13.5 (Fathi)**.** *Suppose that $(X, \mathcal{B}, \mu)$ is a continuous standard probability space. Then $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is simple.*

Proof. Suppose that $N \trianglelefteq \mathrm{Aut}(X, \mathcal{B}, \mu)$ contains the equivalence class of some Borel automorphism $T \in N$ which is not almost everywhere trivial. We will abuse language in the usual fashion and identify N with the group of Borel automorphisms whose equivalence classes are in N.

Lemma 13.6. *There is a measure-preserving involution $I \in N$ such that $\mu(\mathrm{supp}(I)), \mu(\mathrm{fix}(I)) > 0$.*

Proof. Fix a maximal T-discrete Borel set $B \subseteq X$. Fix a Borel set $C \subseteq B$ such that $0 < \mu(C) < \mu(B)$. By Theorem 12.5, there is a Borel involution $I : X \to X$ such that $C = \mathrm{supp}(I)$, and it easily follows that $[T, I]$ is as desired. □

As a consequence, we obtain the following:

Lemma 13.7. *N contains every measure-preserving Borel involution of $(X, \mathcal{B})$.*

Proof. By Theorem 12.5 and the fact that Borel involutions are smooth, any two Borel involutions whose supports are of the same measure are conjugate. Lemma 13.6 therefore implies that N contains every measure-preserving Borel involution whose support is of some fixed measure strictly between 0 and 1. As every measure-preserving Borel involution is easily seen to be a composition of involutions of this form, the lemma follows. □

It is easy to see that if T is measure-preserving, then so too is every element of $[T]$. As Theorem 4.11 implies that every Borel isomorphism of $(X, \mathcal{B})$ is a composition of three involutions in $[T]$, it follows that $N = \mathrm{Aut}(X, \mathcal{B}, \mu)$. $\square$

Exercise 11. Suppose that $(X, \mathcal{B}, \mu)$ is a continuous standard probability space, and let G denote the group of non-singular Borel automorphisms of (X, μ), modulo almost everywhere equality. Show that G is simple.

14 Bergman's property

For each set $\Gamma \subseteq \mathrm{Aut}(X, \mathcal{B})$, we say that a set $B \subseteq X$ is Γ-*large* if there exist $n \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $X = \gamma_1(B) \cup \cdots \cup \gamma_n(B)$.

Proposition 14.1. *Suppose that $(X, \mathcal{B})$ is a Borel space, Γ is countable, and $\Gamma \curvearrowright (X, \mathcal{B})$ is aperiodic separable. Then there is a pairwise disjoint sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ of X of $[\Gamma]$-large sets.*

Proof. Set $A_0 = X$. Given a $[\Gamma]$-large set $A_n \in \mathcal{B}$, appeal to Proposition 7.8 to obtain an involution $I_n \in [\Gamma]$ such that $\mathrm{supp}(I_n) = A_n$. Fix a transversal $B_n \in \mathcal{B}$ of $I_n | A_n$, and set $A_{n+1} = I_n(B_n)$. It is clear that B_n and A_{n+1} are both $[\Gamma]$-large. The resulting sequence $\langle B_n \rangle$ is as desired. $\square$

For $\Gamma \subseteq \mathrm{Aut}(X, \mathcal{B})$ and $B \subseteq X$, let $\Gamma | B = \{\gamma | B : \gamma \in \Gamma \text{ and } \gamma(B) = B\}$.

Proposition 14.2. *Suppose that $(X, \mathcal{B})$ is a Borel space, $\Gamma \leq \mathrm{Aut}(X, \mathcal{B})$ is σ-full, $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ is pairwise disjoint, and $\langle \Gamma_n \rangle \in \mathcal{P}(\Gamma)^{\mathbb{N}}$ is an increasing, exhaustive sequence of subsets of Γ. Then there exists $m \in \mathbb{N}$ such that*

$$\forall n \geq m \ (\Gamma_n | B_n = \Gamma | B_n).$$

Proof. Suppose, towards a contradiction, that there is an increasing sequence $\langle k_n \rangle \in \mathbb{N}^{\mathbb{N}}$ and a sequence $\langle \gamma_n \rangle \in \Gamma^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \ (\gamma_n(B_{k_n}) = B_{k_n} \text{ and } \gamma_n | B_{k_n} \notin \Gamma_{k_n} | B_{k_n}).$$

Fix any $\gamma \in \Gamma$ such that $\forall n \in \mathbb{N} \ (\gamma | B_{k_n} = \gamma_n | B_{k_n})$, and observe that $\gamma \notin \bigcup_{n \in \mathbb{N}} \Gamma_n$, which contradicts our assumption that $\langle \Gamma_n \rangle$ is exhaustive. $\square$

A group Γ is *Bergman* if for every increasing, exhaustive sequence $\langle \Gamma_n \rangle \in \mathcal{P}(\Gamma)^{\mathbb{N}}$ of subsets of Γ, there exists $n \in \mathbb{N}$ such that $\Gamma = (\Gamma_n)^n$. It is easy to see that Γ is Bergman if and only if every left-invariant metric on Γ is bounded. It follows that if Γ is Bergman, then every Cayley graph of Γ has finite diameter.

Theorem 14.3. *Suppose that $(X, \mathcal{B})$ is a Borel space and Γ is a σ-full group of separable automorphisms of $(X, \mathcal{B})$ which has a countable aperiodic subgroup. Then Γ is Bergman.*

Proof. Suppose that $\langle \Gamma_n \rangle \in \mathcal{P}(\Gamma)^{\mathbb{N}}$ is an increasing, exhaustive sequence of subsets of Γ. By Proposition 14.1, there is a sequence $\langle A_n \rangle \in \mathcal{B}^{\mathbb{N}}$ of pairwise disjoint Γ-large sets. By Proposition 14.2, there exists $l \in \mathbb{N}$ such that

$$\forall m \geq l \ (\Gamma_m | A_m = \Gamma | A_m).$$

By Proposition 14.1, there is a sequence $\langle B_n \rangle \in \mathcal{B}^{\mathbb{N}}$ of pairwise disjoint Γ-large subsets of A_l. Let $\Delta = \{\gamma \in \Gamma : \mathrm{supp}(\gamma) \subseteq A_l\}$ and $\Delta_n = \Gamma_n \cap \Delta$, for each $n \in \mathbb{N}$. By applying Proposition 14.2 to $\Delta | A_l$ and $\langle \Delta_n | A_l \rangle$, we obtain $m \geq l$ such that $\forall n \geq m \ (\Delta_n | B_n = \Delta | B_n)$. Put $A = A_l$ and $B = B_m$.

Lemma 14.4. $\forall \gamma \in \Gamma \ (\mathrm{supp}(\gamma) \subseteq B \Rightarrow \gamma \in (\Gamma_m)^4)$.

Proof. By Theorem 7.11, there exist $\delta_1, \delta_2 \in \Gamma | B$ such that $\gamma | B = [\delta_1, \delta_2]$. Extend δ_1, δ_2 to elements of Γ_m such that $\delta_1 | (A \setminus B) = \mathrm{id}$ and $\delta_2 | (X \setminus A) = \mathrm{id}$, and observe that $\gamma = [\delta_1, \delta_2]$. □

By Proposition 7.8, there is an involution $I \in \Gamma$ such that $B = \mathrm{supp}(I)$, a transversal $C \in \mathcal{B}$ of $I | B$, and $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $X = \gamma_1(C) \cup \cdots \cup \gamma_k(C)$. By choosing $n \geq m$ sufficiently large, we can ensure that for all $1 \leq i < j \leq k$, there is an involution $J \in \Gamma_n$ such that $J(C) = \gamma_i(C)$, $J(I(C)) = \gamma_j(C)$. Then every involution in Γ is a composition of n^2 involutions whose supports are contained in a set of the form $\gamma_i(C) \cup \gamma_j(C)$. As each such involution is the composition of 6 elements of Γ_n, it follows from Theorem 4.11 that $\Gamma = (\Gamma_n)^{18n^2}$, thus Γ is Bergman. □

Remark 14.5. In particular, it follows that the group of Borel automorphisms of a standard Borel space is Bergman, as is the group of measure-preserving (or non-singular) automorphisms of a continuous standard probability space.

15 Uniformization

A *σ-ideal* on a set Y is a set $\mathcal{I} \subseteq \mathcal{P}(Y)$ such that:

1. $\emptyset \in \mathcal{I}$;
2. $\forall A \in \mathcal{I} \forall B \subseteq A \ (B \in \mathcal{I})$;
3. $\forall \langle A_n \rangle \in \mathcal{I}^{\mathbb{N}} \ (\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{I})$.

Suppose that X and Y are Polish spaces. We say that an assignment $x \mapsto \mathcal{I}_x$ of σ-ideals on Y to points of X is *Borel on Borel* if for every Borel set $R \subseteq X \times Y$, the corresponding set $\{x \in X : R_x \in \mathcal{I}_x\}$ is Borel. For example, Theorem 2.5 implies that the assignment $x \mapsto \mathcal{I}_x = \{M \subseteq Y : M \text{ is meager}\}$ is Borel on Borel.

A *uniformization* of a set $R \subseteq X \times Y$ is a function $\pi : \mathrm{proj}_X(R) \to Y$ such that $\forall x \in \mathrm{proj}_X(R) \ ((x, \pi(x)) \in R)$.

Theorem 15.1 (Kechris). *Suppose that X and Y are Polish spaces, $x \mapsto \mathcal{I}_x$ is a Borel on Borel assignment of σ-ideals on Y to points of X, and $R \subseteq X \times Y$ is a Borel set such that $\forall x \in X$ $(R_x \notin \mathcal{I}_x)$. Then R has a Borel uniformization.*

Proof. Fix a zero-dimensional Polish topology τ on $X \times Y$, finer than the given one and compatible with its Borel structure, such that R is τ-clopen. Fix a pruned tree T on $\mathbb{N}$ and a sequence $\langle R_t \rangle_{t \in T}$ of τ-clopen subsets of R such that:

1. $R_\emptyset = R$;
2. $\forall t \in T$ $(\{R_{tn}\}_{tn \in T}$ partitions $R_t)$;
3. $\forall t \in T$ $(\mathrm{diam}(R_t) < 1/|t|)$.

For each $n \in \mathbb{N}$, define $t_n : X \to T \cap \mathbb{N}^n$ by

$$t_n(x) = \min_{\mathrm{lex}}\{t \in T \cap \mathbb{N}^n : (R_t)_x \notin \mathcal{I}_x\}.$$

Define $\pi : X \to Y$ by

$$\pi(x) = \text{ the unique } y \in Y \text{ such that } (x, y) \in \bigcap_{n \in \mathbb{N}} R_{t_n(x)}.$$

It is clear that π is a uniformization of R. Noting that

$$\begin{aligned} \pi(x) = y \quad &\Leftrightarrow \quad \forall n \in \mathbb{N}\ ((x, y) \in R_{t_n(x)}) \\ &\Leftrightarrow \quad \forall n \in \mathbb{N}\, \exists t \in T \cap \mathbb{N}^n \\ &\qquad ((x, y) \in R_t \text{ and } (R_t)_x \notin \mathcal{I}_x \text{ and } \forall s <_{\mathrm{lex}} t\ ((R_s)_x \in \mathcal{I}_x)), \end{aligned}$$

it follows that graph(π) is Borel, thus so too is π, by Theorem 11.4. □

We prove next a useful technical fact:

Theorem 15.2 (Lusin). *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then $\{x \in X : |R_x| = 1\}$ is coanalytic.*

Proof (Kechris). By Theorem 12.4, we can assume that $X = Y = \mathbb{N}^\mathbb{N}$. Fix a zero-dimensional Polish topology τ on $X \times Y$, finer than the usual one and compatible with its Borel structure, such that R is τ-clopen. Fix a pruned tree T on $\mathbb{N}$ and a sequence $\langle R_t \rangle_{t \in T}$ of τ-clopen subsets of R such that:

1. $R_\emptyset = R$;
2. $\forall t \in T$ $(\{R_{tn}\}_{tn \in T}$ partitions $R_t)$;
3. $\forall t \in T$ $(\mathrm{diam}(R_t) < 1/|t|)$.

Then the function $\pi : [T] \to X \times Y$ given by $\pi(\alpha) =$ the unique point of $\bigcap_{n \in \mathbb{N}} R_{\alpha|n}$ is continuous, thus the set $R' \subseteq X \times \mathbb{N}^\mathbb{N}$ given by

$$R' = \bigcap_{n \in \mathbb{N}} \{(x, \alpha) \in X \times [T] : x = \mathrm{proj}_X(\pi(\alpha))\}$$

is closed, and $\forall x \in X\ (|R_x| = |R'_x|)$. Fix a tree T' on $\mathbb{N} \times \mathbb{N}$ such that $R' = [T']$. For each $x \in X$, let $T'_x = \bigcup_{n \in \mathbb{N}} \{t \in \mathbb{N}^n : (x|n, t) \in T\}$, and observe that $\forall x \in X\ (|R'_x| = |[T'_x]|)$.

Note that $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ can be identified with $2^{\mathbb{N}^{<\mathbb{N}}}$, and therefore inherits from this space a standard Borel structure in which the set

$$\mathrm{Tr} = \{T \subseteq \mathbb{N}^{<\mathbb{N}} : T \text{ is a tree on } \mathbb{N}\}$$

is Borel. Let UB denote the set of trees T on $\mathbb{N}$ such that $|[T]| = 1$.

Lemma 15.3. *The set* UB *is coanalytic.*

Proof. Let $\mathrm{Tr}' = \{T \in \mathrm{Tr} : \forall n \in \mathbb{N}\ (T \cap \mathbb{N}^n \neq \emptyset)\}$. It is clear that $\mathrm{Tr} \setminus \mathrm{Tr}'$ is a Borel subset of $\mathrm{Tr} \setminus \mathrm{UB}$, so it is enough to show that $\mathrm{Tr}' \cap \mathrm{UB}$ is coanalytic. We say that a function $\pi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is *Lipschitz* if

$$\forall n \in \mathbb{N}\, \forall \alpha, \beta \in \mathbb{N}^{\mathbb{N}}\ (\alpha|n = \beta|n \Rightarrow \pi(\alpha)|n = \pi(\beta)|n),$$

in which case we set $\pi(t) = \pi(\alpha)|n$, where $t = \alpha|n$. It is clearly enough to show that if $T \in \mathrm{Tr}'$, then exactly one of the following holds:

1. $T \in \mathrm{UB}$;
2. There exist $n \in \mathbb{N}$ and a Lipschitz function $\pi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\forall t \in \mathbb{N}^{<\mathbb{N}}\ (t \in T \Rightarrow \pi(t) \in T \text{ and } |t| > n \Rightarrow t \neq \pi(t)).$$

It is clear that $(1) \Rightarrow \neg(2)$, so we will show $\neg(1) \Rightarrow (2)$.

If there are distinct branches $\beta, \gamma \in [T]$, then fix $n \in \mathbb{N}$ least such that $\beta(n) \neq \gamma(n)$ and define $\pi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$[\pi(\alpha)](k) = \begin{cases} \beta(k) & \text{if } k < n \text{ or } \alpha|k \neq \beta|k, \\ \gamma(k) & \text{otherwise.} \end{cases}$$

It is clear that n and π are as desired.

If T has no branches, then we can recursively define a rank function $\rho : T \to \omega_1$ in the usual fashion. Fix $n \in \mathbb{N}$ such that $\rho(\emptyset)$ is of the form $\lambda + n$, where $\lambda < \omega_1$ is a limit ordinal. Fix a sequence $t \in T$ of length n such that $\rho(t) = \lambda$, and let $\pi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be any Lipschitz function such that

$$\forall s \in T\ (|s| \leq n \Rightarrow \pi(s) \subseteq t \text{ and } n < |s| \Rightarrow \rho(s) < \rho(\pi(s))).$$

It is clear that n and π are as desired. □

As the map $x \mapsto T'_x$ is clearly Borel and $|R_x| = 1 \Leftrightarrow |R'_x| = 1 \Leftrightarrow |T'_x| = 1 \Leftrightarrow T'_x \in \mathrm{UB}$, it follows that $\{x \in X : |R_x| = 1\}$ is coanalytic. □

In order to see that Borel sets $R \subseteq X \times Y$ with countable vertical sections have Borel uniformizations, it only remains to check that if R is such a set, then the assignment $x \mapsto \mathcal{I}_x$ given by $\mathcal{I}_x = \{Z \subseteq Y : R_x \cap Z = \emptyset\}$ is Borel on Borel:

Proposition 15.4. *Suppose that X and Y are Polish spaces, $R \subseteq X \times Y$ is Borel, and every vertical section of R is countable. Then $\mathrm{proj}_X(R)$ is Borel.*

Proof. As $\mathrm{proj}_X(R)$ is clearly analytic, it is enough to show that $\mathrm{proj}_X(R)$ is coanalytic, by Theorem 11.3. Fix a zero-dimensional Polish topology τ on $X \times Y$, finer than the given one and compatible with its Borel structure, such that R is τ-clopen. Fix a pruned tree T on $\mathbb{N}$ and a sequence $\langle R_t \rangle_{t \in T}$ of τ-clopen subsets of R such that:

1. $R_\emptyset = R$;
2. $\forall t \in T$ $(\{R_{tn}\}_{tn \in T}$ partitions $R_t)$;
3. $\forall t \in T$ $(\mathrm{diam}(R_t) < 1/|t|)$.

Fix a countable basis $\langle U_n \rangle$ for $(X \times Y, \tau)$. Then

$$\mathrm{proj}_X(R) = \bigcup_n \{x \in X : |(R \cap U_n)_x| = 1\},$$

thus the desired result follows from Theorem 15.2. □

A set $R \subseteq X \times Y$ is a *graph of a function* if $\forall (x, y_1), (x, y_2) \in R$ $(y_1 = y_2)$.

Theorem 15.5 (Lusin-Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set with countable vertical sections. Then R is the union of countably many Borel graphs of functions.*

Proof (Kechris). It is clearly enough to show that R is contained in the union of countably many graphs. We can clearly assume that every vertical section of R is countable. Define $S \subseteq X \times Y^{\mathbb{N}}$ by

$$S = \{(x, \langle y_n \rangle) \in X \times Y^{\mathbb{N}} : R_x = \{y_n\}_{n \in \mathbb{N}}\}.$$

Then $S = S_1 \cap S_2$, where

$$S_1 = \{(x, \langle y_n \rangle) \in X \times Y^{\mathbb{N}} : \{y_n\}_{n \in \mathbb{N}} \subseteq R_x\}$$

and

$$S_2 = \{(x, \langle y_n \rangle) \in X \times Y^{\mathbb{N}} : R_x \subseteq \{y_n\}_{n \in \mathbb{N}}\}.$$

It is clear that S_1 is Borel, so to see that S is Borel, it is enough to check that S_2 is Borel. Towards this end, simply observe that

$$\begin{aligned} (X \times Y) \setminus S_2 &= \{(x, \langle y_n \rangle) : \exists z \in R_x \, \forall n \in \mathbb{N} \; (y_n \neq z)\} \\ &= \mathrm{proj}_{X \times Y^{\mathbb{N}}} \left(\bigcap_{k \in \mathbb{N}} \{((x, \langle y_n \rangle), z) : z \in R_x \setminus \{y_k\}\} \right), \end{aligned}$$

and Proposition 15.4 ensures that the latter set is Borel.

For each $x \in X$, endow R_x with the discrete topology, endow $R_x^{\mathbb{N}}$ with the corresponding product topology, and let $\mathcal{I}_x$ denote the σ-ideal of sets $M \subseteq Y^{\mathbb{N}}$ such that $M \cap R_x^{\mathbb{N}}$ is meager.

Lemma 15.6. *The assignment $x \mapsto \mathcal{I}_x$ is Borel on Borel.*

Proof. Suppose that $B \subseteq X \times Y^{\mathbb{N}}$. Then $B_x \cap R_x^{\mathbb{N}}$ is meager in $R_x^{\mathbb{N}}$ if and only if there is a bijection $\varphi : R_x \to \mathbb{N}$ such that $\{\langle \varphi(x_n) \rangle : \langle x_n \rangle \in B \cap R_x^{\mathbb{N}}\}$ is meager in $\mathbb{N}^{\mathbb{N}}$, thus $\{x \in X : B_x \cap S_x \in \mathcal{I}_x\}$ is analytic. Similarly, $B_x \cap R_x^{\mathbb{N}}$ is meager in $R_x^{\mathbb{N}}$ if and only if for every bijection $\varphi : R_x \to \mathbb{N}$, the set $\{\langle \varphi(x_n) \rangle : \langle x_n \rangle \in B \cap R_x^{\mathbb{N}}\}$ is meager in $\mathbb{N}^{\mathbb{N}}$, thus $\{x \in X : B_x \cap S_x \in \mathcal{I}_x\}$ is coanalytic. Theorem 11.3 now implies that $\{x \in X : B_x \cap R_x \in \mathcal{I}_x\}$ is Borel. □

As $\forall x \in X$ $(S_x \notin \mathcal{I}_x)$, it follows from Theorem 15.1 that there is a Borel uniformization $\pi : X \to Y^{\mathbb{N}}$ of S. For each $n \in \mathbb{N}$, define $f_n : X \to Y$ by $f_n(x) = [\pi(x)](n)$, and observe that $R = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$. □

We close this section with the following useful corollary:

Proposition 15.7. *Suppose that X is a Polish space and T is a countable-to-one partial function on X whose graph is Borel. Then there is a partition $\langle B_n \rangle_{n \in \mathbb{N}}$ of $\mathrm{dom}(T)$ into Borel sets such that $\forall n \in \mathbb{N}$ $(T|B_n$ is injective$)$.*

Proof. By Theorem 15.5, there are Borel partial functions T_n from X to itself such that $\mathrm{graph}(T^{-1}) = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(T_n)$. By Proposition 15.4, the sets $B_n = \mathrm{range}(T_n)$ are as desired. □

16 Selection

Suppose that X is a Polish space. We say that an equivalence relation E on X is *Borel* if it is Borel when thought of as a subset of $X \times X$. We say that E is *countable* if each of its equivalence classes is countable.

Theorem 16.1 (Feldman-Moore)**.** *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$.*

Proof. We will obtain the theorem from the following observation:

Lemma 16.2. *Suppose that $A, B \subseteq X$ are Borel sets and $T : A \to B$ is a Borel injection. Then there is a Borel automorphism $U : X \to X$ such that $\forall x \in X$ $([x]_T = [x]_U)$.*

Proof. We define U on the T-saturation of the set $C = \bigcap_{n \in \mathbb{N}} T^{-n}(A) \setminus B$ by

$$U(x) = \begin{cases} T^{-1}(x) & \text{if } x \in T(C), \\ T^{-2}(x) & \text{if } \exists k \in \mathbb{N}\ (x \in T^{2k+3}(C)), \\ T^{2}(x) & \text{if } \exists k \in \mathbb{N}\ (x \in T^{2k}(C)). \end{cases}$$

We define U on the T-saturation of the set $D = \bigcap_{n \in \mathbb{N}} T^{n}(B) \setminus A$ by

$$U(x) = \begin{cases} T(x) & \text{if } x \in T^{-1}(D), \\ T^{2}(x) & \text{if } \exists k \in \mathbb{N}\ (x \in T^{-(2k+3)}(D)), \\ T^{-2}(x) & \text{if } \exists k \in \mathbb{N}\ (x \in T^{-2k}(D)). \end{cases}$$

On the rest of X, we simply set $U(x) = T(x)$. □

By Theorem 15.5, we can write E as the union of countably many Borel graphs. By Proposition 15.7, we can assume that these are graphs of Borel injections $T_n : A_n \to B_n$. By Lemma 16.2, we can assume that $X = A_n = B_n$, thus the group generated by these automorphisms is as desired. □

The original proof of Theorem 16.1 actually showed the somewhat stronger fact that there is a countable sequence $\langle I_n \rangle$ of Borel involutions of X such that $E = \bigcup_n \text{graph}(I_n)$, which follows from Proposition 7.6 and Theorem 16.1. We will now discuss a useful rephrasing of this fact.

A *graph* on X is an irreflexive, symmetric subset of $X \times X$. A *coloring* of a graph $\mathcal{G}$ on X is a function $c : X \to I$ such that

$$\forall (x, y) \in \mathcal{G} \ (c(x) \neq c(y)).$$

We say that a coloring is *Borel* if both X and I are Polish spaces and $c : X \to I$ is Borel. The *Borel chromatic number* of $\mathcal{G}$, or $\chi_B(\mathcal{G})$, is the least cardinal κ such that there is a Polish space I of cardinality κ and a Borel coloring $c : X \to I$ of $\mathcal{G}$. The following simple fact will be useful later on:

Proposition 16.3 (Dougherty-Jackson-Kechris)**.** *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $\mathcal{G}$ is a locally finite Borel graph on X. Then $\chi_B(\mathcal{G}) \leq \aleph_0$.*

Proof. Fix a countably sequence $\langle U_n \rangle \in \mathcal{B}^{\mathbb{N}}$ which separates points and is closed under finite intersection. For each $x \in X$, let $c(x)$ be the least natural number n such that $x \in U_n$ and $\mathcal{G}_x \cap U_n = \emptyset$. It is clear that c is the desired coloring. □

For each $n \in \mathbb{N}$, let $[X]^n = \{S \subseteq X : |S| = n\}$ and $[E]^n = \{S \in [X]^n : \forall x, y \in S \ (xEy)\}$, and define $[E]^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} [E]^n$. These sets all inherit standard Borel structures from $X^{<\mathbb{N}}$. Let $\mathcal{G}_E$ denote the graph on $[E]^{<\mathbb{N}}$ given by

$$(S, T) \in \mathcal{G}_E \Leftrightarrow S \neq T \text{ and } S \cap T \neq \emptyset.$$

It is not hard to see that $\chi_B(\mathcal{G}_E | [E]^2) \leq \aleph_0$ means exactly that E is the union of the graphs of countably many Borel involutions. Strengthening this, we have:

Proposition 16.4. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then $\chi_B(\mathcal{G}_E) \leq \aleph_0$.*

Proof. Fix Borel involutions $I_n : X \to X$ such that

$$E = \bigcup_{n \in \mathbb{N}} \text{graph}(I_n).$$

Let $\leq_X$ be a Borel linear ordering of X, and for each $S \in [E]^{<\mathbb{N}}$, let $\langle x_i^{(S)} \rangle_{i < n(S)}$ be the $\leq_X$-increasing enumeration of S. Define $c : [E]^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by letting $c(S)$ be the lexicographically least sequence $\langle k_{ij} \rangle_{i,j < |S|}$ of natural numbers such that

$$\forall i, j < |S| \ (I_{k_{ij}}(x_i^{(S)}) = x_j^{(S)}).$$

Now suppose, towards contradiction, that c is not a coloring. Fix $(S, T) \in \mathcal{G}_E$ such that $c(S) = c(T) = \langle k_{ij} \rangle$, put $n = |S| = |T|$, and fix $i, j < n$ such that

$$x_i^{(S)} = x_j^{(T)}.$$

Then

$$\begin{aligned} i < j &\Leftrightarrow x_i^{(S)} <_X x_j^{(S)} \\ &\Leftrightarrow x_i^{(S)} <_X I_{k_{ij}}(x_i^{(S)}) \\ &\Leftrightarrow x_j^{(T)} <_X I_{k_{ij}}(x_j^{(T)}) \\ &\Leftrightarrow x_j^{(T)} <_X x_i^{(T)} \\ &\Leftrightarrow j < i, \end{aligned}$$

thus $i = j$, so $x_i^{(S)} = x_i^{(T)}$. It follows that for all $m < n$,

$$\begin{aligned} x_m^{(S)} &= I_{k_{im}}(x_i^{(S)}) \\ &= I_{k_{im}}(x_i^{(T)}) \\ &= x_m^{(T)}, \end{aligned}$$

thus $S = T$, which contradicts our assumption that $(S, T) \in \mathcal{G}$. □

We say $\Psi \subseteq [X]^{<\mathbb{N}}$ is *pairwise disjoint* if $\forall S, T \in \Psi \ (S \neq T \Rightarrow S \cap T = \emptyset)$.

Proposition 16.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\Phi \subseteq [E]^{<\mathbb{N}}$ is Borel. Then there is a maximal pairwise disjoint Borel subset of Φ.*

Proof. Fix a Borel coloring $c : \Phi \to \mathbb{N}$ of $\mathcal{G}$, define $\Phi_n \subseteq \Phi$ by

$$\Phi_n = \{S \in \Phi : c(S) = n\},$$

and recursively define $\Psi_0 \subseteq \Psi_1 \subseteq \cdots$ by setting $\Psi_0 = \Phi_0$ and

$$\Psi_{n+1} = \Psi_n \cup \{S \in \Phi_{n+1} : \forall T \in \Psi_n \ (S \cap T = \emptyset)\}.$$

It can easily be shown that each Ψ_n is Borel, for example, by appealing to the fact that E is of the form E_Γ^X. As each Ψ_n is pairwise disjoint, so too is the set

$$\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n.$$

It remains to check that Ψ is a maximal pairwise disjoint subset of Φ. To see this, simply note that if $S \in \Phi \setminus \Psi$ then $S \notin \Psi_{c(S)}$, thus there exists $T \in \Psi_{c(S)}$ such that $S \cap T \neq \emptyset$. □

In practice, it is useful to think of Borel sets $\Phi \subseteq [E]^{<\mathbb{N}}$ as properties of finite, pairwise E-related subsets of X. As a result, we will use "$\Phi(S)$" synonymously with "$S \in \Phi$". Associated with each pairwise disjoint set $\Psi \subseteq [E]^{<\mathbb{N}}$ is the equivalence relation $F \subseteq E$ on $\operatorname{dom}(F) = \bigcup \Psi$, whose classes are the elements of Ψ. We will refer to such equivalence relations as *fsr's of* E. This abbreviation is meant to distinguish such equivalence relations from *finite subequivalence relations of* E, which are fsr's whose domain is all of X.

Given a set $\Phi \subseteq [E]^{<\mathbb{N}}$, we say that an fsr $F \subseteq E$ is Φ-*satisfying* if

$$\forall x \in \operatorname{dom}(F)\ \Phi([x]_F),$$

and we say that F is Φ-*maximal* if

$$\forall S \in [E]^{<\mathbb{N}}\ (S \cap \operatorname{dom}(F) = \emptyset \Rightarrow \neg\Phi(S)),$$

or equivalently, if there is no Φ-satisfying fsr $F' \supset F$ such that

$$\forall x \in \operatorname{dom}(F)\ ([x]_F = [x]_{F'}).$$

Still one more equivalent condition is that $\Psi = \{[x]_F : x \in \operatorname{dom}(F)\}$ is a maximal pairwise disjoint subset of Φ, thus Proposition 16.5 yields the following:

Proposition 16.6. *Suppose that E is a countable Borel equivalence relation on a Polish space and $\Phi \subseteq [E]^{<\mathbb{N}}$ is Borel. Then E admits a Φ-maximal fsr.*

Proposition 16.6 is a quite useful tool in the study of countable Borel equivalence relations. Here is a simple example of its application:

Proposition 16.7. *Suppose that X is a Polish space, E is an aperiodic countable Borel equivalence relation on X, and n is a positive natural number. Then there is a Borel subequivalence relation F of E whose classes have cardinality n.*

Proof. First we will handle the case that E is smooth. Fix a Borel transversal $B \subseteq X$, as well as a countable group $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms such that $E = E_\Gamma^X$. Let $k_0 : B \to \mathbb{N}$ be the function with constant value 0, recursively define by $k_{m+1} : B \to \mathbb{N}$ by putting

$$k_{m+1}(x) = \min\{k \in \mathbb{N} : \forall \ell \leq m\ (\gamma_k \cdot x \neq \gamma_{k_\ell(x)} \cdot x)\},$$

and define $f_m : B \to X$ by

$$f_m(x) = \gamma_{k_m(x)} \cdot x.$$

For each $x \in X$, let $m(x)$ be the unique $m \in \mathbb{N}$ with $x \in f_m(B)$, and note that

$$xFy \Leftrightarrow xEy \text{ and } \lfloor m(x)/n \rfloor = \lfloor m(y)/n \rfloor,$$

where $\lfloor z \rfloor$ denotes the integer part of $z \in \mathbb{R}$, is a subequivalence relation of E whose classes are all of cardinality n.

For the general case, define $\Phi \subseteq [E]^{<\mathbb{N}}$ by

$$\Phi(S) \Leftrightarrow |S| = n,$$

and let $F \subseteq E$ be a Φ-maximal fsr. Clearly $X \setminus \mathrm{dom}(F)$ intersects each E-class in at most $n-1$ points, thus E is smooth off of an E-invariant Borel set on which F is as desired, and we have already seen how to handle the smooth case. $\square$

As can be seen by considering $\Phi = [E]^{<\mathbb{N}}$, a Φ-maximal fsr can be properly contained in another Φ-maximal fsr. As a result, one naturally wonders if there are stronger notions of maximality which satisfy an analog of Proposition 16.6.

A Φ-maximal fsr $F \subseteq E$ is *strongly Φ-maximal* if

$$\forall x \in \mathrm{dom}(F)\, \forall S \in [E]^{<\mathbb{N}}\ (S \cap \mathrm{dom}(F) = \emptyset \Rightarrow \neg\Phi([x]_F \cup S)),$$

or equivalently, if there is no Φ-satisfying fsr $F' \supset F$ such that

$$\forall x \in \mathrm{dom}(F)\ ([x]_F = [x]_{F'} \cap \mathrm{dom}(F)).$$

There are simple obstructions to the existence of such fsr's:

Example 16.8. Set $X = \mathbb{N}$ and $E = \mathbb{N} \times \mathbb{N}$, and define $\Phi \subseteq [E]^{<\mathbb{N}}$ by

$$\Phi(S) \Leftrightarrow 0 \in S.$$

Clearly E does not admit a strongly Φ-maximal fsr.

Nevertheless, a version of Proposition 16.6 goes through in the measure-theoretic context. First, we need some definitions. The *full group* of E is the group $[E]$ of Borel automorphisms $T : X \to X$ such that $\forall x \in X\ (xET(x))$. We say that μ is *E-invariant* if every element of $[E]$ is μ-preserving.

Proposition 16.9. *Suppose that E is a countable Borel equivalence relation on a Polish space X and $\Phi \subseteq [E]^{<\mathbb{N}}$ is Borel. Then there is an E-invariant Borel set $B \subseteq X$ such that:*

1. *$E|B$ admits no invariant probability measures.*
2. *$E|(X \setminus B)$ admits a strongly Φ-maximal fsr.*

Proof. We recursively define an increasing sequence $F_0 \subseteq F_1 \subseteq \cdots$ of maximal fsr's. Fix a Φ-maximal fsr $F_0 \subseteq E$. Suppose now that F_n has been defined, set

$$\Phi_n = \{S \in \Phi : S \cap \mathrm{dom}(F_n) \text{ is an } F_n\text{-class and } S \setminus \mathrm{dom}(F_n) \neq \emptyset\},$$

let F'_n be a Φ_n-maximal fsr, and define $F_{n+1} = F_n \cup F'_n$.

Define $F = \bigcup_{n\in\mathbb{N}} F_n$, let $A \subseteq X$ be the aperiodic part of F, and set $B = [A]_E$. The maximality of each F'_n ensures that $F|(X \setminus B)$ is a strongly Φ-maximal fsr of $E|(X \setminus B)$. As the intersection of each F-class with $\mathrm{dom}(F_0)$ is a single equivalence class of F_0, it follows that $F|A$ is a smooth aperiodic subequivalence relation of $E|A$, thus $E|B$ admits no invariant probability measures. $\square$

There is another direction in which one can improve Proposition 16.6. Given a sequence of Borel sets $\Phi_n \subseteq [E]^{<\mathbb{N}}$, we say that an fsr $F \subseteq E$ is *$\langle \Phi_n \rangle$-satisfying* if it is $(\bigcup_n \Phi_n)$-satisfying and for each $n \in \mathbb{N}$, every equivalence class of E contains an equivalence class of F which satisfies Φ_n. Although it is once again straightforward to see that such an fsr need not always exist, we do have:

Proposition 16.10. *Suppose that X is a Polish space, E is an aperiodic countable Borel equivalence relation on X, and $\langle \Phi_n \rangle_{n \in \mathbb{N}}$ is a sequence of E-complete Borel subsets of $[E]^{<\mathbb{N}}$. Then there is an E-invariant Borel set $B \subseteq X$ such that:*

1. *$E|B$ admits no invariant probability measures;*

2. *$E|(X \setminus B)$ admits a $\langle \Phi_n \rangle$-satisfying fsr.*

Proof. Fix a group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms such that $E = E_\Gamma^X$. We recursively define Borel sets $\Psi_n \subseteq [E]^{<\mathbb{N}}$, fsr's $F_n \subseteq E$, and Borel sets $A_n \subseteq X$. We begin by setting

$$\Psi_n(S) \Leftrightarrow \exists T \subseteq S \ (\Phi_n(T) \text{ and } \forall m < n \ (|[T]_{F_m}|/|A_m \cap S| < 1/2^n)),$$

and letting F_n be a Ψ_n-maximal fsr. Let $B \subseteq X$ be a Borel transversal of F_n, and associate with each $x \in B$ the lexicographically minimal sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that the set

$$T_{[x]_{F_n}} = \{\gamma_{s_k} \cdot x\}_{k<|s|}$$

is contained in $[x]_{F_n}$, satisfies Φ_n, and

$$\forall m < n \ (|[T_{[x]_{F_n}}]_{F_m}| < |A_m \cap S|/2^n).$$

Set $A_n = \bigcup_{x \in B} T_x$.

Once the recursion is complete, define $B_n \subseteq A_n$ by

$$B_n = A_n \setminus \bigcup_{m>n} [A_m]_{F_n},$$

and set $F = \bigcup_{n \in \mathbb{N}} F_n|B_n$. As the sets B_n are pairwise disjoint and $(F_n|A_n)$-invariant, it follows that F is simultaneously Φ_n-satisfying on the set

$$B = \bigcap_{n \in \mathbb{N}} [B_n]_E.$$

It remains to show that $E|(X \setminus B)$ admits no invariant probability measures. Suppose, towards a contradiction, that there exists $n \in \mathbb{N}$ and an invariant probability measure μ such that $A = X \setminus [B_n]_E$ is of positive measure, thus

$$\mu(A \cap \operatorname{dom}(F_n)) > 0.$$

For $m > n$, set $A'_m = A \cap [A_m]_{F_n}$, and observe that

$$\begin{aligned}
\mu(A'_m) &= \int_{A\cap \mathrm{dom}(F_n)} |[T_{[x]_{F_m}}]_{F_n}|/|[x]_{F_m}| \; d\mu(x) \\
&= \int_{A\cap \mathrm{dom}(F_m)} (|[T_{[x]_{F_m}}]_{F_n}|/|A_n \cap [x]_{F_m}|)(|A_n \cap [x]_{F_m}|/|[x]_{F_m}|) \; d\mu(x) \\
&< \int_{A\cap \mathrm{dom}(F_m)} (|A_n \cap [x]_{F_m}|/|[x]_{F_m}|)/2^m \; d\mu(x) \\
&= \mu(A \cap A_n \cap \mathrm{dom}(F_m))/2^m \\
&\leq \mu(A \cap A_n)/2^m.
\end{aligned}$$

It follows that

$$\mu(A \cap A_n) > \sum_{m>0} \mu(A'_m),$$

thus $\mu(A \cap B_n) > 0$, the desired contradiction. □

17 Measures

Here we review some basic facts about probability measures on Polish spaces. We say that μ is *regular* if for every $\epsilon > 0$ and Borel set $B \subseteq X$, there is an open set $U \supseteq B$ such that $\mu(U \setminus B) < \epsilon$.

Proposition 17.1. *Suppose that X is a Polish space and μ is a probability measure on X. Then μ is regular.*

Proof. We will show that each Borel set $B \subseteq X$ satisfies the following:

1. $\forall \epsilon > 0 \, \exists C \subseteq B$ closed $(\mu(B \setminus C) < \epsilon)$;
2. $\forall \epsilon > 0 \, \exists U \supseteq B$ open $(\mu(U \setminus B) < \epsilon)$.

When $B \subseteq X$ is closed, it is clear that (1) holds. To see that (2) holds, fix a compatible Polish metric d on X, and for each $n \in \mathbb{N}$, define $U_n \supseteq B$ by

$$U_n = \{x \in X : d(x, B) < 1/n\}.$$

It is clear that $\langle U_n \rangle$ is a decreasing sequence of open sets whose intersection is B, and (2) follows.

As conditions (1) and (2) hold of B if and only if they hold of $X \setminus B$, it only remains to show that the family of sets which satisfy (1) and (2) is closed under countable unions. Towards this end, suppose that $\langle B_n \rangle_{n\in\mathbb{N}}$ is a sequence of Borel sets for which we have already verified (1) and (2), and set $B = \bigcup_{n\in\mathbb{N}} B_n$. To see (1), suppose that $\epsilon > 0$, fix $n \in \mathbb{N}$ sufficiently large that $\mu(B_n) > \mu(B) - \epsilon/2$, fix $C \subseteq B_n$ closed such that $\mu(C) > \mu(B_n) - \epsilon/2$, and observe that $\mu(B \setminus C) < \epsilon$. To see (2), suppose that $\epsilon > 0$, and for each $n \in \mathbb{N}$, fix $U_n \supseteq B_n$ open such that $\mu(U_n) < \mu(B_n) + \epsilon/2^{n+1}$, set $U = \bigcup_{n\in\mathbb{N}} U_n$, and observe that $\mu(U \setminus B) < \epsilon$. □

We say that μ is *tight* if for every $\epsilon > 0$ and Borel set $B \subseteq X$, there is a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \epsilon$.

Proposition 17.2. *Suppose that X is a Polish space and μ is a probability measure on X. Then μ is tight.*

Proof. By Proposition 17.1, it is enough to show that if $\epsilon > 0$ and $C \subseteq X$ is closed, then there is a compact set $K \subseteq C$ such that $\mu(C \setminus K) < \epsilon$. By Proposition 10.1, it is enough to handle the case that $C = X$. Towards this end, fix a compatible Polish metric d on X, and for each $n \in \mathbb{N}$, fix a sequence $\langle B_{nk} \rangle_{k<k_n}$ of closed balls of diameter at most $1/2^n$ such that

$$\mu\left(\bigcup_{k<k_n} B_{nk}\right) > 1 - \epsilon/2^{n+1},$$

and set $K = \bigcap_{n\in\mathbb{N}} \bigcup_{k<k_n} B_{nk}$. It is clear that $\mu(K) > 1 - \epsilon$ and K is closed. To see that K is compact, it is enough to show that it is sequentially compact. Towards this end, observe that if $\langle x_n \rangle \in K^{\mathbb{N}}$, then we can recursively find a strictly increasing sequence $\langle i_n \rangle \in \mathbb{N}^{\mathbb{N}}$ and a sequence $\langle l_n \rangle \in \mathbb{N}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ (x_{i_n} \in \bigcap_{m \leq n} B_{ml_m})$. Then $x_{i_n} \to x$, for some $x \in K$. $\square$

We next establish a version of the Lebesgue density theorem:

Theorem 17.3. *Suppose that (X, d) is a Polish ultrametric space, μ is a probability measure on X, and $\varphi \in L^1(\mu)$. Then*

$$\varphi(x) = \lim_{\epsilon\to 0} \frac{\int_{B_d(x,\epsilon)} \varphi(y)\ d\mu(y)}{\mu(B_d(x,\epsilon))}\ \mu\text{-a.e.}$$

Proof. We handle first the case that $\varphi = \chi_B$, for some set $B \subseteq X$. Suppose, towards a contradiction, that there exists $\delta > 0$ such that the clopen set

$$C = \{x \in B : \liminf_{\epsilon\to 0} \mu(B \cap B_d(x,\epsilon))/\mu(B_d(x,\epsilon)) \leq 1 - \delta\}$$

is of positive measure. Since μ is tight, there is a compact set $K \subseteq C$ of positive measure and an open set $U \supseteq K$ such that $\mu(K)/\mu(U) > 1-\delta$. For each $x \in K$, fix $\epsilon_x > 0$ such that $B_d(x, \epsilon_x) \subseteq U$ and

$$\mu(B \cap B_d(x, \epsilon_x))/\mu(B_d(x, \epsilon_x)) \leq 1 - \delta.$$

Since K is compact, there exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{1\leq i\leq n} B_d(x_i, \epsilon_i)$. As d is an ultrametric, by removing the appropriate x_i's we can ensure that the sets $B_d(x_i, \epsilon_i)$ partition their union $V = \bigcup_{1\leq i\leq n} B_d(x_i, \epsilon_i)$. Then

$$\begin{aligned} \mu(B \cap V) &= \sum_{1\leq i\leq n} \mu(B \cap B_d(x_i, \epsilon_i)) \\ &\leq (1-\delta) \sum_{1\leq i\leq n} \mu(B_d(x_i, \epsilon_i)) \\ &= (1-\delta)\mu(V), \end{aligned}$$

thus $1-\delta < \mu(K)/\mu(U) \leq \mu(B \cap V)/\mu(V) \leq 1-\delta$, the desired contradiction.

The theorem immediately follows for simple functions, and we leave the straightforward extension to $L^1(\mu)$ to the reader. □

We say that μ is *absolutely continuous* with respect to ν, or $\mu \ll \nu$, if every ν-null set is μ-null. Recall the following basic measure-theoretic fact:

Theorem 17.4 (Radon-Nikodym)**.** *Suppose that $(X, \mathcal{B})$ is a standard Borel space and $\mu \ll \nu$ are probability measures on $(X, \mathcal{B})$. Then there is a Borel function $d\mu/d\nu \in L^1(\mu)$ such that*

$$\int \varphi(x)\ d\mu(x) = \int \varphi(x)(d\mu/d\nu)(x)\ d\nu(x),$$

for all $\varphi \in L^1(\mu)$. Moreover, the function $d\mu/d\nu$ is unique ν-a.e.

We immediately obtain the following corollary:

Proposition 17.5. *Suppose that $(X, \mathcal{B})$ is a standard Borel space, d is a Polish ultrametric on $(X, \mathcal{B})$, and $\mu \ll \nu$ are probability measures on $(X, \mathcal{B})$. Then*

$$(d\mu/d\nu)(x) = \lim_{\epsilon \to 0} \frac{\int_{B_d(x,\epsilon)} (d\mu/d\nu)(y)\ d\nu(y)}{\nu(B_d(x,\epsilon))} = \lim_{\epsilon \to 0} \frac{\mu(B_d(x,\epsilon))}{\nu(B_d(x,\epsilon))}\ \nu\text{-a.e.}$$

We say that μ is *E-quasi-invariant* if $\forall B \in \mathcal{B}\ (\mu(B) = 0 \Rightarrow \mu([B]_E) = 0)$.

Proposition 17.6 (essentially Woodin)**.** *Suppose that $(X, \mathcal{B})$ is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a probability measure on $(X, \mathcal{B})$. Then there is a μ-conull Borel E-complete section $B \subseteq X$ such that $\mu|B$ is $(E|B)$-quasi-invariant.*

Proof. Fix a Polish ultrametric d on $(X, \mathcal{B})$, as well as a countable dense set $C = \{x_n\}_{n \in \mathbb{N}}$. Fix a countable group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of automorphisms of $(X, \mathcal{B})$ such that $E = E_\Gamma^X$. For each triple $(l, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ for which it is possible, fix a Borel set $B_{lmn} \subseteq B_d(x_m, 1/l)$ such that

$$\mu(B_{lmn}) > \mu(B_d(x_m, 1/l))/2 \text{ and } \mu(\gamma_n^{-1}(B_{lmn})) = 0.$$

Each map $\gamma_n^{-1}(B_{lmn}) \xrightarrow{\gamma_n} B_{lmn}$ is a witness to the failure of quasi-invariance. We now remove these witnesses by restricting our attention to the set

$$A = X \setminus \bigcup_{l,m,n} \gamma_n^{-1}(B_{lmn}).$$

Lemma 17.7. *$\mu|A$ is $(E|A)$-quasi-invariant.*

Proof. Suppose, towards a contradiction, that there is a null set $A' \subseteq A$ such that $[A']_{E|A}$ is non-null. Set $A_n = A' \cap \gamma_n^{-1}(A)$, and note that

$$[A']_{E|A} = \bigcup_{n \in \mathbb{N}} \gamma_n(A_n).$$

In particular, it follows that there exists $n \in \mathbb{N}$ such that $\mu(\gamma_n(A_n)) > 0$. By Theorem 17.3, there exist $l, m \in \mathbb{N}$ such that

$$\mu(\gamma_n(A_n) \cap B_d(x_m, 1/l)) > \mu(B_d(x_m, 1/l))/2.$$

Then B_{lmn} exists, and since $\mu(B_{lmn}) > \mu(B_d(x_m, 1/l)/2$, it follows that

$$\gamma_n(A_n) \cap B_{lmn} \neq \emptyset,$$

thus $A_n \cap \gamma_n^{-1}(B_{lmn}) \neq \emptyset$, the desired contradiction. □

It now follows that the set $B = A \cup (X \setminus [A]_E)$ is a conull Borel E-complete section and $\mu|B$ is $(E|B)$-quasi-invariant. □

A function $\rho : E \to \mathbb{R}^+$ is a *cocycle* if $xEyEz \Rightarrow \rho(x, z) = \rho(x, y)\rho(y, z)$.

Proposition 17.8. *Suppose that $(X, \mathcal{B})$ is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant probability measure on $(X, \mathcal{B})$. Then there is a Borel cocycle $\rho : E \to \mathbb{R}^+$ such that*

$$\int \varphi(x)\ dT_*\mu(x) = \int \varphi(x)\rho(T^{-1}(x), x)\ d\mu(x),$$

for all $T \in [E]$ and $\varphi \in L^1(\mu)$.

Proof. By Proposition 10.8, there is a zero-dimensional Polish topology τ on $(X, \mathcal{B})$. Fix a Polish ultrametric d on (X, τ), as well as a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. By Proposition 17.5, the map

$$d_\gamma(x) = \lim_{\epsilon \to 0} \frac{\mu(\gamma^{-1}(B_d(x, \epsilon)))}{\mu(B_d(x, \epsilon))}$$

is equal to $d\gamma_*\mu/d\mu$ on a set of full measure. By throwing out the E-saturation of the complement of the domain of each d_γ, we can assume that d_γ is defined everywhere. Define $A_{\gamma\delta} \subseteq X$ by

$$A_{\gamma\delta} = \{x \in X : \gamma^{-1} \cdot x = \delta^{-1} \cdot x \text{ and } d_\gamma(x) \neq d_\delta(x)\}.$$

As Radon-Nikodym derivatives are unique modulo null sets and $\gamma|A_{\gamma\delta} = \delta|A_{\gamma\delta}$, it follows that each of the sets A_{mn} is null. By throwing out the E-saturation of each of these sets, we can therefore assume that $\gamma^{-1}\cdot x = \delta^{-1}\cdot x \Rightarrow d_\gamma(x) = d_\delta(x)$. We can now unambiguously define ρ on E by putting

$$\rho(y, x) = d_\gamma(x),$$

for any $\gamma \in \Gamma$ such that $y = \gamma^{-1} \cdot x$.

Given $T \in [E]$, fix a partition $\langle B_\gamma \rangle_{\gamma \in \Gamma}$ of X into Borel sets such that $T|B_\gamma = \gamma|B_\gamma$, and observe that

$$
\begin{aligned}
\int \varphi(x)\ dT_*\mu(x) &= \sum_{\gamma \in \Gamma} \int_{T(B_\gamma)} \varphi(x)\ dT_*\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma(B_\gamma)} \varphi(x)\ d\gamma_*\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma(B_\gamma)} \varphi(x) d_\gamma(x)\ d\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma(B_\gamma)} \varphi(x)\rho(\gamma^{-1} \cdot x, x)\ d\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{T(B_\gamma)} \varphi(x)\rho(T^{-1}(x), x)\ d\mu(x) \\
&= \int \varphi(x)\rho(T^{-1}(x), x)\ d\mu(x).
\end{aligned}
$$

Next, we will show that ρ is a cocycle almost everywhere:

Lemma 17.9. *Suppose that $\gamma, \delta \in \Gamma$. Then*

$$\rho(\gamma^{-1}\delta^{-1} \cdot x, x) = \rho(\gamma^{-1}\delta^{-1} \cdot x, \delta^{-1} \cdot x)\rho(\delta^{-1} \cdot x, x)\ \mu\text{-a.e.}$$

Proof. Note first that

$$
\begin{aligned}
\rho(\gamma^{-1}\delta^{-1} \cdot x, \delta^{-1} \cdot x) &= d_\gamma(\delta^{-1} \cdot x) \\
&= \lim_{\epsilon \to 0} \frac{\mu(\gamma^{-1}(B_d(\delta^{-1} \cdot x, \epsilon)))}{\mu(B_d(\delta^{-1} \cdot x, \epsilon))} \\
&= \lim_{\epsilon \to 0} \frac{\mu(\gamma^{-1}\delta^{-1}(B_d(x, \epsilon)))}{\mu(\delta^{-1}(B_d(x, \epsilon)))}\ \mu\text{-a.e.},
\end{aligned}
$$

where the last equality follows from the fact that the pullback of d through the action of γ is also a Polish ultrametric on X which is compatible with its underlying Borel structure. Observing that

$$\rho(\delta^{-1} \cdot x, x) = d_\delta(x) = \lim_{\epsilon \to 0} \frac{\mu(\delta^{-1}(B_d(x, \epsilon)))}{\mu(B_d(x, \epsilon))},$$

it follows that

$$
\begin{aligned}
\rho(\gamma^{-1}\delta^{-1} \cdot x, x) &= \rho(\gamma^{-1}\delta^{-1} \cdot x, \delta^{-1} \cdot x)\rho(\delta^{-1} \cdot x, x) \\
&= \lim_{\epsilon \to 0} \left(\frac{\mu(\gamma^{-1}\delta^{-1}(B_d(x, \epsilon)))}{\mu(\delta^{-1}(B_d(x, \epsilon)))} \right) \left(\frac{\mu(\delta^{-1}(B_d(x, \epsilon)))}{\mu(B_d(x, \epsilon))} \right)\ \mu\text{-a.e.} \\
&= \lim_{\epsilon \to 0} \frac{\mu(\gamma^{-1}\delta^{-1}(B_d(x, \epsilon)))}{\mu(B_d(x, \epsilon))} \\
&= \rho(\gamma^{-1}\delta^{-1} \cdot x, x).
\end{aligned}
$$

which completes the proof of the lemma. □

The E-quasi-invariance of μ now ensures that ρ is a cocycle off of an E-invariant null Borel set $N \subseteq X$, thus by setting $\rho|(E|N) = 1$ be identically 1 on this set, we obtain the desired cocycle. $\square$

We say that μ is *ρ-invariant* if it satisfies the conclusion of Proposition 17.8.

Remark 17.10. Our primary use of ρ-invariance will be when φ is the characteristic function of some Borel set $B \subseteq X$, in which case we obtain that

$$\forall T \in [E] \ (\mu(T^{-1}(B)) = \int_B \rho(T^{-1}(x), x) \ d\mu(x)).$$

Note that when $\rho : E \to \mathbb{R}^+$ is the constant cocycle, this just says that the elements of $[E]$ are all measure-preserving, that is, the measure μ is E-invariant.

18 The existence of σ-finite measures

In this section, we study ρ-invariant, σ-finite measures. We note first that the family of such measures is essentially unaffected if we pass to complete sets:

Proposition 18.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and $B \subseteq X$ is a Borel E-complete set. Then every $(\rho|B)$-invariant, σ-finite measure has a unique extension to a ρ-invariant, σ-finite measure.*

Proof. Suppose that μ is a $(\rho|B)$-invariant, σ-finite measure, fix a countable sequence $\langle T_n \rangle_{n \in \mathbb{N}}$ of elements of $[E]$ and a countable sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of B such that $\langle T_n(B_n) \rangle_{n \in \mathbb{N}}$ partitions X, and define ν on X by putting

$$\nu(B) = \sum_{n \in \mathbb{N}} \int_{B_n \cap T_n^{-1}(B)} \rho(T_n(x), x) \ d\mu(x).$$

It is clear that ν is the only possible ρ-invariant extension of μ, and it is straightforward to check that ν is indeed a ρ-invariant, σ-finite measure. $\square$

In particular, this implies that every σ-finite measure on a Borel partial transversal of E extends to a ρ-invariant, σ-finite measure. Here we will study the circumstances under which there is a ρ-invariant, σ-finite measure which concentrates off of sets on which E is smooth.

A measure μ is *E-ergodic* if every E-invariant Borel set is null or conull.

Exercise 12. Let E_0 denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N} \, \forall m \geq n \ (\alpha(m) = \beta(m)).$$

Show that the $(1/2, 1/2)$ product measure μ on $2^{\mathbb{N}}$ is E_0-ergodic.

We can now spell out the relevance of ergodicity to the problem at hand:

Proposition 18.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and μ is an atomless, E-ergodic, σ-finite measure. Then μ concentrates off of sets on which E is smooth.*

Proof. Suppose, towards a contradiction, that there is a Borel partial transversal $A \subseteq X$ of E such that $0 < \mu(A) < \infty$. Fix a countable sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of A which separates points of A, and set

$$B = \bigcap_{\mu(A_n)=\mu(A)} A_n \setminus \bigcup_{\mu(A_n)=0} A_n.$$

Then $\mu(B) = \mu(A)$, but $|B| \leq 1$, the desired contradiction. □

Exercise 13. Show that the assumptions of ergodicity and atomlessness are necessary in the statement of Proposition 18.2.

In particular, if we wish to find ρ-invariant, σ-finite measures which concentrate off of smooth sets, it is sufficient to find atomless, E-ergodic, ρ-invariant, σ-finite measures. Before saying more about this, we must first discuss some basic observations about cocycles.

Suppose that $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Intuitively, we can think of each E-class as being a single mass which has been divided into countably many pieces. When xEy, we think of $\rho(x, y)$ as the ratio of the mass of x to that of y. For each set $S \subseteq [x]_E$, we use

$$|S|_x = \sum_{y \in S} \rho(y, x)$$

to denote the ratio of the mass of S to that of x.

Let $\tilde{E}$ denote the equivalence relation on $[E]^{<\mathbb{N}}$ given by

$$S\tilde{E}T \Leftrightarrow [S]_E = [T]_E.$$

Note that if $S, T \subseteq [x]_E$, then the quantity $|S|_y/|T|_y$ is independent of the choice of $y \in [x]_E$. We can therefore extend ρ to a cocycle $\tilde{\rho} : \tilde{E} \to \mathbb{R}^+$ by setting

$$\tilde{\rho}(S, T) = |S|_x/|T|_x,$$

for any $x \in [S]_E$.

Although $|S|_x$ depends on the choice of x, whether or not $|S|_x$ is finite does not. We say that S is *ρ-finite* if $|S|_x$ is finite for some $x \in [S]_E$, and we say that S is *ρ-infinite* otherwise. We say that E is *ρ-finite* if all of its classes are ρ-finite, and we say that E is *ρ-aperiodic* if all of its classes are ρ-infinite.

We say that a set $B \subseteq X$ is *ρ-negligible* if it is null with respect to every ρ-invariant probability measure on X, and we say that a statement holds *ρ-almost everywhere* if it is true off of a ρ-negligible set.

Proposition 18.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle.*

1. *If E is ρ-finite, then E is smooth.*

2. *If E is smooth, then ρ-almost every equivalence class of E is ρ-finite.*

Proof. To see (1), note that if E is ρ-finite, then the set

$$B = \{x \in X : \forall y \in [x]_E \ (\rho(y, x) \leq 1)\}$$

has a non-empty, finite intersection with every equivalence class of E. Fix a Borel linear ordering $\leq_X$ of X, and observe that the set

$$A = \{x \in B : \forall y \in B \cap [x]_E \ (y \leq_X x)\}$$

is a Borel transversal of E, thus E is smooth.

To see (2), suppose that E is smooth and μ is a ρ-invariant probability measure on X. The *aperiodic part* of ρ is given by

$$\mathrm{Aper}(\rho) = \{x \in X : [x]_E \text{ is } \rho\text{-infinite}\}.$$

We must show that $\mu(\mathrm{Aper}(\rho)) = 0$. Towards this end, fix a countable group $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms such that $E = E_\Gamma^X$, as well as a Borel transversal B_0 of $E|\mathrm{Aper}(\rho)$. Suppose now that we have found Borel $(E|\mathrm{Aper}(\rho))$-complete sets $B_0, \ldots, B_n$ which intersect every equivalence class of E in only finitely many points. Then for each $x \in B_0$, there is a sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that the corresponding set

$$S_{x,s} = \{\gamma_{s(i)} \cdot x : i < |s|\}$$

is disjoint from $B_0 \cup \cdots \cup B_n$ and $|S_{x,s}|_x \geq 1$. For each $x \in B_0$, let s_x be the lexicographically least such sequence, define $B_{n+1} \subseteq \mathrm{Aper}(\rho)$ by

$$B_{n+1} = \bigcup_{x\in B_0} S_{x,s_x},$$

and observe that $\mu(B_0) \leq \int_{B_0} |S_{x,s_x}|_x \ d\mu(x) = \mu(B_{n+1})$.

As $\mu(X) \geq \sum_{n\in\mathbb{N}} \mu(B_n) \geq \sum_{n\in\mathbb{N}} \mu(B_0)$, it follows that $\mu(B_0) = 0$, and the ρ-quasi-invariance of μ implies that $\mu(\mathrm{Aper}(\rho)) = 0$. □

Given a topological group G, a set $U \subseteq G$, and a cocycle $\rho : E \to G$, we say that a set $B \subseteq X$ is (ρ, U)-*discrete* if

$$\forall (x, y) \in E|B \ (x \neq y \Rightarrow \rho(x, y) \notin U).$$

We say that B is ρ-*discrete* if there is an open neighborhood U of 1_G such that B is (ρ, U)-discrete.

Proposition 18.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and E is ρ-aperiodic. Then every ρ-discrete Borel set is ρ-negligible.*

Proof. We must show that if $B \subseteq X$ is a ρ-discrete Borel set and μ is a ρ-invariant probability measure, then $\mu(B) = 0$. Fix $\epsilon > 0$ such that

$$\forall (x, y) \in E|B \ (x \neq y \Rightarrow \rho(x, y) \notin (1 - \epsilon, 1 + \epsilon)).$$

Proposition 18.3 allows us to assume that for all $x \in B$,

$$\forall \epsilon > 0 \, \exists y, z \in B \cap [x]_E \ (\rho(y, x) < \epsilon \text{ and } \rho(z, x) > 1/\epsilon),$$

since E is smooth on the set of $x \in B$ for which this fails. Define $T : B \to B$ by

$$T(x) = y \Leftrightarrow \rho(y, x) > 1 \text{ and } \forall z \in [x]_E \ (\rho(z, x) > 1 \Rightarrow \rho(z, y) \geq 1),$$

and observe that

$$\mu(B) = \mu(T(B)) = \int_B \rho(T(x), x) \ d\mu(x) \geq (1 + \epsilon)\mu(B),$$

thus $\mu(B) = 0$. □

As a corollary, we see that discreteness rules out the measures we seek:

Proposition 18.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and μ is a ρ-invariant, σ-finite measure on X. Then there is a μ-conull Borel set $B \subseteq X$ such that $E|B$ is smooth.*

Proof. Fix a partition $\langle B_n \rangle_{n \in \mathbb{N}}$ of X into Borel sets of finite measure. By Proposition 18.4, the aperiodic parts of $\rho|(E|B_n)$ are μ-null, thus it follows from Proposition 18.3 that the set $B = \bigcup_{n \in \mathbb{N}} \mathrm{Per}(\rho|(E|B_n))$ is as desired. □

Let $\mathcal{I}_{(\rho,U)}$ denote the σ-ideal generated by the (ρ, U)-discrete Borel sets.

Proposition 18.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, $U \subseteq G$ is an open neighborhood of 1_G with compact closure, and $\rho : E \to G$ is a Borel cocycle. Then $\mathcal{I}_\rho = \mathcal{I}_{(\rho,U)}$.*

Proof. It is enough to show that if V is an open neighborhood of 1_G, then every (ρ, V)-discrete Borel set $B \subseteq X$ can be covered by countably many (ρ, U)-discrete Borel sets. By replacing U with $U \cup U^{-1}$, we can assume that U is symmetric. Let $\mathcal{G}$ denote the graph on B given by

$$\mathcal{G} = \{(x, y) \in E|B : x \neq y \text{ and } \rho(x, y) \in U\}.$$

Lemma 18.7. *$\mathcal{G}$ is locally finite.*

Proof. Fix an open neighborhood W of 1_G such that $WW^{-1} \subseteq V$, as well as $g_1, \ldots, g_n \in G$ such that $U \subseteq g_1 W \cup \cdots \cup g_n W$. If $\mathcal{G}$ is not locally finite, then there exist $x \in B$, $1 \leq i \leq n$, and distinct points $y, z \in B \cap [x]_E$ such that $\rho(x, y), \rho(x, z) \in g_i W$. Then $\rho(y, z) = \rho(y, x)\rho(x, z) \in W^{-1} g_i^{-1} g_i W = W^{-1} W \subseteq V$, which contradicts our assumption that B is (ρ, V)-discrete. □

As $\mathcal{G}$ is locally finite, Proposition 16.3 ensures that $\chi_B(\mathcal{G}) \leq \aleph_0$, thus B is the union of countably many $\mathcal{G}$-discrete Borel sets. As every $\mathcal{G}$-discrete set is necessarily (ρ, U)-discrete, the proposition follows. $\square$

Let $\mathcal{I}_\rho$ denote the σ-ideal generated by the ρ-discrete Borel sets. We say that ρ is *discrete* if $X \in \mathcal{I}_\rho$.

Proposition 18.8. *Suppose X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, and $\rho : E \to G$ is a Borel cocycle. Then the following are equivalent:*

1. *ρ is discrete;*
2. *E admits a ρ-discrete Borel complete set.*

Proof. To see (1) $\Rightarrow$ (2), suppose that ρ is discrete, and fix an open neighborhood U of 1_G with compact closure. By Proposition 18.6, there is a cover of X by countably many (ρ, U)-discrete Borel sets $A_n \subseteq X$. Set $B_n = A_n \setminus \bigcup_{m<n} [A_m]_E$, and observe that the set $B = \bigcup_{n \in \mathbb{N}} B_n$ is as desired.

To see (2) $\Rightarrow$ (1), it is enough to show that if $B \subseteq X$ is (ρ, U)-discrete and $T \in [E]$, then $T(B)$ can be covered with countably many ρ-discrete Borel sets. Towards this end, fix a basis $\langle U_n \rangle_{n \in \mathbb{N}}$ for G, and for all $m, n \in \mathbb{N}$ such that $U_n^{-1} U_m U_n \subseteq U$, define $B_{mn} \subseteq X$ by

$$B_{mn} = \{T(x) : x \in B \text{ and } \rho(T(x), x) \in U_n\}.$$

Lemma 18.9. *B_{mn} is (ρ, U_m)-discrete.*

Proof. Suppose that $T(x)$ and $T(y)$ are distinct E-related points of B_{mn} such that $\rho(T(x), T(y)) \in U_m$, and observe that

$$\rho(x, y) = \rho(x, T(x))\rho(T(x), T(y))\rho(T(y), y) \in U_n^{-1} U_m U_n \subseteq U,$$

which contradicts our assumption that B is (ρ, U)-discrete. $\square$

Now note that for each $x \in B$, there is an open neighborhood U_m of 1_G and an open neighborhood U_n of $\rho(T(x), x)$ such that $U_n^{-1} U_m U_n \subseteq U$, so $T(x) \in B_{mn}$, thus $T(B) \subseteq \bigcup_{m,n} B_{mn}$. $\square$

We say that ρ *is dense around* x if for every open neighborhood U of 1_G, there are infinitely many $y \in [x]_E$ such that $\rho(y, x) \in U$.

Proposition 18.10. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, and $\rho : E \to G$ is a Borel cocycle. Then there is an E-invariant Borel set $B \in \mathcal{I}_\rho$ such that ρ is dense around every point of $X \setminus B$.*

Proof. Fix a neighborhood basis $\langle U_n \rangle_{n\in\mathbb{N}}$ of 1_G. For each $n \in \mathbb{N}$, set

$$B_n = \{x \in X : \forall^\infty y \in [x]_E \ (\rho(y,x) \notin U_n)\},$$

and define a graph $\mathcal{G}_n$ on B_n by putting

$$\mathcal{G}_n = \{(x,y) \in E|B_n : x \neq y \text{ and } \rho(x,y) \in U_n\}.$$

Then $\mathcal{G}_n$ is locally finite, so Proposition 16.3 ensures that $\chi_B(\mathcal{G}_n) \leq \aleph_0$, thus $B_n \in \mathcal{I}_\rho$. Proposition 18.8 implies that $B = \bigcup_{n\in\mathbb{N}} [B_n]_E$ is as desired. □

We say that a set $B \subseteq X$ is *(ρ, U)-bounded* if

$$\forall (x,y) \in E|B \ (\rho(x,y) \in U).$$

We say that B is *ρ-bounded* if there is an open neighborhood U of 1_G with compact closure such that B is (ρ, U)-bounded. Let $\mathcal{I}_\rho^\perp$ denote the σ-ideal generated by the ρ-bounded Borel sets. We say that ρ is *bounded* if $X \in \mathcal{I}_\rho^\perp$.

Proposition 18.11. *Suppose X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, and $\rho : E \to G$ is a Borel cocycle. Then the following are equivalent:*

1. *E is smooth;*

2. *ρ is bounded and discrete.*

Proof. It is clear that (1) ⇒ (2). To see (2) ⇒ 1, observe that if ρ is bounded, then there are countably many open neighborhoods U_n of 1_G with compact closure such that X is the union of (ρ, U_n)-bounded Borel sets B_n. As ρ is discrete, Proposition 18.6 ensures that each B_n can be covered with countably many Borel sets B_{mn} which are (ρ, U_n)-discrete. As any such set is clearly a partial transversal of E, it follows that E is smooth. □

We say that $\rho : E \to \mathbb{R}^+$ is a *Borel coboundary* if there is a Borel function $w : X \to \mathbb{R}^+$ such that $\forall (x,y) \in E \ (\rho(x,y) = w(x)/w(y))$.

Proposition 18.12. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then the following are equivalent:*

1. *ρ is bounded;*

2. *ρ is a Borel coboundary.*

Proof. To see (1) ⇒ (2), suppose that $\langle B_n \rangle_{n\in\mathbb{N}}$ is a sequence of ρ-bounded Borel sets which covers X, and for each $n \in \mathbb{N}$, fix an open neighborhood U_n of 1_G with compact closure such that B_n is (ρ, U_n)-bounded. Set

$$n(x) = \min\{n \in \mathbb{N} : B_n \cap [x]_E \neq \emptyset\},$$

and define $w : X \to \mathbb{R}^+$ by

$$w(x) = \sup\{\rho(x, y) : y \in B_{n(x)} \cap [x]_E\}.$$

Suppose now that xEy. Given $\epsilon > 0$, fix $z \in [x]_E$ such that $w(x) \leq \rho(x, z) \leq w(x) + \epsilon$ and $w(y) \leq \rho(y, z) \leq w(y) + \epsilon$, and observe that

$$\frac{w(x)}{w(y) + \epsilon} \leq \frac{\rho(x, z)}{\rho(y, z)} \leq \frac{w(x) + \epsilon}{w(y)},$$

so $\rho(x, y) = \rho(x, z)/\rho(y, z) = w(x)/w(y)$, thus ρ is a Borel coboundary.

To see (2) $\Rightarrow$ (1), suppose that $w : X \to \mathbb{R}^+$ is a Borel function, and for each $n \in \mathbb{N}$, observe that the set $B_n = w^{-1}(-1/n, n)$ is $(\rho, (-1/n^2, n^2))$-bounded. As $X = \bigcup_{n \in \mathbb{N}} B_n$, it follows that ρ is bounded. □

The following observation reduces the problem of finding atomless, E-ergodic, ρ-invariant, σ-finite measures to that of finding atomless, E-ergodic, E-invariant, σ-finite measures which concentrate on ρ-bounded sets:

Proposition 18.13. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel coboundary, and μ is an invariant, σ-finite measure. Then there is a ρ-invariant, σ-finite measure $\nu \sim \mu$.*

Proof. Fix a Borel function $w : X \to \mathbb{R}^+$ such that $\forall (x, y) \in E$ $(\rho(x, y) = w(x)/w(y))$, and define ν on X by

$$\nu(B) = \int_B w(x)\ d\mu(x).$$

It is clear that $\mu \sim \nu$. To see that ν is ρ-invariant, suppose that $B \subseteq X$ is Borel and $T \in [E]$, and observe that

$$\begin{aligned} \nu(T(B)) &= \int_{T(B)} w(x)\ d\mu(x) \\ &= \int_B w(T(x))\ d\mu(x) \\ &= \int_B w(x)\rho(T(x), x)\ d\mu(x) \\ &= \int_B \rho(T(x), x)\ d\nu(x), \end{aligned}$$

which completes the proof of the proposition. □

An *embedding* of E_0 into E is an injection $\pi : 2^{\mathbb{N}} \to X$ such that

$$\forall \alpha, \beta \in 2^{\mathbb{N}}\ (\alpha E_0 \beta \Leftrightarrow \pi(\alpha) E \pi(\beta)).$$

We say that such an embedding is (ρ, U)-*bounded* if $\pi[2^{\mathbb{N}}]$ is (ρ, U)-bounded.

Theorem 18.14. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, $U \subseteq G$ is an open neighborhood of 1_G, and $\rho : E \to G$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is discrete;*
2. *There is a continuous (ρ, U)-bounded embedding of E_0 into E.*

Proof. To see that conditions (1) and (2) are mutually exclusive, simply note that if ρ is discrete and $\pi : 2^{\mathbb{N}} \to X$ is a Borel (ρ, U)-bounded embedding of E_0 into E, then $\rho|\pi[2^{\mathbb{N}}]$ is bounded and discrete, so Proposition 18.11 implies that $E|\pi[2^{\mathbb{N}}]$ is smooth, thus E_0 is smooth, a contradiction.

It remains to show $\neg(1) \Rightarrow (2)$. Fix a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. By Proposition 10.9 and Exercise 9, there is a finer zero-dimensional Polish topology τ, compatible with the underlying Borel structure of X, with respect to which Γ acts by homeomorphisms and each map of the form $x \mapsto \rho(x, \gamma \cdot x)$ is continuous. Fix a sequence of symmetric open neighborhoods U_n of 1_G with compact closures such that

$$\forall n \in \mathbb{N} \; ((U_0 \cdots U_n)(U_0 \cdots U_n)^{-1} \subseteq U),$$

and fix an increasing, exhaustive sequence of symmetric finite sets $\Gamma_n \subseteq \Gamma$ containing 1_Γ.

We will recursively find clopen subsets $B_0 \supseteq B_1 \supseteq \cdots$ of X and homeomorphisms $\gamma_n \in \Gamma$ such that for all $n \in \mathbb{N}$, the following conditions are satisfied:

1. $B_n \notin \mathcal{I}_\rho$;
2. $\forall x \in B_{n+1} \; (\rho(\gamma_n \cdot x, x) \in U_n)$;
3. $\forall s \in 2^{n+1} \; (\mathrm{diam}(\gamma_s(B_{n+1})) \leq 1/n)$, where $\gamma_s = \gamma_0^{s_0} \cdots \gamma_n^{s_n}$;
4. $\forall s, t \in 2^n \, \forall \gamma \in \Gamma_n \; (\gamma_n(B_{n+1}) \cap \gamma_t^{-1} \gamma \gamma_s(B_{n+1}) = \emptyset)$.

We begin by setting $B_0 = X$. Now suppose that we have found $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_n$ and $\gamma_0, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ which satisfy conditions $(1) - (4)$ below n. For each $\delta \in \Gamma$, let $A_\delta \subseteq X$ be the open set consisting of all $x \in X$ such that:

(a) $x, \delta \cdot x \in B_n$;

(b) $\forall s, t \in 2^n \, \forall \gamma \in \Gamma_n \; (\delta \cdot x \neq \gamma_s^{-1} \gamma \gamma_t \cdot x)$;

(c) $\rho(\delta \cdot x, x) \in U_n$.

Lemma 18.15. *There exists $\delta \in \Gamma$ such that $A_\delta \notin \mathcal{I}_\rho$.*

Proof. Suppose, towards a contradiction, that each A_δ is in $\mathcal{I}_\rho$. Then the set

$$A = B_n \setminus \bigcup_{\delta \in \Gamma} A_\delta$$

is not in $\mathcal{I}_\rho$. Now, for each $x \in A$, the only points $y \in [x]_{E|A}$ for which $\rho(y,x) \in U_n$ are those of the form $y = \gamma_s^{-1}\gamma\gamma_t \cdot x$, where $s,t \in 2^n$ and $\gamma \in \Gamma_n$. In particular, there can only be finitely many such y. It follows that the restriction of ρ to $E|A$ is not dense around any point of A, thus $A \in \mathcal{I}_\rho$ by Proposition 18.10, which is the desired contradiction. □

Fix $\delta \in \Gamma$ such that $A_\delta \notin \mathcal{I}_\rho$ and set $\gamma_n = \delta$. As the set A_δ is τ-open, the continuity of the γ's ensure that A_δ can be covered by countably many τ-clopen sets $C_k \subseteq A_\delta$ such that:

(i) $\forall s \in 2^{n+1}$ $(\mathrm{diam}(\gamma_s(C_k)) < 1/n)$;

(ii) $\forall s,t \in 2^n \, \forall \gamma \in \Gamma_n$ $(\delta(C_k) \cap \gamma_t^{-1}\gamma\gamma_s(C_k) = \emptyset)$.

As $\mathcal{I}_\rho$ is a σ-ideal, there exists $k \in \mathbb{N}$ such that $C_k \notin \mathcal{I}_\rho$. Put $B_{n+1} = C_k$.

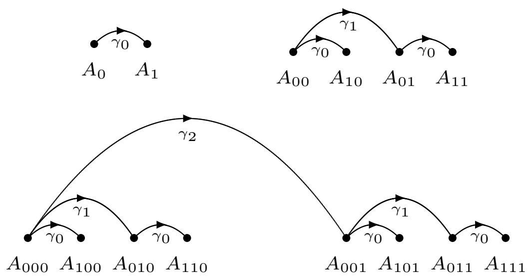

Figure 2: The first three stages of the construction of $\pi : \mathcal{C} \hookrightarrow X$.

Once the recursion is complete, we associate with each $s \in 2^n$ the set

$$A_s = \gamma_s(B_n).$$

For each $\alpha \in 2^{\mathbb{N}}$, conditions (1) and (2) ensure that the sets $A_{\alpha|0}, A_{\alpha|1}, \ldots$ form a decreasing sequence of clopen sets with vanishing diameter. It follows that their intersection consists of a single point $\pi(\alpha)$. By (1) and (2), the function $\pi : 2^{\mathbb{N}} \to X$ is a continuous injection.

To see that $\alpha E_0 \beta \Rightarrow \pi(\alpha) E \pi(\beta)$, it is enough to check the following:

Lemma 18.16. *If $k \in \mathbb{N}$, $s \in 2^k$, and $\alpha \in 2^{\mathbb{N}}$, then $\pi(s\alpha) = \gamma_s \cdot \pi(0^k\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}
\{\pi(s\alpha)\} &= \bigcap_{n\in\mathbb{N}} A_{(s\alpha)|n} \\
&= \bigcap_{n\in\mathbb{N}} A_{s(\alpha|n)} \\
&= \bigcap_{n\in\mathbb{N}} \gamma_s\gamma_{0^k(\alpha|n)}(B_{k+n}) \\
&= \gamma_s\left(\bigcap_{n\in\mathbb{N}} \gamma_{0^k(\alpha|n)}(B_{k+n})\right) \\
&= \gamma_s\left(\bigcap_{n\in\mathbb{N}} A_{0^k(\alpha|n)}\right) \\
&= \gamma_s\left(\bigcap_{n\in\mathbb{N}} A_{(0^k\alpha)|n}\right) \\
&= \gamma_s(\{\pi(0^k\alpha)\}),
\end{aligned}$$

thus $\pi(s\alpha) = \gamma_s \cdot \pi(0^k\alpha)$. □

To see that $(\alpha, \beta) \notin E_0 \Rightarrow (\pi(\alpha), \pi(\beta)) \notin E$, it is enough to check the following:

Lemma 18.17. *Suppose that* $\alpha(n) \neq \beta(n)$. *Then* $\forall\gamma \in \Gamma_n$ $(\gamma \cdot \pi(\alpha) \neq \pi(\beta))$.

Proof. We can assume that $\alpha(n) = 0$ and $\beta(n) = 1$. Suppose, towards a contradiction, that there exists $\gamma \in \Gamma_n$ with $\gamma \cdot \pi(x) = \pi(y)$. Set $s = x|n$ and $t = y|n$, and put

$$x' = \gamma_s^{-1} \cdot \pi(x) \text{ and } y' = \gamma_n^{-1}\gamma_t^{-1} \cdot \pi(y),$$

noting that these are both elements of B_{n+1}. As

$$\gamma\gamma_s \cdot x' = \gamma_t\gamma_n \cdot y',$$

it follows that

$$\gamma_n \cdot y' = \gamma_t^{-1}\gamma\gamma_s \cdot x',$$

which contradicts the fact that $\gamma_n(B_{n+1}) \cap \gamma_t^{-1}\gamma\gamma_s(B_{n+1}) = \emptyset$. □

It only remains to check that π is (ρ, U)-bounded. To see this, note that if $\alpha E_0 \beta$, then there exists $n \in \mathbb{N}$ such that $\forall m > n$ $(\alpha_m = \beta_m)$. Set $s = \alpha(0)\ldots\alpha(n)$ and $t = \beta(0)\ldots\beta(n)$, noting that $\gamma_s^{-1} \cdot \pi(\alpha) = \gamma_t^{-1} \cdot \pi(\beta)$, by Lemma 18.16.

Observe now that

$$\begin{aligned}
\rho(\gamma_s^{-1} \cdot \pi(\alpha), \pi(\alpha)) &= \prod_{i\leq n} \rho(\gamma_{s|(i+1)}^{-1} \cdot \pi(\alpha), \gamma_{s|i}^{-1} \cdot \pi(\alpha)) \\
&\in U_n^{-1} \cdots U_0^{-1},
\end{aligned}$$

and similarly, $\rho(\gamma_t^{-1} \cdot \pi(\beta), \pi(\beta)) \in U_n^{-1} \cdots U_0^{-1}$, thus

$$\begin{aligned} \rho(\pi(\alpha), \pi(\beta)) &= \rho(\pi(\alpha), \gamma_s^{-1} \cdot \pi(\alpha))\rho(\gamma_t^{-1} \cdot \pi(\beta), \pi(\beta)) \\ &\in (U_0 \cdots U_n)(U_0 \cdots U_n)^{-1} \subseteq U, \end{aligned}$$

which completes the proof of the theorem. □

We are now prepared to give the promised characterization:

Theorem 18.18. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then the following are equivalent:*

1. *There is an atomless, ρ-invariant, E-ergodic, σ-finite measure;*
2. *There is a ρ-invariant σ-finite measure on X which concentrates off sets on which E is smooth;*
3. *ρ is not discrete.*

Proof. Propositions 18.2 and 18.5 give (1) ⇒ (2) ⇒ (3), so it only remains to show (3) ⇒ (1). By Theorem 18.14, there is a $(\rho, (1/2, 3/2))$-bounded embedding $\pi : 2^{\mathbb{N}} \to X$ of E_0 into E. Set $B = \pi(2^{\mathbb{N}})$, and note that we can push through the product measure on $2^{\mathbb{N}}$ to an $(E|B)$-invariant probability measure μ. By Propositions 18.12 and 18.13, this is equivalent to a $\rho|(E|B)$-invariant, σ-finite measure ν, and Proposition 18.1 ensures that ν extends to the desired ρ-invariant, σ-finite measure. □

Exercise 14. Show that if X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, then the following are equivalent:

1. There is an atomless, ρ-invariant, E-ergodic, σ-finite measure;
2. There is a family of $\mathfrak{c}$-many atomless, ρ-invariant, E-ergodic, σ-finite measure with pairwise disjoint supports.

Exercise 15. Show that if X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, then the following are equivalent:

1. There is an atomless, ρ-invariant, E-ergodic, σ-finite measure;
2. There is a Polish topology τ on X, compatible with the underlying Borel structure of X, such that for every τ-comeager Borel set $C \subseteq X$, there is an atomless $\rho|(E|C)$-invariant, $(E|C)$-ergodic, σ-finite measure.

Exercise 16. Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Show that the following are equivalent:

1. There is a Borel automorphism $T : X \to X$ such that $E = E_T^X$;
2. There is a Borel cocycle $\rho : E \to \mathbb{R}^+$ such that there are no atomless, ρ-invariant, E-ergodic, σ-finite measures.

19 Building measures

Given a set X, an algebra $\mathcal{U}$ of subsets of X, and a finitely additive probability measure $\mu : \mathcal{U} \to [0,1]$, define $\mu^* : \mathcal{P}(X) \to [0,1]$ by

$$\mu^*(B) = \inf_{\mathcal{V} \subseteq \mathcal{U} \text{ covers } B} \sum_{V \in \mathcal{V}} \mu(V).$$

Proposition 19.1. *Suppose that $\mathcal{U}$ is an algebra of subsets of X and μ is a finitely additive probability measure on $(X, \mathcal{U})$. Then μ^* is an outer measure.*

Proof. It is clear that $\mu^*(\emptyset) \leq \mu(\emptyset) = 0$, and if $A \subseteq B$, then every cover of B is a cover of A, thus $\mu^*(A) \leq \mu^*(B)$. It only remains to check that μ^* is subadditive. Towards this end, suppose that $B = \bigcup_{n \in \mathbb{N}} B_n$, and given $\epsilon > 0$, fix covers $\mathcal{U}_n \subseteq \mathcal{U}$ of B_n such that

$$\mu^*(B_n) + \epsilon/2^{n+1} \geq \sum_{U \in \mathcal{U}_n} \mu(U).$$

It now follows that

$$\begin{aligned} \epsilon + \sum_{n \in \mathbb{N}} \mu^*(B_n) &= \sum_{n \in \mathbb{N}} \mu^*(B_n) + \epsilon/2^{n+1} \\ &\geq \sum_{n \in \mathbb{N}} \sum_{U \in \mathcal{U}_n} \mu(U) \\ &\geq \mu^*(B), \end{aligned}$$

since $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ covers B, and this implies that $\mu^*(B) \leq \sum_{n \in \mathbb{N}} \mu^*(B_n)$. □

As a consequence, we obtain the following:

Proposition 19.2. *Suppose that $\mathcal{U}$ is an algebra of subsets of X, μ is a finitely additive probability measure on $(X, \mathcal{U})$, and $\mathcal{B}$ is the σ-algebra generated by $\mathcal{U}$. Then μ^* induces a measure on $(X, \mathcal{B})$.*

Proof. By Carathéodory's Theorem, it is enough to show that

$$\forall U \in \mathcal{U} \, \forall B \subseteq X \; (\mu^*(B \cap U) + \mu^*(B \setminus U) \leq \mu^*(B)).$$

Towards this end, suppose that $U \in \mathcal{U}$ and $B \subseteq X$, and given $\epsilon > 0$, fix a cover $\mathcal{V} \subseteq \mathcal{U}$ of B such that $\mu^*(B) + \epsilon \geq \sum_{V \in \mathcal{V}} \mu(V)$, and observe that

$$\begin{aligned} \mu^*(B \cap U) + \mu^*(B \setminus U) &\leq \sum_{V \in \mathcal{V}} \mu(V \cap U) + \sum_{V \in \mathcal{V}} \mu(V \setminus U) \\ &= \sum_{V \in \mathcal{V}} \mu(V) \\ &\leq \mu^*(B) + \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\mu^*(B \cap U) + \mu^*(B \setminus U) \leq \mu^*(B)$. □

We say that μ is *strongly regular* if for every $U \in \mathcal{U}$ and $\epsilon > 0$, there is a sequence $\langle U_n \rangle \in \mathcal{U}^{\mathbb{N}}$ of subsets of U of diameter $\leq \epsilon$ such that

$$\mu(U) = \lim_{n \to \infty} \mu\Big(\bigcup_{m \leq n} U_m \Big).$$

Proposition 19.3. *Suppose that (X, d) is a Polish metric space, $\mathcal{U}$ is an algebra of sets which is a basis for (X, d), and μ is a strongly regular finitely additive probability measure on (X, d). Then $\mu = \mu^* | \mathcal{U}$.*

Proof. Suppose, towards a contradiction, that there exists $U \in \mathcal{U}$ and $\epsilon > 0$ such that $\mu^*(U) < \mu(U) - \epsilon$. For each $j \in \mathbb{N}$, fix a sequence $\langle U_{ij} \rangle \in \mathcal{U}^{\mathbb{N}}$ of subsets of U of diameter $\leq \epsilon$ such that

$$\mu(U) = \lim_{k \to \infty} \mu\Big(\bigcup_{i<k} U_{ij} \Big),$$

as well as $k_j \in \mathbb{N}$ with $\mu(U) - \epsilon/2^{j+1} \leq \mu(\bigcup_{i<k_j} U_{ij})$. Set $K = \bigcap_{j \in \mathbb{N}} \bigcup_{i<k_j} U_{ij}$.

Lemma 19.4. *K is compact.*

Proof. It is enough to show that K is sequentially compact. Towards this end, observe that if $\langle x_n \rangle \in K^{\mathbb{N}}$, then we can recursively find a sequence $\langle i_n \rangle \in \prod_{n \in \mathbb{N}} k_n$ and a strictly increasing sequence $\langle j_n \rangle \in \mathbb{N}^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \, \Big(x_{j_n} \in \bigcap_{m \leq n} U_{i_m m} \Big).$$

It follows that $\langle x_{j_n} \rangle$ is Cauchy, thus the point $x = \lim_{n \to \infty} x_{j_n}$ is in K. □

As $K \subseteq U$, it follows that there is a finite cover $\mathcal{V} \subseteq \mathcal{U}$ of K such that

$$\sum_{V \in \mathcal{V}} \mu(V) \leq \mu(U) - \epsilon.$$

Lemma 19.5. *There exists $l \in \mathbb{N}$ such that $\bigcap_{j<l} \bigcup_{i<k_j} U_{ij} \subseteq \bigcup \mathcal{V}$.*

Proof. Suppose, towards a contradiction, that for each $j \in \mathbb{N}$, there exists

$$x_j \in \bigcap_{j<l} \bigcup_{i<k_j} U_{ij} \setminus \bigcup \mathcal{V}.$$

Then there is a sequence $\langle i_n \rangle \in \prod_{n \in \mathbb{N}} k_n$ as well as an increasing sequence $\langle j_n \rangle \in \mathbb{N}^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \, \Big(x_{j_n} \in \bigcap_{m \leq n} U_{i_m m} \Big).$$

It follows that $\langle x_{j_n} \rangle$ is Cauchy, thus the point $x = \lim_{n \to \infty} x_{j_n}$ is in $K \setminus \bigcup \mathcal{V}$, the desired contradiction. □

It now follows that $\mu(U) - \epsilon < \mu(\bigcap_{j<l} \bigcup_{i<k_j} U_{ij}) \leq \sum_{V \in \mathcal{V}} \mu(V) \leq \mu(U) - \epsilon$, which is the desired contradiction. ☐

Let $C_b(X)$ denote the space of continuous, bounded functions $\varphi : X \to \mathbb{R}$. We say that a linear space $\Phi \subseteq C_b(X)$ *contains* a set $\mathcal{U} \subseteq \mathcal{P}(X)$ if $\forall U \in \mathcal{U}\ (\mathbf{1}_U \in \Phi)$. A *mean* on Φ is a positive linear functional $I : \Phi \to \mathbb{R}$ such that $I(\mathbf{1}) = 1$. We say that I is *strongly regular* if Φ contains an algebra of sets $\mathcal{U}$ which is a basis for X, and the finitely additive probability measure on $(X, \mathcal{U})$ given by $\mu(U) = I(\mathbf{1}_U)$ is strongly regular. Associated with each strongly regular mean I on Φ is the mean I^* on $C_b(X)$ given by

$$I^*(\varphi) = \int \varphi \ d\mu^*.$$

Proposition 19.6. *Suppose that (X, d) is a Polish metric space, Φ is a linear subspace of $C_b(X)$, and I is a strongly regular mean on Φ. Then $I = I^*|\Phi$.*

Proof. Fix a set $\mathcal{U}$ which witnesses the strong regularity of I. Given $\varphi \in \Phi$ and $\epsilon > 0$, fix a partition $\mathcal{V} \subseteq \mathcal{U}$ of X and a function $\psi : \mathcal{V} \to \mathbb{R}$ such that

$$\forall V \in \mathcal{V}\, \forall x \in V\ (\psi(V) \leq \varphi(x) \leq \psi(V) + \epsilon),$$

as well as a finite set $\mathcal{W} \subseteq \mathcal{V}$ such that $\sum_{W \in \mathcal{W}} \mu(W) \geq 1 - \epsilon$. Set $W' = X \setminus \bigcup \mathcal{W}$ and $b = \sup_{x \in X} \varphi(x)$, and observe that

$$\begin{aligned} I(\varphi) &\leq I\Big(b\mathbf{1}_{W'} + \sum_{W \in \mathcal{W}} (\psi(W) + \epsilon)\mathbf{1}_W\Big) \\ &= I^*\Big(b\mathbf{1}_{W'} + \sum_{W \in \mathcal{W}} (\psi(W) + \epsilon)\mathbf{1}_W\Big) \\ &\leq b\epsilon + \epsilon + I^*\Big(\sum_{W \in \mathcal{W}} \psi(W)\mathbf{1}_W\Big) \\ &\leq (b+1)\epsilon + I^*(\varphi). \end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $I(\varphi) \leq I^*(\varphi)$. A similar argument shows that $I(\varphi) \geq I^*(\varphi)$, and the proposition follows. ☐

20 The existence of probability measures

Given $\varphi : X \to \mathbb{R}$, $x \in X$, and a ρ-finite set $S \subseteq [x]_E$, let

$$I_S(\varphi) = \frac{\sum_{y \in S} \varphi(y)\rho(y, x)}{\sum_{y \in S} \rho(y, x)}.$$

The definition of cocycle ensures that $I_S(\varphi)$ does not depend on the choice of $x \in [S]_E$. The quantity $I_S(\varphi)$ is the "best guess" at the value of $\int \varphi \ d\mu$, where μ is ρ-invariant, from the data $\varphi|S$ and $\rho|(E|S)$. Given $B \subseteq X$, we set also

$$\mu_S(B) = I_S(\mathbf{1}_B) = \frac{\sum_{y \in B \cap S} \rho(y, x)}{\sum_{y \in S} \rho(y, x)} = \tilde{\rho}(B \cap S, S).$$

Again, the quantity $\mu_S(B)$ can be thought of as the "best guess" at $\mu(B)$.

Proposition 20.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, μ is a ρ-invariant probability measure on X, $F \subseteq E$ is a ρ-finite Borel equivalence relation, and $\varphi : X \to \mathbb{R}$ is a Borel function. Then*

$$\int \varphi(x)\ d\mu(x) = \int I_{[x]_F}(\varphi)\ d\mu(x).$$

Proof. We begin with the following often useful fact:

Lemma 20.2. *Suppose that $B \subseteq X$ is a Borel transversal of F. Then*

$$\int \varphi(x)\ d\mu(x) = \int_B I_{[x]_F}(\varphi)|[x]_F|_x\ d\mu(x).$$

Proof. Fix a countable group $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms such that $F = E_\Gamma^X$, define $B_n \subseteq B$ by

$$B_n = \{x \in B : \forall i < n\ (\gamma_n \cdot x \neq \gamma_i \cdot x)\},$$

noting that the sets $\gamma_n(B_n)$ partition X, and observe that

$$\begin{aligned}
\int \varphi(x)\ d\mu(x) &= \int \sum_{n\in\mathbb{N}} \varphi(x)\chi_{\gamma_n(B_n)}(x)\ d\mu(x) \\
&= \int \sum_{n\in\mathbb{N}} \varphi(\gamma_n \cdot x)\chi_{B_n}(x)\ d(\gamma_n^{-1})_*\mu(x) \\
&= \int \sum_{n\in\mathbb{N}} \varphi(\gamma_n \cdot x)\chi_{B_n}(x)\rho(\gamma_n \cdot x, x)\ d\mu(x) \\
&= \int_B \sum_{y\in[x]_F} \varphi(y)\rho(y,x)\ d\mu(x).
\end{aligned}$$

The latter integrand is clearly equal to $I_{[x]_F}(\varphi)|[x]_F|_x$. ☐

By Proposition 18.3, there is a Borel transversal $B \subseteq X$ of F. Then

$$\begin{aligned}
\int \varphi(x)\ d\mu(x) &= \int_B I_{[x]_F}(\varphi)|[x]_F|_x\ d\mu(x) \\
&= \int_B I_{[x]_F}(I_{[x]_F}(\varphi))|[x]_F|_x\ d\mu(x) \\
&= \int I_{[x]_F}(\varphi)\ d\mu(x),
\end{aligned}$$

which completes the proof of the proposition. ☐

As a consequence, we obtain a version of the Hurewicz ergodic theorem:

Proposition 20.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, μ is a ρ-invariant probability measure on X, and $F_0 \subseteq F_1 \subseteq \cdots$ is an increasing sequence of ρ-finite Borel equivalence relations whose union is E. Then for every Borel set $B \subseteq X$, the limit $\mu_x(B) = \lim_{n\to\infty} \mu_{[x]_{F_n}}(B)$ exists μ-almost everywhere, and*

$$\mu(B) = \int \mu_x(B)\ d\mu(x).$$

In particular, if μ is E-ergodic, then $\mu_x(B) = \mu(B)$ almost everywhere.

Proof. First, we will show that $\mu(B) \geq \int \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x)$. Fix $\epsilon > 0$, and choose $k \in \mathbb{N}$ sufficiently large that the set

$$A = \{x \in X : \exists i < k\ (\mu_{[x]_{F_i}}(B) \geq \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B) - \epsilon)\}$$

is of measure at least $1 - \epsilon$. For each $x \in A$, let $i(x) < k$ be maximal such that

$$\mu_{[x]_{F_{i(x)}}}(B) \geq \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B) - \epsilon,$$

and define an equivalence relation $F' \subseteq F_k$ by

$$xF'y \Leftrightarrow (x, y \in A \text{ and } i(x) = i(y) \text{ and } xF_iy),$$

where $i = i(x) = i(y)$. Proposition 20.1 ensures that

$$\begin{aligned} \mu(B) &= \int_A \mu_{[x]_{F'}}(B)\ d\mu(x) + \int_{X\setminus A} \mu_{[x]_{F'}}(B)\ d\mu(x) \\ &\geq \int_A \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B) - \epsilon\ d\mu(x) \\ &\geq \int \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x) - 2\epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\mu(B) \geq \int \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x)$.

The same argument shows that $\mu(B) \leq \int \liminf_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x)$, thus

$$\mu(B) = \int \liminf_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x) = \int \limsup_{n\to\infty} \mu_{[x]_{F_n}}(B)\ d\mu(x).$$

It follows that $\mu_x(B) = \lim_{n\to\infty} \mu_{[x]_{F_n}}(B)$ exists almost everywhere, and

$$\mu(B) = \int \mu_x(B)\ d\mu(x).$$

It is clear that the map $x \mapsto \mu_x(B)$ is E-invariant, thus if μ is E-ergodic, then $\mu_x(B) = \mu(B)$ almost everywhere. □

We say that an E-invariant Borel set $B \subseteq X$ is *ρ-negligible of type* 1 if there is an increasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E and a partition $\langle B_k \rangle_{k \in \mathbb{N}}$ of B into Borel sets such that

$$\forall x \in B \left(\sum_{k \in \mathbb{N}} \limsup_{n \to \infty} \mu_{[x]_{F_n}}(B_k) < 1 \right).$$

Let $\mathcal{J}_\rho$ denote the σ-ideal generated by sets which are ρ-negligible of type 1.

Proposition 20.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then every set which is ρ-negligible of type 1 is ρ-negligible.*

Proof. Suppose that $B \subseteq X$ is ρ-negligible of type 1, as witnessed by $(\langle F_n \rangle, \langle B_k \rangle)$. If μ is a ρ-invariant probability measure, then Proposition 20.3 ensures that

$$\begin{aligned} \mu(B) &= \sum_{k \in \mathbb{N}} \mu(B_k) \\ &= \int \sum_{k \in \mathbb{N}} \limsup_{n \to \infty} \mu_{[x]_{F_n}}(B_k) \ d\mu(x) \\ &= \int_B \sum_{k \in \mathbb{N}} \limsup_{n \to \infty} \mu_{[x]_{F_n}}(B_k) \ d\mu(x), \end{aligned}$$

thus $\mu(B) = 0$. □

We say that an E-invariant Borel set $B \subseteq X$ is *ρ-negligible of type* 2 if there is a finite Borel subequivalence relation F of E, an automorphism $T \in [E]$, and a Borel $(E|B)$-complete set $A \subseteq B$ such that

$$\forall x \in B \ (\tilde{\rho}(T(A \cap [x]_F), T(A) \cap [x]_F) > 1).$$

Let $\mathcal{K}_\rho$ denote the σ-ideal generated by sets which are ρ-negligible of type 2.

Proposition 20.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then every set which is ρ-negligible of type 2 is ρ-negligible.*

Proof. Suppose that $B \subseteq X$ is ρ-negligible of type 2, as witnessed by (F, T, A). If μ is a ρ-invariant probability measure, then

$$\begin{aligned} \mu(T(A)) &= \int_A \rho(T(x), x) \ d\mu(x) \\ &= \int_B \frac{\sum_{y \in A \cap [x]_F} \rho(T(y), y) \rho(y, x)}{\sum_{y \in [x]_F} \rho(y, x)} \ d\mu(x) \\ &= \int_B \tilde{\rho}(T(A \cap [x]_F), [x]_F) \ d\mu(x), \end{aligned}$$

and similarly,

$$\begin{aligned}\mu(T(A)) &= \int_B \mu_{[x]_F}(T(A))\ d\mu(x) \\ &= \int_B \tilde{\rho}(T(A)\cap[x]_F,[x]_F)\ d\mu(x).\end{aligned}$$

As $\forall x \in B\ (\tilde{\rho}(T(A \cap [x]_F), T(A) \cap [x]_F) > 1)$, it follows that $\mu(B) = 0$. □

Our main goal in this section is to show that membership in $\mathcal{J}_\rho \vee \mathcal{K}_\rho$ is the only obstacle to the existence of ρ-invariant probability measures. In view of our earlier results, we must certainly prove the following:

Proposition 20.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and E is ρ-aperiodic. Then $\mathcal{I}_\rho \subseteq \mathcal{J}_\rho \vee \mathcal{K}_\rho$.*

Proof. It is enough to show that if $B \subseteq X$ is an E-invariant Borel set in $\mathcal{I}_\rho$, then B is in $\mathcal{J}_\rho \vee \mathcal{K}_\rho$. We deal first with the special case that $E|B$ is smooth. To handle this, simply partition B into Borel partial transversals $B_k \subseteq X$ of $E|B$, put $k_0(x) = 0$, recursively define $k_n : B \to \mathbb{N}$ by

$$k_{n+1}(x) = \min\{k : \tilde{\rho}((B_0 \cup \cdots \cup B_{k_n(x)}) \cap [x]_E, (B_0 \cup \cdots \cup B_k) \cap [x]_E) \leq 1\},$$

and define a finite Borel equivalence relation $F_n \subseteq E$ by setting

$$xF_ny \Leftrightarrow x = y \text{ or } (xEy \text{ and } x, y \in B_0 \cup \cdots \cup B_{k_n(x)}).$$

It is clear that $\mu_{[x]_{F_n}}(B_k) \to 0$, for all $x \in B$ and $k \in \mathbb{N}$, thus $B \in \mathcal{J}_\rho$.

We will now handle the general case. Fix $\epsilon > 0$ such that

$$\forall (x, y) \in E|B\ (x \neq y \Rightarrow \rho(x, y) \notin (1 - \epsilon, 1 + \epsilon)).$$

By our previous observation, we can assume that for all $x \in B$,

$$\forall \epsilon > 0\, \exists y, z \in B \cap [x]_E\ (\rho(y, x) < \epsilon \text{ and } \rho(z, x) > 1/\epsilon),$$

since E is smooth on the set of $x \in B$ for which this fails. Set $A = B$ and $F = \Delta(X)$, and define $T : B \to B$ by

$$T(x) = y \Leftrightarrow \rho(y, x) > 1 \text{ and } \forall z \in [x]_E\ (\rho(z, x) > 1 \Rightarrow \rho(z, y) \geq 1).$$

It is clear that $\tilde{\rho}(T(A \cap [x]_F), T(A) \cap [x]_F) > 1$ for all $x \in B$, thus $B \in \mathcal{K}_\rho$. □

Given a Borel function $\varphi : X \to \mathbb{R}$ and $\epsilon > 0$, we say that a finite Borel equivalence relation $F \subseteq E$ is (φ, ϵ)-*approximating* if

$$\forall x \in X\, \forall y, z \in [x]_E\ \left(|I_{[y]_F}(\varphi) - I_{[z]_F}(\varphi)| \leq \epsilon\right).$$

That is, within every equivalence class $[x]_E$, all of the local approximations $I_{[y]_F}(\varphi)$ to $\int \varphi(y)\ d\mu(y)$ are within ϵ of one another. Driving the results here is the fact that such approximations can always be refined:

Proposition 20.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, E is ρ-aperiodic, $\varphi : X \to \mathbb{R}$ is Borel, $\epsilon > 0$, and $F \subseteq E$ is a finite Borel equivalence relation which is (φ, ϵ)-approximating. Then there is an E-invariant Borel set $B \in \mathcal{I}_\rho$ and a finite Borel equivalence relation $F \subseteq F' \subseteq E$ such that $F'|(X \setminus B)$ is $(\varphi|(X \setminus B), 3\epsilon/4)$-approximating.*

Proof. We associate with each equivalence class C of E the quantity

$$I_C(\varphi) = \frac{1}{2}\left(\inf_{x \in C} I_{[x]_F}(\varphi) + \sup_{x \in C} I_{[x]_F}(\varphi)\right).$$

Define $\Phi \subseteq [E]^{<\infty}$ by

$$\Phi(S) \Leftrightarrow (S \text{ is } F\text{-invariant and } |I_S(\varphi) - I_{[S]_E}(\varphi)| \leq \epsilon/4).$$

By Proposition 16.6, there is a Φ-maximal fsr $F'' \subseteq E$. Set $F' = F \cup F''$ and

$$B = \{x \in X : \exists y, z \in [x]_E \ (|I_{[y]_{F'}}(\varphi) - I_{[z]_{F'}}(\varphi)| > 3\epsilon/4)\}.$$

Then B is E-invariant and $F'|(X \setminus B)$ is $(\varphi|(X \setminus B), 3\epsilon/4)$-approximating, so it only remains to prove that $B \in \mathcal{I}_\rho$.

Suppose, towards a contradiction, that $B \notin \mathcal{I}_\rho$. Fix a Borel transversal A of $F|B$, and define a Borel cocycle $\rho' : E|A \to \mathbb{R}^+$ by setting

$$\rho'(x, y) = \tilde{\rho}([x]_F, [y]_F).$$

Lemma 20.8. $B \notin \mathcal{I}_{\rho'}$.

Proof. Suppose, towards a contradiction, that $B \in \mathcal{I}_{\rho'}$. For each $n \in \mathbb{N}$, define

$$A_n = \{x \in A : |[x]_F|_x \leq n)\}.$$

Observe now that if $x, y \in A_n$, then

$$\rho(x, y) = \tilde{\rho}(\{x\}, [x]_F)\tilde{\rho}([x]_F, [y]_F)\tilde{\rho}([y]_F, \{y\}) = (1/|[x]_F|_x)\rho'(x, y)|[y]_F|_y,$$

thus $(1/n)\rho'(x, y) \leq \rho(x, y) \leq n\rho'(x, y)$. As $A_n \in \mathcal{I}_{\rho'}$, it follows from Proposition 18.10 that $A_n \in \mathcal{I}_\rho$. As $A = \bigcup_{n \in \mathbb{N}} A_n$, this means that $A \in \mathcal{I}_\rho$. Proposition 18.8 then implies that $B \in \mathcal{I}_\rho$, the desired contradiction. □

Now define $(E|B)$-complete sections $Y, Z \subseteq A$ by setting

$$Y = \{y \in A : I_{[y]_{F'}}(\varphi) < I_{[y]_E}(\varphi) - \epsilon/4\}$$

and

$$Z = \{z \in A : I_{[z]_{F'}}(\varphi) > I_{[z]_E}(\varphi) + \epsilon/4\},$$

noting that Y and Z are disjoint from $\mathrm{dom}(F'')$, thus $F|(Y \cup Z) = F'|(Y \cup Z)$. It follows from Proposition 18.10 and Lemma 20.8 that there exists $x \in A$ such that the restrictions of ρ' to $E|Y$ and $E|Z$ are dense around every point of $Y \cap [x]_E$ and $Z \cap [x]_E$.

Fix $y \in Y \cap [x]_E$ and $z \in Z \cap [x]_E$, and choose $m, n \in \mathbb{N}$ such that

$$2/3 \leq (m/n)\,\rho'(y,z) \leq 3/2,$$

as well as $\delta > 0$ sufficiently small that

$$\delta/(m|[y]_F|_x), \delta/(n|[z]_F|_x) < 1/2.$$

Fix pairwise distinct points $y_i \in [x]_E \cap Y$ and $z_j \in [x]_E \cap Z$ such that

$$\forall i,j \in \mathbb{N}\ (1 \leq \rho'(y_i, y), \rho'(z_j, z) \leq 1 + \delta).$$

Set $Y' = \bigcup_{i<m} [y_i]_F$ and $Z' = \bigcup_{j<n} [z_j]_F$, and note that

$$m|[y]_F|_x \leq |Y'|_x \leq m(|[y]_F|_x + \delta)$$

and

$$n|[z]_F|_x \leq |Z'|_x \leq n(|[z]_F|_x + \delta),$$

thus

$$\frac{m|[y]_F|_x}{n(|[z]_F|_x + \delta)} \leq \frac{|Y'|_x}{|Z'|_x} \leq \frac{m(|[y]_F|_x + \delta)}{n|[z]_F|_x}.$$

As the middle quantity is by definition $\tilde{\rho}(Y', Z')$, it follows that

$$\tilde{\rho}(Y', Z') \leq (m/n)\rho'(y,z) + \delta/(n|[z]_F|_x) \leq 2$$

and

$$\tilde{\rho}(Z', Y') \leq (n/m)\rho'(z,y) + \delta/(m|[y]_F|_x) \leq 2,$$

so $\tilde{\rho}(Y' \cup Z', Y'), \tilde{\rho}(Y' \cup Z', Z') \leq 3$.

Now observe that

$$\begin{aligned}
I_{Y' \cup Z'}(\varphi) &= \frac{\sum_{y' \in Y'} \varphi(y')\rho(y',x) + \sum_{z' \in Z'} \varphi(z')\rho(z',x)}{\sum_{w' \in Y' \cup Z'} \rho(w',x)} \\
&= \left(\frac{\sum_{y' \in Y'} \varphi(y')\rho(y',x)}{\sum_{y' \in Y'} \rho(y',x)}\right)\left(\frac{\sum_{y' \in Y'} \rho(y',x)}{\sum_{w' \in Y' \cup Z'} \rho(w',x)}\right) + \\
&\quad \left(\frac{\sum_{z' \in Z'} \varphi(z')\rho(z',x)}{\sum_{z' \in Z'} \rho(z',x)}\right)\left(\frac{\sum_{z' \in Z'} \rho(z',x)}{\sum_{w' \in Y' \cup Z'} \rho(w',x)}\right) \\
&= \tilde{\rho}(Y', Y' \cup Z') I_{Y'}(\varphi) + \tilde{\rho}(Z', Y' \cup Z') I_{Z'}(\varphi).
\end{aligned}$$

As $\tilde{\rho}(Y', Y' \cup Z'), \tilde{\rho}(Z', Y' \cup Z') \geq 1/3$ and $\tilde{\rho}(Y', Y' \cup Z') + \tilde{\rho}(Z', Y' \cup Z') = 1$, it follows that

$$\begin{aligned}
I_{Y' \cup Z'}(\varphi) &= \tilde{\rho}(Y', Y' \cup Z') I_{Y'}(\varphi) + \tilde{\rho}(Z', Y' \cup Z') I_{Z'}(\varphi) \\
&\leq (1/3) I_{Y'}(\varphi) + (2/3) I_{Z'}(\varphi) \\
&\leq (1/3)(I_{[x]_E}(\varphi) - \epsilon/4) + (2/3)(I_{[x]_E}(\varphi) + \epsilon/2) \\
&= I_{[x]_E}(\varphi) + \epsilon/4,
\end{aligned}$$

and similarly,

$$\begin{aligned} I_{Y'\cup Z'}(\varphi) &= \tilde{\rho}(Y', Y'\cup Z')I_{Y'}(f) + \tilde{\rho}(Z', Y'\cup Z')I_{Z'}(\varphi) \\ &\geq (2/3)I_{Y'}(\varphi) + (1/3)I_{Z'}(\varphi) \\ &\geq (2/3)(I_{[x]_E}(\varphi) - \epsilon/2) + (1/3)(I_{[x]_E}(\varphi) + \epsilon/4) \\ &= I_{[x]_E}(\varphi) - \epsilon/4, \end{aligned}$$

thus $|I_{Y'\cup Z'}(\varphi) - I_{[x]_E}(\varphi)| \leq \epsilon/4$, which contradicts the Φ-maximality of F''. $\square$

We are now ready to characterize the existence of probability measures:

Theorem 20.9. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then exactly one of the following holds:*

1. *$X \in \mathcal{J}_\rho \vee \mathcal{K}_\rho$;*
2. *There is a ρ-invariant probability measure.*

Proof. Propositions 20.4 and 20.5 imply that conditions (1) and (2) are mutually exclusive, so it is enough to show $\neg(1) \Rightarrow (2)$. By Theorem 16.1, there is a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. For each $\gamma \in \Gamma$, define $\rho_\gamma : X \to X$ by $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$. By Proposition 10.9 and Exercise 9, there is a finer Polish topology τ on X, compatible with the Borel structure of X, for which there is a Γ-invariant countable algebra $\mathcal{U}$ of sets which is a basis for (X, τ) and contains every set of the form $\rho_\gamma^{-1}(I)$, where $I \subseteq (0, \infty)$ is an open interval with rational endpoints. Fix an enumeration $\langle \varphi_n \rangle_{n \in \mathbb{N}}$ of the bounded functions of the form $\rho_\gamma \mathbf{1}_U$, where $\gamma \in \Gamma$ and $U \in \mathcal{U}$, and let Φ denote the linear subspace of $C_b(X, \tau)$ spanned by $\langle \varphi_n \rangle$.

We will now construct an increasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E. We begin by setting $F_0 = \Delta(X)$. Given F_n, by applying Proposition 20.7 finitely many times and throwing out the corresponding E-invariant Borel sets in $\mathcal{I}_\rho$, we can assume that there is a finite Borel subequivalence relation $F \supseteq F_n$ of E such that $\forall m \leq n$ (F is $(\varphi_m, 1/n)$-approximating). Set $F_{n+1} = F$.

Define $I_x : \Phi \to \mathbb{R}$ by

$$I_x(\varphi) = \lim_{n \to \infty} I_{[x]_{F_n}}(\varphi).$$

Then I_x is a mean on Φ. Define $\mu : \mathcal{U} \to [0, 1]$ by

$$\mu_x(U) = I_x(\mathbf{1}_U) = \lim_{n \to \infty} \mu_{[x]_{F_n}}(U).$$

By Propositions 19.1 and 19.2, each μ_x^* is a measure.

For each $U \in \mathcal{U}$ and $n \in \mathbb{N}$, fix a partition $\langle U_n \rangle \in \mathcal{U}^{\mathbb{N}}$ of U into sets of diameter less than $1/n$. Then the set

$$A_{U,n} = \{x \in X : \mu_x(U) \neq \lim_{n \to \infty} \mu_x(U_n)\}$$

is in $\mathcal{J}_\rho$. By throwing out all such sets, we can therefore assume that each μ_x is strongly regular. It then follows from Proposition 19.6 that $\forall x \in X$ $(I_x = I_x^*|\Phi)$.

For each $\gamma \in \Gamma$, $U \in \mathcal{U}$, and $n \in \mathbb{N}$ such that $\rho_\gamma|U$ is bounded, the set

$$B_{\gamma,U,n} = \{x \in X : \forall y \in [x]_E \ (\tilde{\rho}(\gamma(V) \cap [y]_{F_n}, \gamma(V \cap [y]_{F_n})) > 1)\}$$

is in $\mathcal{K}_\rho$, as is the set

$$C_{\gamma,U,n} = \{x \in X : \forall y \in [x]_E \ (\tilde{\rho}(\gamma(V) \cap [y]_{F_n}, \gamma(V \cap [y]_{F_n})) < 1)\}.$$

We will complete the proof of the theorem by showing that if x is not in the union of all such sets, then μ_x^* is ρ-invariant. Suppose, towards a contradiction, that there exists $\gamma \in \Gamma$ and $B \subseteq X$ Borel such that

$$\mu_x^*(\gamma(B)) \neq \int_B \rho(\gamma \cdot y, y) \ d\mu_x^*(y).$$

By regularity, we can assume that $B = U$, for some $U \in \mathcal{U}$. Clearly, we can also assume that $\rho_\gamma|U$ is bounded, and it follows that

$$\mu_x(\gamma(V)) \neq I_x(\rho_\gamma \mathbf{1}_V).$$

If $\mu_x(\gamma(V)) > I_x(\rho_\gamma \mathbf{1}_V)$, then there exists $n \in \mathbb{N}$ such that $x \in B_{\gamma,U,n}$, a contradiction. If $\mu_x(\gamma(V)) < I_x(\rho_\gamma \mathbf{1}_V)$, then there exists $n \in \mathbb{N}$ such that $x \in C_{\gamma,U,n}$, another contradiction. □

By slightly modifying our technique, we obtain the following:

Theorem 20.10. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then there is an E-invariant Borel set $B \subseteq X$ and a Borel function $\pi : B \to P(X)$ such that:*

1. $X \setminus B \in \mathcal{J}_\rho \vee \mathcal{K}_\rho$;
2. $\pi[B] = \mathrm{ErgInv}_\rho$;
3. $\forall \mu \in \mathrm{ErgInv}_\rho$ $(\mu(\pi^{-1}(\mu)) = 1)$;
4. $\forall \mu \in \mathrm{Inv}_\rho$ $(\mu = \int \pi \ d\mu)$.

Proof. We will assume that E is ρ-aperiodic, as it is clear how to proceed in the ρ-finite case. Let $\pi(x) = \mu_x^*$, where μ_x is defined as in the proof of Theorem 20.9 (note that μ_x^* is defined on an E-invariant Borel set $B \subseteq X$ whose complement is in $\mathcal{J}_\rho \vee \mathcal{K}_\rho$.) Proposition 20.3 immediately implies that conditions (3) and (4) are satisfied, so it only remains to check that each of the measures μ_x^* is E-ergodic. While it is possible to verify this using a little functional analysis, we give here a purely descriptive proof.

Let F denote the equivalence relation on B given by

$$xFy \Leftrightarrow \mu_x = \mu_y.$$

It is clear that F is Borel and contains $E|B$.

Lemma 20.11. *Suppose that $\mu_x^*([x]_F) = 1$. Then μ_x^* is E-ergodic.*

Proof. Suppose that $C \subseteq X$ is an E-invariant Borel set of positive μ_x^*-measure. Fix $0 < \epsilon < \mu_x^*(C)$. By the regularity of μ_x^*, there is a set $U \in \mathcal{U}$ such that $\mu_x^*(U) > \epsilon$ and $\mu_x^*(U \setminus C) \leq \epsilon^2$. Using the notation of Proposition 20.3, put

$$D = \{y \in [x]_F : \mu_y(C), \mu_y(U \setminus C) \text{ exist and } \mu_y(U \setminus C) \leq \epsilon\}.$$

Proposition 20.3 ensures that $\mu_y(C), \mu_y(U \setminus C)$ exist μ_x^*-almost everywhere, and

$$\begin{aligned} \epsilon^2 &\geq \mu_x^*(U \setminus C) \\ &= \int_{[x]_F} \mu_y(U \setminus C)\ d\mu_x^*(y) \\ &= \int_D \mu_y(U \setminus C)\ d\mu_x^*(y) + \int_{[x]_F \setminus D} \mu_y(U \setminus C)\ d\mu_x^*(y) \\ &\geq \epsilon(1 - \mu_x^*(D)), \end{aligned}$$

thus $\mu_x^*(D) \geq 1 - \epsilon$. Observe now that if $y \in D$, then

$$\begin{aligned} \mu_y(C) &\geq \mu_y(U) - \mu_y(U \setminus C) \\ &= \mu_x^*(U) - \mu_y(U \setminus C) > 0, \end{aligned}$$

so $C \cap [y]_E \neq \emptyset$, thus $y \in C$. As $y \in D$ was arbitrary, it follows that $D \subseteq C$, hence $\mu_x^*(C) \geq \mu_x^*(D) \geq 1 - \epsilon$. As $0 < \epsilon < \mu_x^*(C)$ was arbitrary, it follows that $\mu_x^*(C) = 1$, therefore μ_x^* is E-ergodic. □

It only remains to modify our construction of $\langle F_n \rangle$ so as to ensure that $\mu_x^*([x]_F) = 1$. This time, we will simultaneously construct Polish topologies τ_n, and linear spaces $\Phi_n \subseteq C_b(X, \tau_n)$ spanned by sequences $\langle \varphi_{mn} \rangle_{m \in \mathbb{N}}$ which contain bases $\mathcal{U}_n$ for (X, τ_n). We begin by setting $F_0 = \Delta(X)$ and fixing $\varphi_{m0} = \varphi_m$, $\Phi_0 = \Phi$, and $\mathcal{U}_n = \mathcal{U}$ as before. Given $(F_n, \langle \varphi_{mn} \rangle, \mathcal{U}_n)$, we can again apply Proposition 20.7 finitely many times so as to ensure that $\forall i, j \leq n$ (F is $(\varphi_{ij}, 1/n)$-approximating). This time, however, we shall approximate more sets. For $i, j, k, l \leq n$, define

$$X_{ijkl} = \{x \in X : \forall y \in [x]_E\ (I_{[y]_{F_i}}(\varphi_{jk}) \in I_l)\}.$$

By applying Proposition 20.7 finitely many times and throwing out the corresponding E-invariant Borel sets in $\mathcal{I}_\rho$, we can construct a finite Borel subequivalence relation $F' \supseteq F$ of E such that $\forall i, j, k, l \leq n$ (F' is $(\mathbf{1}_{X_{ijkl}}, 1/n)$-approximating). Set $F_{n+1} = F'$. Fix now an extension τ_{n+1} of τ_n for which there is a Γ-invariant countable algebra $\mathcal{U}_{n+1}$ of sets which is a basis for (X, τ_{n+1}) and contains each of the sets X_{ijkl}, and fix an enumeration $\langle \varphi_{m(n+1)} \rangle$ of the bounded functions of the form $\rho_\gamma \mathbf{1}_U$, where $\gamma \in \Gamma$ and $U \in \mathcal{U}_{n+1}$.

As before, by throwing out an E-invariant Borel set in $\mathcal{J}_\rho \vee \mathcal{K}_\rho$, we can assume that each of the corresponding maps μ_x^* is a ρ-invariant probability measure. Observe now that if $(x, y) \notin F$, then there exist $i, j, k, l \in \mathbb{N}$ such that $x \in X_{ijkl}$ and $y \notin X_{ijkl}$. It follows that $[x]_F$ is the intersection of the sets of the form X_{ijkl} which contain x. As X_{ijkl} is E-invariant, it follows that $\mu_x^*(X_{ijkl}) = \mu_x(X_{ijkl}) = 1$, thus $\mu_x^*([x]_F) = 1$. □

A *compression* of E is a Borel injection $T : X \to X$ such that graph$(T) \subseteq E$ and $X \setminus T[X]$ is E-complete.

Exercise 17. Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is the trivial cocycle. Show that if X is in $\mathcal{J}_\rho \cup \mathcal{K}_\rho$, then E is compressible.

Theorem 20.12 (Nadkarni)**.** *Suppose that E is a countable Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

1. *E is compressible;*
2. *There is an E-invariant probability measure.*

Proof. This follows easily from Theorem 20.9 and Exercise 17. □

This theorem has a much older measure-theoretic analog:

Theorem 20.13 (Hopf)**.** *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant probability measure. Then exactly one of the following holds:*

1. *There is a Borel set $B \subseteq X$ such that $\mu(B) > 0$ and $E|B$ is compressible;*
2. *There is an E-invariant probability measure $\nu \sim \mu$.*

Proof. It is clear that (1) and (2) are mutually exclusive, so it is enough to show $\neg(1) \Rightarrow (2)$. Proceeding as in the proof of Theorem 20.9, after throwing out E-invariant Borel sets on which E is compressible, we obtain a map $x \mapsto \mu_x$ such that each μ_x^* is an E-invariant probability measure. Then the map

$$\nu(B) = \int \mu_x^*(B)\ d\mu(x)$$

is as desired. □

We say that a group Γ is *strongly Bergman* if there is a natural number $k \in \mathbb{N}$ such that for every increasing, exhaustive sequence $\langle \Gamma_n \rangle \in \mathcal{P}(\Gamma)^{\mathbb{N}}$ of subsets of Γ, there exists $n \in \mathbb{N}$ such that $\Gamma = (\Gamma_n)^k$.

Exercise 18. Show that an aperiodic countable Borel equivalence relation E is compressible if and only if $[E]$ is strongly Bergman. Use this to characterize the existence of E-invariant probability measures in terms of a purely algebraic property of $[E]$.

A *ρ-compression* of E is a Borel function $\varphi : E \to [0, 1]$ which satisfies the following conditions:

1. $\forall x \in X \left(\sum_{y \in [x]_E} \varphi(x,y) = 1 \right)$;

2. $\forall y \in Y \left(\sum_{x \in [y]_E} \varphi(x,y)\rho(x,y) \leq 1 \right)$;

3. $\forall x \in X \, \exists y \in [x]_E \left(\sum_{z \in [y]_E} \varphi(z,y)\rho(z,y) = 0 \right)$.

Exercise 19. Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Show that if X is in $\mathcal{J}_\rho \cup \mathcal{K}_\rho$, then E is ρ-compressible.

Theorem 20.14. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to \mathbb{R}^+$ is a Borel cocycle. Then exactly one of the following holds:*

1. *E is ρ-compressible;*

2. *There is a ρ-invariant probability measure.*

Proof. This follows easily from Theorem 20.9 and Exercise 19. □

Again, this theorem has a measure-theoretic analog:

Theorem 20.15. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and μ is an E-quasi-invariant probability measure. Then exactly one of the following holds:*

1. *There is a Borel set $B \subseteq X$ such that $\mu(B) > 0$ and $E|B$ is ρ-compressible;*

2. *There is a ρ-invariant probability measure $\nu \sim \mu$.*

Proof. It is clear that (1) and (2) are mutually exclusive, so it is enough to show $\neg(1) \Rightarrow (2)$. Proceeding as in the proof of Theorem 20.9, after throwing out countably many E-invariant Borel sets on which E is ρ-compressible, we can obtain a map $x \mapsto \mu_x$ such that each μ_x^* is a ρ-invariant probability measure. Then the map

$$\nu(B) = \int \mu_x^*(B) \, d\mu(x)$$

is as desired. □

21 Generic compressibility

Suppose that $R \subseteq X \times X$. We say that a set $B \subseteq X$ is *R-complete* if

$$\forall x \in X \, \exists y \in B \, ((x,y) \in R).$$

A *marker sequence* for R is a decreasing, vanishing sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of R-complete Borel sets.

Proposition 21.1. *Suppose that X is a Polish space and R is a transitive Borel binary relation on X whose vertical sections are all countably infinite. Then R admits a marker sequence.*

Proof. Fix an enumeration $\langle B_n \rangle_{n\in\mathbb{N}}$ of a countable family of Borel subsets of X which separates points, and for each $s \in 2^{<\mathbb{N}}$, define $B_s \subseteq X$ by

$$B_s = \left(\bigcap_{s(i)=0} X \setminus B_i \right) \cap \left(\bigcap_{s(i)=1} B_i \right).$$

For each $n \in \mathbb{N}$, define $S_n : X \to \mathcal{P}(2^n)$ by

$$S_n(x) = \{s \in 2^n : |B_s \cap R_x| = \aleph_0\}.$$

For each $s \in 2^n$, define $C_s \subseteq X$ by

$$C_s = \{x \in X : \forall y \in R_x \ (s = \min_{\mathrm{lex}} S_n(y))\},$$

and for each $n \in \mathbb{N}$, define $D_n \subseteq X$ by

$$D_n = \bigcup_{s\in 2^n} B_s \cap C_s.$$

Lemma 21.2. $\forall n \in \mathbb{N}\ (D_{n+1} \subseteq D_n)$.

Proof. Fix $n \in \mathbb{N}$ and suppose that $x \in D_{n+1}$. Then there exists $s \in 2^n$ and $i \in \{0, 1\}$ such that $x \in B_{si} \cap C_{si}$. Given $y \in R_x$, the fact that $x \in C_{si}$ ensures that $si = \min S_{n+1}(y)$. In particular, it follows that $s \in S_n(y)$, and if $t <_{\mathrm{lex}} s$, then $t0, t1 \notin S_{n+1}(y)$, so $t \notin S_n(y)$, thus $s = \min S_n(y)$. It now follows that $x \in C_s$, and since $x \in B_s$, this implies that $x \in D_n$. □

Lemma 21.3. *Each D_n is an R-complete section.*

Proof. For each $x \in X$, set $x_0 = x$, and given x_i, let x_{i+1} be any element of R_{x_i} such that $S_n(x_{i+1}) \subset S_n(x_i)$, if such an element exists. Otherwise, set $x_{i+1} = x_i$. Let $y = x_{2^n}$ and $s = \min S_n(y)$, and observe that $y \in C_s$, thus $B_s \cap C_s \cap R_y \subseteq D_n \cap R_y \subseteq D_n \cap R_x$ is infinite. □

Define $D = \bigcap_{n\in\mathbb{N}} D_n$.

Lemma 21.4. $(D \times D) \cap R \subseteq \Delta(X)$.

Proof. Suppose that $x \in D$, $(x, y) \in R$, and $x \neq y$, and fix $n \in \mathbb{N}$ and $s \in 2^n$ such that $x \in B_s$ and $y \notin B_s$. As $x \in D_n$, it follows that $s = \min S_n(x) = \min S_n(y)$, so $y \notin D_n$, thus $y \notin D$. □

Now define $A_n = D_n \setminus D$. Lemma 21.2 implies that these sets are decreasing, and they clearly have empty intersection, so it only remains to check that each A_n is R-complete. Towards this end, fix $x \in X$, and observe that two applications of Lemma 21.3 ensure that there are distinct points $y \in D_n \cap R_x$ and $z \in D_n \cap R_y$. Lemma 21.4 then ensures that $y \notin A_n \Rightarrow y \in D \Rightarrow z \notin D \Rightarrow z \in A_n$, and the transitivity of R then implies that $A_n \cap R_x \neq \emptyset$. □

Remark 21.5. It is not difficult to produce examples which show that the assumption of transitivity in Proposition 21.1 is necessary.

We next prove a generic compressibility result for quasi-orders:

Proposition 21.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\leq$ is a Borel quasi-order on X such that*

$$\forall x \in X\, \exists^\infty y \in [x]_E \ (x \leq y).$$

Then there is an E-invariant comeager Borel set $C \subseteq X$ and a $\leq$-decreasing, infinite-to-one Borel function $\pi : C \to C$.

Proof. Fix a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. By Proposition 21.1, there is a marker sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ for $\leq$. For $\alpha \in \Gamma^{\leq \mathbb{N}}$, let

$$D_\alpha = \bigcup_{n < |\alpha|} \{x \in X \setminus B_n : \gamma_{\alpha(n)} \cdot x \leq x\}.$$

Define $n_\alpha : D_\alpha \to \mathbb{N}$ by

$$n_\alpha(x) = \min\{n < |\alpha| : x \in X \setminus B_n \text{ and } \gamma_{\alpha(n)} \cdot x \leq x\},$$

and define $\pi_\alpha : D_\alpha \to X$ by $\pi_\alpha(x) = \gamma_{\alpha(n_\alpha(x))} \cdot x$.

Lemma 21.7. $\forall x \in X\, \forall s \in \Gamma^{<\mathbb{N}}\, \exists t \supseteq s\ (\pi_s^{-1}(x) \subset \pi_t^{-1}(x))$.

Proof. Given $x \in X$ and $s \in \Gamma^{<\mathbb{N}}$, fix $y \in \bigcap_{n<|s|} B_n$ such that $x \leq y$, as well as $\gamma \in \Gamma$ such that $\gamma \cdot y = x$, and $n \in \mathbb{N}$ least such that $y \notin B_n$. Then any $t \supseteq s$ such that $t(n) = \gamma$ is as desired. □

A straightforward induction now shows that

$$\forall x \in X\, \forall^* \alpha \in \Gamma^{\mathbb{N}}\ (|\pi_\alpha^{-1}(x)| = \aleph_0).$$

This immediately implies that

$$\forall x \in X\, \forall^* \alpha \in \Gamma^{\mathbb{N}}\, \forall \gamma \in \Gamma (|\pi_\alpha^{-1}(\gamma \cdot x)| = \aleph_0),$$

and Exercise 3 then yields that

$$\forall^* \alpha \in \Gamma^{\mathbb{N}}\, \forall^* x \in X\, \forall \gamma \in \Gamma\ (|\pi_\alpha^{-1}(\gamma \cdot x)| = \aleph_0).$$

In particular, there exists $\alpha \in \Gamma^{\mathbb{N}}$ such that the set

$$C = \{x \in X : \pi_\alpha | [x]_E \text{ is infinite-to-one}\}$$

is comeager. Then the function $\pi = \pi_\alpha \cup \mathrm{id}|(C_\alpha \setminus D_\alpha)$ is as desired. □

We are now ready for the main result of this section:

Theorem 21.8 (Kechris-Miller)**.** *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to \mathbb{R}^+$ is a Borel cocycle, and E is ρ-aperiodic. Then there is a comeager E-invariant Borel set $C \subseteq X$ and a smooth Borel equivalence relation $F \subseteq E|C$ on C which is $(\rho|F)$-aperiodic. In particular, there are no $\rho|(E|C)$-invariant probability measures.*

Proof. Define $B \subseteq X$ by

$$B = \{x \in X : \exists y \in [x]_E \, \forall^\infty z \in [y]_E \ (z < y)\}.$$

Set $x \leq_\rho y \Leftrightarrow \rho(x, y) \leq 1$, and let $\leq$ be the union of $E|B$ with $\leq_\rho$. By Proposition 21.6, there is a comeager E-invariant Borel set $C \subseteq X$ and a $\leq$-decreasing, infinite-to-one Borel function $\pi : C \to C$. Define F on C by

$$xFy \Leftrightarrow (xEy \text{ and } (x, y \in B \text{ or } \pi(x) = \pi(y))).$$

It is clear that $F \subseteq E$ is ρ-aperiodic and smooth. $\square$