DEGREE THEOREMS AND LIPSCHITZ SIMPLICIAL VOLUME FOR NON-POSITIVELY CURVED MANIFOLDS OF FINITE VOLUME

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ABSTRACT. We study a metric version of the simplicial volume on Riemannian manifolds, the Lipschitz simplicial volume, with applications to degree theorems in mind. We establish a proportionality principle and a product inequality from which we derive an extension of Gromov's volume comparison theorem to products of negatively curved manifolds or locally symmetric spaces of non-compact type. In contrast, we provide vanishing results for the ordinary simplicial volume; for instance, we show that the ordinary simplicial volume of non-compact locally symmetric spaces with finite volume of $\mathbb{Q}$-rank at least 3 is zero.

1. Introduction and statement of results

The prototypical *degree theorem* bounds the degree $\deg f$ of a proper, continuous map $f : N \to M$ between n-dimensional Riemannian manifolds of finite volume by

$$\deg(f) \leq \mathrm{const}_n \cdot \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)}.$$

For example, Gromov's volume comparison theorem [16, p. 13] is a degree theorem where the target M has negative sectional curvature and the domain N satisfies a lower Ricci curvature bound. In *loc. cit.* Gromov also pioneered the use of the *simplicial volume* to prove theorems of this kind. Recall that the simplicial volume $\|M\|$ of a manifold M without boundary is defined by

$$\|M\| = \inf\bigl\{|c|_1;\ c \text{ fundamental cycle of } M \text{ with } \mathbb{R}\text{-coefficients}\bigr\}.$$

Here $|c|_1$ denotes the ℓ^1-norm with respect to the basis given by the singular simplices. If M is non-compact then one takes locally finite fundamental cycles in the above definition. Under the given curvature assumptions, Gromov's comparison theorem is proved by the following three steps (of which the third one is elementary):

(1) Upper volume estimate for target: $\|M\| \geq \mathrm{const}_n \mathrm{vol}(M)$.
(2) Lower volume estimate for domain: $\|N\| \leq \mathrm{const}_n \mathrm{vol}(N)$.
(3) Degree estimate: $\deg(f) \leq \mathrm{const}_n \|N\|/\|M\|$.

Unless stated otherwise, all manifolds in this text are assumed to be connected and without boundary. As Riemannian metrics on locally symmetric spaces of non-compact type we always choose the standard metric, i.e., the one given by the Killing form [12, Section 2.3.11].

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1.1. **Main results.** In this article, we prove degree theorems where the target is non-positively curved and has finite volume. More specifically, we consider the case where the target is a product of negatively curved manifolds of finite volume or locally symmetric spaces of finite volume. To this end, we study a variant of the simplicial volume, the *Lipschitz simplicial volume*, and pursue a Lipschitz version of the three step strategy above. The properties of the Lipschitz simplicial volume we show en route are also of independent interest.

Before introducing the Lipschitz simplicial volume, we give a brief overview of the properties of the ordinary simplicial volume of non-compact locally symmetric spaces of finite volume: On the one hand by a classic result of Thurston [29, Chapter 6] the simplicial volume of finite volume hyperbolic manifolds is proportional to the Riemannian volume. According to Gromov and Thurston the simplicial volume of complete Riemannian manifolds with pinched negative curvature and finite volume is positive [16, Section 0.3]. In addition, we proved by different means that the simplicial volume of Hilbert modular varieties is positive [23] (see also Theorem 1.14 below). In accordance with these examples we expect positivity for all locally symmetric spaces of $\mathbb{Q}$-rank 1.

On the other hand, in Section 5 we show that the simplicial volume of locally symmetric spaces of $\mathbb{Q}$-rank at least 3 vanishes – in particular, the ordinary simplicial volume does not give rise to the desired degree theorems.

Theorem 1.1. *Let Γ be a torsion-free, arithmetic lattice of a semi-simple, center-free $\mathbb{Q}$-group $\mathbf{G}$ with no compact factors. Let $X = \mathbf{G}(\mathbb{R})/K$ be the associated symmetric space where K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. If Γ has $\mathbb{Q}$-rank at least 3, then $\|\Gamma\backslash X\| = 0$.*

This result is based on a more general vanishing theorem (Corollary 5.4) derived from Gromov's vanishing-finiteness theorem [16, Corollary (A) on p. 58] by constructing suitable amenable coverings for manifolds with nice boundary and whose fundamental groups admit small classifying spaces.

Gromov's original applications of the vanishing-finiteness theorem contain the surprising fact that the simplicial volume of any product of three open manifolds is zero [16, p. 59]. However, there are products of two open manifolds whose simplicial volume is non-zero (see Example 5.5), and Gromov's argument fails for products of two open surfaces. In particular, the $\mathbb{Q}$-rank 2 case is still open.

In contrast to Theorem 1.1, Lafont and Schmidt showed the following positivity result in the closed case [18]; the proof is based on work of Connell-Farb [9], as well as – for the exceptional cases – Thurston, Savage, and Bucher-Karlsson:

Theorem 1.2 (Lafont, Schmidt)**.** *Let M be a closed locally symmetric space of non-compact type. Then $\|M\| > 0$.*

In view of the fact that the simplicial volume of non-compact manifolds is zero in a large number of cases, Gromov studied geometric variants of the simplicial volume [16, Section 4.4f], i.e., simplicial volumes where the simplices allowed in fundamental cycles respect a geometric condition. In this article, we consider the following Lipschitz version of simplicial volume:

Definition 1.3. Let M be an n-dimensional, oriented Riemannian manifold. For a locally finite chain $c \in C_n^{\mathrm{lf}}(M)$ we denote the supremum of the Lipschitz constants of the simplices occurring in c by $\mathrm{Lip}(c) \in [0, \infty]$. The *Lipschitz simplicial* *volume* $\|M\|_{\mathrm{Lip}} \in [0,\infty]$ of M is defined by

$$\|M\|_{\mathrm{Lip}} = \inf\bigl\{|c|_1;\ c \in C_n^{\mathrm{lf}}(M) \text{ fundamental cycle of } M \text{ with } \mathrm{Lip}(c) < \infty\bigr\}.$$

By definition, we have the obvious inequality $\|M\| \le \|M\|_{\mathrm{Lip}}$. It is easy to see that if $f : N \to M$ is a proper Lipschitz map between Riemannian manifolds, then

$$\deg(f) \cdot \|M\|_{\mathrm{Lip}} \le \|N\|_{\mathrm{Lip}}.$$

Remark 1.4. If M is a *closed* Riemannian manifold, then $\|M\| = \|M\|_{\mathrm{Lip}}$; each fundamental cycle involves only finitely many simplices, and hence this equality is implied by the fact that singular homology and smooth singular homology are isometrically isomorphic [21, Proposition 5.3].

In Section 4 we prove the following theorem, which leads to a degree theorem for locally symmetric spaces of finite volume.

Theorem 1.5 (Proportionality principle)**.** *Let M and N be complete, non-positively curved Riemannian manifolds of finite volume. Assume that their universal covers are isometric. Then*

$$\frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)} = \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)}.$$

The proportionality principle for closed Riemannian manifolds is a classical theorem of Gromov [16, Section 0.4; 28, Chapter 5; 29, pp. 6.6–6.10]. The proportionality principle in the closed case does not require a curvature condition, and our proof in the non-closed case uses non-positive curvature in a light way. It might be possible to weaken the curvature condition in the non-compact case.

By Theorem 1.2 the proportionality principle for the ordinary simplicial volume cannot hold in general since for every locally symmetric space of finite volume there is always a *compact* one such that their universal covers are isometric [4]. For the same reason, Theorems 1.5 and 1.2 and Remark 1.4 imply the following corollary.

Corollary 1.6. *The Lipschitz simplicial volume of locally symmetric spaces of finite volume and non-compact type is non-zero.*

Gromov [16, Section 4.5] states also a proportionality principle for non-compact manifolds for geometric invariants related to the Lipschitz simplicial volume. Unraveling his definitions, one sees that it implies a proportionality principle for finite volume manifolds without a curvature assumption (which we need) provided one of the manifolds is compact (which we do not need). This would be sufficient for the previous corollary. Gromov's proof, which is unfortunately not very detailed, and ours seem to be independent.

The simplicial volume of a product of oriented, closed, connected manifolds can be estimated from above as well as from below in terms of the simplicial volume of both factors [1, Theorem F.2.5; 16, p. 17f]. While the upper bound continues to hold for the locally finite simplicial volume in the case of non-compact manifolds [22, Theorem C.7], the lower bound in general does not.

The Lipschitz simplicial volume on the other hand is better behaved with respect to products. In addition to the estimate $\|M \times N\|_{\mathrm{Lip}} \le c(\dim M + \dim N) \cdot \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}$, the presence of non-positive curvature enables us to derive also the non-trivial lower bound:

Theorem 1.7 (Product inequality for non-positively curved manifolds)**.** *Let M and N be two complete, non-positively curved Riemannian manifolds. Then*

$$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leq \|M \times N\|_{\mathrm{Lip}}.$$

On a technical level, we mention two issues that often prevent one from extending properties of the simplicial volume for compact manifolds to non-compact ones, and thus force one to work with the Lipschitz simplicial volume instead. Firstly, there is no straightening (see Section 2.2) for locally finite chains: The straightening of a locally finite chain c is not necessarily locally finite. However, it is locally finite provided $\mathrm{Lip}(c) < \infty$, which motivates a Lipschitz condition. Secondly, there is no well-defined cup product for compactly supported cochains. This is an issue arising in the proof of the product inequality. We circumvent this difficulty by introducing the complex of cochains with Lipschitz compact support (see Definition 3.6), which carries a natural cup-product.

1.2. **Degree theorems.** To apply the theorems of the previous section to degree theorems, we need upper and lower estimates of the volume by the Lipschitz simplicial volume.

For the (locally finite) simplicial volume and all complete n-dimensional Riemannian manifolds, Gromov gives the bound $\|M\| \leq (n-1)^n n! \operatorname{vol}(M)$ provided $\mathrm{Ricci}(M) \geq -(n-1)$ [16]. The latter stands for $\mathrm{Ricci}(M)(v,v) \geq -(n-1)\|v\|^2$ for all $v \in TM$. One can extract from *loc. cit.* a similar estimate for the Lipschitz simplicial volume:

Theorem 1.8 (Gromov)**.** *For every $n \geq 1$ there is a constant $C_n > 0$ such that every complete n-dimensional Riemannian manifold M with sectional curvature* $\sec(M) \leq 1$ *and Ricci curvature* $\mathrm{Ricci}(M) \geq -(n-1)$ *satisfies*

$$\|M\|_{\mathrm{Lip}} \leq C_n \cdot \operatorname{vol}(M).$$

Proof. For $\sec(M) \leq 0$ this follows from [16, Theorem (A),(4), in Section 4.3] by applying it to $U = M$, $R = 1$, a fundamental cycle c, and $\varepsilon \to 0$: One obtains a fundamental cycle c' made out of straight simplices whose diameter is less than $R+\varepsilon$. In particular, $\mathrm{Lip}(c') < \infty$ by Proposition 2.4. Further, the estimate $\|M\|_{\mathrm{Lip}} \leq \|c'\| \leq C_n \operatorname{vol}(M)$ follows from (4) in *loc. cit.* and the Bishop-Gromov inequality, which provides a bound of $l'_v(R)$ in terms of n [15, Theorem 4.19 on p. 214; 16, (C) in Section 4.3].

Gromov also explains why these arguments carry over to the general case that $\sec(M) \leq 1$ [16, Remarks (B) and (C) in Section 4.3]. In this case, c' is made out of straight simplices of diameter less than $\pi/2$ (Section 2.1), and $\mathrm{Lip}(c') < \infty$ follows from Proposition 2.6. □

Corollary 1.9. *Any complete Riemannian manifold of finite volume that has an upper sectional curvature and lower Ricci curvature bound has finite Lipschitz simplicial volume.*

Connell and Farb [9] prove, building upon techniques of Besson-Courtois-Gallot, a degree theorem where the target M is a locally symmetric space (closed or finite volume) with no local $\mathbb{R}$, $\mathbb{H}^2$, or $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$-factor. For non-compact M they have to assume that $f : N \to M$ is (coarse) Lipschitz. Using the simplicial volume (and the work by Connell-Farb, Thurston, Savage, and Bucher-Karlsson), Lafont and Schmidt [18] prove degree theorems for closed locally symmetric spaces including the exceptional cases. The following theorem includes also the non-compact exceptional cases.

Theorem 1.10 (Degree theorem, complementing [9, 18])**.** *For every $n \in \mathbb{N}$ there is a constant $C_n > 0$ with the following property: Let M be an n-dimensional locally symmetric space of non-compact type with finite volume. Let N be an n-dimensional complete Riemannian manifold of finite volume with* $\operatorname{Ricci}(N) \geq -(n-1)$ *and* $\sec(N) \leq 1$*, and let $f : N \to M$ be a proper Lipschitz map. Then*

$$\deg(f) \leq C_n \cdot \frac{\operatorname{vol}(N)}{\operatorname{vol}(M)}.$$

Proof. By Theorem 1.5 and Corollary 1.6 we know that $\|M\|_{\mathrm{Lip}} = \mathrm{const}_n \operatorname{vol}(M)$ where $\mathrm{const}_n > 0$ depends only on the symmetric space $\widetilde{M}$. Because there are only finitely many symmetric spaces (with the standard metric) in each dimension, there is $D_n > 0$ depending only on n such that $\|M\|_{\mathrm{Lip}} \geq D_n \operatorname{vol}(M)$. So Theorem 1.8 applied to N and $\|N\|_{\mathrm{Lip}} \geq \deg(f)\|M\|_{\mathrm{Lip}}$ yield the assertion. □

Unfortunately, the Lipschitz simplicial volume cannot be used to prove positivity of Gromov's *minimal volume* $\operatorname{minvol}(M)$ of a smooth manifold M; the minimal volume is defined as the infimum of volumes $\operatorname{vol}(M, g)$ over all complete Riemannian metrics g on M whose sectional curvature is pinched between -1 and 1.

Next we describe the appropriate modification of $\operatorname{minvol}(M)$ in our setting: The *Lipschitz class* $[g]$ of a complete Riemannian metric g on M is defined as the set of all complete Riemannian metrics g' such that the identity $\mathrm{id} : (M, g') \to (M, g)$ is Lipschitz. Then we define the *minimal volume of* $[g]$ as

$$\operatorname{minvol}_{\mathrm{Lip}}(M, [g]) = \bigl\{\operatorname{vol}(M, g');\ -1 \leq \sec(g') \leq 1 \text{ and } g' \in [g]\bigr\}.$$

Of course, we have $\operatorname{minvol}_{\mathrm{Lip}}(M, [g]) = \operatorname{minvol}(M)$ whenever M is compact. Theorem 1.10, applied to the identity map and varying metrics, implies:

Theorem 1.11. *The minimal volume of the Lipschitz class of the standard metric of a locally symmetric space of non-compact type and finite volume is positive.*

Excluding certain local factors, Connell and Farb have the following stronger statement for the minimal volume instead of the Lipschitz minimal volume.

Theorem 1.12 (Connell-Farb)**.** *The minimal volume of a locally symmetric space of non-compact type and finite volume that has no local $\mathbb{H}^2$- or $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3, \mathbb{R})$-factors is positive.*

A little caveat: Connell and Farb state this theorem erroneously as a corollary of a degree theorem for which they have to assume a Lipschitz condition. This would only give the positivity of the Lipschitz minimal volume. However, Chris Connell explained to us how to modify their proof to get the positivity of the minimal volume.

As an application of the product inequality we obtain a new degree theorem for products of manifolds with (variable) negative curvature or locally symmetric spaces.

Theorem 1.13 (Degree theorem for products)**.** *For every $n \in \mathbb{N}$ there is a constant $C_n > 0$ with the following property: Let M be a Riemannian n-manifold* *of finite volume that decomposes as a product $M = M_1 \times \cdots \times M_m$ of Riemannian manifolds, where for every $i \in \{1, \ldots, m\}$ the manifold M_i is either negatively curved with $-\infty < -k < \sec(M_i) \leq -1$ or a locally symmetric space of non-compact type. Let N be an n-dimensional, complete Riemannian manifold of finite volume with $\sec(N) \leq 1$ and $\mathrm{Ricci}(N) \geq -(n-1)$. Then for every proper Lipschitz map $f : N \to M$ we have*

$$\deg(f) \leq C_n \cdot \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)}.$$

Proof. In the sequel, D_i, D_i', E_n, and C_n stand for constants depending only on n. If M_i is negatively curved then Thurston's theorem [16, Section 0.3; 29] yields

$$\mathrm{vol}(M_i) \leq D_n \|M_i\| \leq D_n \|M_i\|_{\mathrm{Lip}}.$$

If M_i is locally symmetric of non-compact type then, as in the proof of Theorem 1.10, we also obtain $\mathrm{vol}(M_i) \leq D_n' \|M_i\|_{\mathrm{Lip}}$. By the product inequality (Theorem 1.7),

$$\mathrm{vol}(M) \leq \max_{i \in \{1,\ldots,m\}} (D_i, D_i')^m \|M\|_{\mathrm{Lip}}.$$

On the other hand, by Theorem 1.8, we have $\|N\|_{\mathrm{Lip}} \leq E_n \,\mathrm{vol}(N)$. Combining everything with $\|N\|_{\mathrm{Lip}} \geq \deg(f) \|M\|_{\mathrm{Lip}}$, proves the theorem with the constant $C_n = E_n / \max_{i \in \{1,\ldots,m\}} (D_i, D_i')^m$. □

As a concluding remark, we mention a computational application of the proportionality principle. We proved that $\|M\| = \|M\|_{\mathrm{Lip}}$ for Hilbert modular varieties [23]. This fact combined with the proportionality principle 1.5 and work of Bucher-Karlsson [8] leads then to the following computation [23]:

Theorem 1.14. *Let Σ be a non-singular Hilbert modular surface. Then*

$$\|\Sigma\| = \frac{3}{2\pi^2} \,\mathrm{vol}(\Sigma).$$

Conversely, the proportionality principle 1.5 together with Thurston's computation of the simplicial volume of hyperbolic manifolds shows that the simplicial volume of hyperbolic manifolds of finite volume equals the Lipschitz simplicial volume. More generally, this holds true for locally symmetric spaces of $\mathbb{R}$-rank 1 [23, Theorem 1.5; see also beginning of Section 1.5]. However, in the general $\mathbb{Q}$-rank 1 case, the relation between the simplicial volume and the Lipschitz simplicial volume remains open.

Organization of this work. Section 2 reviews the basic properties of geodesic simplices and Thurston's straightening. The product inequality (Theorem 1.7) is proved in Section 3. Section 4 contains the proof of the proportionality principle (Theorem 1.5). Finally, Section 5 is devoted to the proof of the vanishing result (Theorem 1.1).

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2. Straightening and Lipschitz estimates of straight simplices

In Section 2.1, we collect some basic properties of geodesic simplices. We recall the technique of straightening singular chains for non-positively curved manifolds in Section 2.2. Variations of this straightening play an important role in the proofs of the proportionality principle (Theorem 1.5) and the product inequality (Theorem 1.7).

2.1. Geodesic simplices.

Let M be a simply connected, complete Riemannian manifold. Firstly assume that M has non-positive sectional curvature. For points x, x' in M, we denote by $[x, x'] : [0, 1] \to M$ the unique geodesic joining x and x'. The *geodesic join* of two maps f and $g : X \to M$ from a space X to M is the map defined by

$$[f, g] : X \times [0, 1] \to M, (x, t) \mapsto \big[f(x), g(x)\big](t).$$

We recall the notion of geodesic simplex: The standard simplex Δ^n is given by $\Delta^n = \{(z_0, \dots, z_n) \in \mathbb{R}^{n+1}_{\geq 0};\ \sum_i z_i = 1\}$, and we identify Δ^{n-1} with the subset $\{(z_0, \dots, z_n) \in \Delta^n;\ z_n = 0\}$. Moreover, the standard simplex is always equipped with the induced Euclidean metric. Let $x_0, \dots, x_n \in M$. The *geodesic simplex* $[x_0, \dots, x_n] : \Delta^n \to M$ with vertices $x_0, \dots, x_n$ is defined inductively as

$$[x_0, \dots, x_n]\big((1-t)s + t(0, \dots, 0, 1)\big) = \big[[x_0, \dots, x_{n-1}](s), x_n\big](t)$$

for $s \in \Delta^{n-1}$ and $t \in [0, 1]$.

More generally, if M admits an upper bound $K_0 \in (0, \infty)$ of the sectional curvature, then every pair of points with distance less than $K_0^{-1/2}\pi/2$ in M is joined by a unique geodesic. Thus we can define the *geodesic simplex* with vertices $x_0, \dots, x_n$ as before whenever $\{x_0, \dots, x_n\}$ has diameter less than $K_0^{-1/2}\pi/2$ [16, 4.3 (B)].

In the following two sections (Sections 2.1.1 and 2.1.2), we provide uniform estimates for Lipschitz constants of geodesic joins and simplices.

2.1.1. *Lipschitz estimates for geodesic joins.*

Proposition 2.1. *Let M be a simply connected, complete Riemannian manifold of non-positive sectional curvature, and let $n \in \mathbb{N}$. Let f, $g \in \operatorname{map}(\Delta^n, M)$ be smooth maps. Then $[f, g]$ is smooth and has a Lipschitz constant that depends only on the Lipschitz constants for f and g.*

Proof. Using the exponential map we can rewrite $[f, g]$ as

$$[f, g](x, t) = \exp_{f(x)}\big(t \cdot \exp^{-1}_{f(x)}(g(x))\big). \tag{2.2}$$

Since the exponential map viewed as a map $TM \to M \times M$ is a diffeomorphism, $[f, g]$ is smooth. The assertion about the Lipschitz constant is a consequence of the following lemma. □

Lemma 2.3. *Let X be a compact metric space and M as above. If f and $g : X \to M$ are two Lipschitz maps, then the geodesic join $[f, g] : X \times [0, 1] \to M$ is also a Lipschitz map, and we have*

$$\operatorname{Lip}[f, g] \leq 2 \cdot \big(\operatorname{Lip} f + \operatorname{Lip} g + \operatorname{diam}(\operatorname{im} f \cup \operatorname{im} g)\big).$$

Proof. Let (x,t), $(x',t') \in X \times [0,1]$. The triangle inequality yields

$$\begin{aligned} d_M\big([f,g](x,t),[f,g](x',t')\big) &\leq d_M\big([f(x),g(x)](t),[f(x),g(x)](t')\big) \\ &\quad + d_M\big([f(x),g(x)](t'),[f(x'),g(x')](t')\big). \end{aligned}$$

Because $\operatorname{Lip}[f(x),g(x)] = d_M(f(x),g(x))$, the first term satisfies

$$\begin{aligned} d_M\big([f(x),g(x)](t),[f(x),g(x)](t')\big) &\leq |t-t'| \cdot d_M\big(f(x),g(x)\big) \\ &\leq |t-t'| \cdot \operatorname{diam}\big(\operatorname{im} f \cup \operatorname{im} g\big); \end{aligned}$$

notice that $\operatorname{diam}(\operatorname{im} f \cup \operatorname{im} g)$ is finite because X is compact. The $\operatorname{CAT}(0)$-inequality allows us to simplify the second term as follows

$$\begin{aligned} d_M\big([f(x),g(x)](t'),[f(x'),g(x')](t')\big) &\leq d_M\big([f(x),g(x)](t'),[f(x),g(x')](t')\big) \\ &\quad + d_M\big([f(x),g(x')](t'),[f(x'),g(x')](t')\big) \\ &\leq (1-t') \cdot d_M\big(g(x),g(x')\big) \\ &\quad + t' \cdot d_M\big(f(x),f(x')\big) \\ &\leq d_M\big(g(x),g(x')\big) + d_M\big(f(x),f(x')\big) \\ &\leq \operatorname{Lip} f \cdot d_X(x,x') + \operatorname{Lip} g \cdot d_X(x,x'). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d_M\big([f,g](x,t),[f,g](x',t')\big) &\leq \big(\operatorname{Lip} f + \operatorname{Lip} g + \operatorname{diam}(\operatorname{im} f \cup \operatorname{im} g)\big) \\ &\quad \cdot 2 \cdot d_{X\times[0,1]}\big((x,t),(x',t')\big). \end{aligned}$$ □

2.1.2. *Lipschitz estimates for geodesic simplices.* Similarly to geodesic joins also geodesic simplices admit a uniform Lipschitz estimate and analogous smoothness properties.

Proposition 2.4. *Let M be a complete, simply connected, non-positively curved Riemannian manifold. Then every geodesic simplex in M is smooth. Moreover, for every $D > 0$ and $k \in \mathbb{N}$ there is $L > 0$ such that every geodesic k-simplex σ of diameter less than D satisfies $\|T_x\sigma\| < L$ for every $x \in \Delta^k$.*

Remark 2.5. Let M be a simply connected, complete Riemannian manifold of non-positive sectional curvature. If $x_0,\dots,x_k \in M$, then applying the triangle inequality inductively shows that

$$\forall_{y\in\Delta^k} \quad d_M\big([x_0,\dots,x_k](y),x_k\big) \leq k \cdot \max_{i,\ j\in\{0,\dots,k\}} d_M(x_i,x_j)$$

and hence that

$$\operatorname{diam}\big(\operatorname{im}[x_0,\dots,x_k]\big) \leq 2\cdot k \cdot \max_{i,\ j\in\{0,\dots,k\}} d_M(x_i,x_j).$$

In the proof of Theorem 1.8, it is necessary to have a more general version of Proposition 2.4 dealing with a positive upper sectional curvature bound. In this case, locally, the same arguments apply:

Proposition 2.6. *Let M be a complete, simply connected Riemannian manifold whose sectional curvature is bounded from above by $K_0 \in (0,\infty)$. Then every geodesic simplex σ of diameter less than $K_0^{-1/2}\pi/2$ is smooth. Further, there is a constant $L > 0$ such that every geodesic k-simplex σ of diameter less than $K_0^{-1/2}\pi/2$ satisfies $\|T_x\sigma\| < L$ for every $x \in \Delta^k$.*

The proofs of the following two lemmas used to prove Propositions 2.4 and 2.6 are elementary and thus omitted. The proof of the first one is very similar to Lee's proof of the Sturm comparison theorem [20, Proof of Theorem 11.1].

Lemma 2.7. *Let $u : [0,1] \to \mathbb{R}_{\geq 0}$ be a smooth function such that $u(0) = 0$, and $u(t) > 0$ for $t \in (0,1]$, as well as*

$$\forall_{t\in[0,1]} \quad \frac{d^2}{dt^2}u(t) + \frac{\pi^2}{4} \cdot u(t) \geq 0.$$

Then for all $t \in [0,1]$ we have

$$u(t) \leq u(1) \cdot \sin\bigl(t \cdot \pi/2\bigr).$$

Lemma 2.8. *Let $f : V \to W$ be a linear map between finite-dimensional vector spaces with inner products. Let $H \subset V$ be a subspace of co-dimension 1, and let $z \in V$ be a vector such that z and H span V. Let $\{y_1, \ldots, y_{k-1}\}$ be an orthonormal basis of H. Assume that for some $C > 0$*

$$\forall_{w\in\{z,y_1,\ldots,y_{k-1}\}} \quad \|f(w)\| \leq C \cdot \|w\|.$$

Further, assume that the angle α between z and H lies in $[\varepsilon, \pi/2]$ with $0 < \varepsilon \leq \pi/2$. Then there is a constant $L > 0$ that depends only on $\dim(V)$, C, and ε such that

$$\|f\| < L.$$

Proof of Proposition 2.4 and 2.6. That geodesic simplices are smooth is easily seen using the fact that the exponential map is a diffeomorphism. Let $K_0 \geq 0$ be an upper bound for the sectional curvature of M. By normalizing the metric we may assume that either $K_0 = 0$ or $K_0 = 1$. In the case $K_0 = 1$, it is understood that $D = \pi/2$. Led by the inductive definition of geodesic simplices, we prove the proposition by induction over k: For $k = 0$ or $k = 1$ there is nothing to show.

We now assume that there is an $L' > 0$ such that every geodesic $(k-1)$-simplex of diameter less than D is smooth and that the norm of its differential is less than L'. Let $\sigma := [x_0, \ldots, x_k] : \Delta^k \to M$ be a geodesic k-simplex of diameter less than D. By the induction hypothesis,

$$\forall_{p\in\Delta^{k-1}} \quad \bigl\|T_p[x_0, \ldots, x_{k-1}]\bigr\| < L'. \tag{2.9}$$

In the following, we write $v_0, \ldots, v_k$ for the vertices of Δ^k. Let $p \in \Delta^{k-1}$, and let $\gamma : [0,1] \to M$ denote the geodesic from x_k to $[x_0, \ldots, x_{k-1}](p)$. Choose an orthonormal basis $\{X_1, \ldots, X_{k-1}\}$ of the hyperplane in $\mathbb{R}^k$ spanned by Δ^{k-1}; then we can view $\{X_1, \ldots, X_{k-1}\}$ as an orthonormal frame of $T\Delta^{k-1}$. For $i \in \{1, \ldots, k-1\}$ we consider the following variation of γ:

$$H_i : (-\varepsilon_i, \varepsilon_i) \times [0,1] \to M, \ (s,t) \mapsto \bigl[x_k, \sigma(s \cdot X_i + p)\bigr](t).$$

Let $X_i(t) := \frac{d}{ds}H_i(s,t)|_{s=0} \in TM$. By definition, X_i is a Jacobi field along γ. Moreover, we have at each point $p(t) := [v_k, p](t)$ of Δ^k the relation

$$T_{p(t)}\sigma(X_i) = t \cdot X_i(t). \tag{2.10}$$

In order to obtain the desired bounds for $\|T_{p(t)}\sigma\|$ we first give estimates for $\|X_i(t)\|$ and then apply Lemma 2.8 to conclude the proof.

For the following computation, let D_t denote the covariant derivative along γ at $\gamma(t)$, and let K and R denote the sectional curvature and the curvature tensor, respectively. Straightforward differentiation and the Jacobi equation yield

$$\begin{aligned}\frac{d^2}{dt^2}\|X_i(t)\|^2 &= 2\cdot\|D_tX_i(t)\|^2 - \Big\langle R\Big(X_i(t),\frac{d}{dt}\gamma\Big)\frac{d}{dt}\gamma, X_i(t)\Big\rangle\\ &\geq -K_0\cdot\|X_i(t)\|^2\cdot\Big\|\frac{d}{dt}\gamma\Big\|^2\\ &\geq -K_0\cdot\|X_i(t)\|^2\cdot D^2.\end{aligned}$$

By definition, $X_i(0)=0$, and by (2.10) and (2.9),

$$\|X_i(1)\| = \|T_p\sigma(X_i)\| = \|T_p[x_0,\dots,x_{k-1}](X_i)\| < L'.$$

First assume that $K_0=0$. Then the smooth function $t\mapsto\|X_i(t)\|^2$ starts with the value 0, is non-negative, and convex. So it is non-decreasing. This implies that

$$\forall_{t\in[0,1]}\quad \|X_i(t)\|\leq\|X_i(1)\|<L'. \tag{2.11}$$

Next assume that $K_0=1$, thus $D=\pi/2$. Lemma 2.7 yields (2.11). Thus, in both cases $K_0=0$ or $K_0=1$ we see that $\|X_i(t)\|\leq L'$ for all $t\in[0,1]$. Further note that

$$\Big\|T_{p(t)}\sigma\Big(\frac{d}{dt}p\Big)\Big\| = \Big\|\frac{d}{dt}\gamma\Big\|\leq D.$$

So Lemma 2.8 implies that there is a constant $L>0$ that depends only on L', D, and k such that

$$\|T_{\gamma(t)}\sigma\|<L$$

because the angle between the line $p(t)$ and Δ^{k-1} is at least $\varepsilon>0$ with ε depending only on Δ^k. □

2.2. **Geodesic straightening.** In the following, we recall the definition of the geodesic straightening map on the level of chain complexes, as introduced by Thurston [29, p. 6.2f].

Let M be a connected, complete Riemannian manifold of non-positive sectional curvature. A singular simplex on M is *straight* if it is of the form $p_M\circ\sigma$ for some geodesic simplex σ on $\widetilde{M}$, where $p_M:\widetilde{M}\to M$ is the universal covering map. The subcomplex of the singular complex $C_*(M)$ generated by the straight simplices is denoted by $\operatorname{Str}_*(M)$; the elements of $\operatorname{Str}_*(M)$ are called *straight chains*. Every straight simplex is uniquely determined by the (ordered set of) vertices of its lift to the universal cover.

The *straightening* $s_M:C_*(M)\to\operatorname{Str}_*(M)$ is defined by

$$s_M(\sigma) := p_M\circ\big[\widetilde{\sigma}(v_0),\dots,\widetilde{\sigma}(v_*)\big]\text{ for }\sigma\in\operatorname{map}(\Delta^*,M),$$

where $p_M:\widetilde{M}\to M$ is the universal covering map, $v_0,\dots,v_*$ are the vertices of Δ^*, and $\widetilde{\sigma}$ is some p_M-lift of σ.

Notice that the definition of $s_M(\sigma)$ is independent of the chosen lift $\widetilde{\sigma}$ because the fundamental group $\pi_1(M)$ acts isometrically on $\widetilde{M}$.

Proposition 2.12 (Thurston)**.** *Let M be a connected, complete Riemannian manifold of non-positive sectional curvature. Then the straightening $s_M:C_*(M)\to\operatorname{Str}_*(M)$ and the inclusion $\operatorname{Str}_*(M)\to C_*(M)$ are mutually inverse chain homotopy equivalences.*

The easy proof is based on Lemma 2.13 below, which is a standard device for constructing chain homotopies. Because we need this lemma later, we reproduce the short argument for Proposition 2.12 here.

Proof of Proposition 2.12. For each singular simplex $\sigma : \Delta^n \to M$ on M, we define

$$H_\sigma := p_M \circ \left[\widetilde{\sigma}, [\widetilde{\sigma}(v_0), \dots, \widetilde{\sigma}(v_n)]\right] : \Delta^n \times [0,1] \to M,$$

where $v_0, \dots, v_n$ are the vertices of Δ^n, and $\widetilde{\sigma}$ is a lift of σ with respect to the universal covering map p_M. It is not difficult to see that H_σ is independent of the chosen lift $\widetilde{\sigma}$ and that H_σ satisfies the hypotheses of Lemma 2.13 below.

Therefore, Lemma 2.13 provides us with a chain homotopy between $\mathrm{id}_{C_*(M)}$ and the straightening map s_M. □

Lemma 2.13. *Let X be a topological space. For each $i \in \mathbb{N}$ and each singular i-simplex $\sigma : \Delta^i \to X$ let $H_\sigma : \Delta^i \times I \to X$ be a homotopy such that for each face map $\partial_k : \Delta^{i-1} \to \Delta^i$ we have*

$$H_{\sigma\circ\partial_k} = H_\sigma \circ (\partial_k \times \mathrm{id}_I).$$

Then $f^{(0)}$ and $f^{(1)} : C_*(X) \to C_*(X)$, defined by $f^{(m)}(\sigma) = H_\sigma \circ i_m$ for $m \in \{0,1\}$, are chain maps. For every $i \in \mathbb{N}$ there are $i+1$ affine simplices $G_{k,i} : \Delta^{i+1} \to \Delta^i \times I$ such that

$$H : C_i(X) \to C_{i+1}(X),\ h(\sigma) = \sum_{k=0}^{i} H_\sigma \circ G_{k,i}$$

defines a chain homotopy $f^{(0)} \simeq f^{(1)}$.

Proof. This is literally proved in Lee's book [19, Proof of Theorem 16.6, p. 422-424] although the lemma above is not stated as such. □

Remark 2.14. The simplices $G_{k,i}$ in the previous lemma arise from decomposing the prism $\Delta^i \times I$ into $(i+1)$-simplices.

3. Product inequality for the Lipschitz simplicial volume

This section is devoted to the proof of the product inequality (Theorem 1.7).

The corresponding statement in the compact case is proved by first showing that the simplicial volume can be computed in terms of bounded cohomology and then exploiting the fact that the cohomological cross-product is compatible with the semi-norm on bounded cohomology [1, Theorem F.2.5; 16, p. 17f]. In a similar fashion, the product inequality for the locally finite simplicial volume can be shown if one of the factors is compact [16, p. 17f; 22, Appendix C].

To prove the Lipschitz version, we proceed in the following steps:

(1) We show that the Lipschitz simplicial volume can be computed in terms of a suitable semi-norm on cohomology with Lipschitz compact supports; this semi-norm is a variant of the supremum norm parametrized by locally finite supports (Sections 3.1, 3.2, and 3.3).
(2) The failure of the product inequality for the locally finite simplicial volume is linked to the fact that there is no well-defined cross product on compactly supported cochains. In contrast, we show in Lemma 3.15 in Section 3.4 that there is a cross-product for cochains with Lipschitz compact support (Definition 3.6), and we analyze the interaction between this semi-norm and the cross-product on cohomology with compact supports (Section 3.4).
(3) Finally, we prove that the presence of non-positive curvature allows us to restrict attention to locally finite fundamental cycles of the product that have nice supports (Section 3.5). This enables us to use the information

on cohomology with Lipschitz compact supports to derive the product inequality (Section 3.6).

3.1. **Locally finite homology with a Lipschitz constraint.** The locally finite simplicial volume is defined in terms of the locally finite chain complex. In the same way, the Lipschitz simplicial volume is related to the chain complex of chains with *Lipschitz locally finite support*.

Definition 3.1. For a topological space X, we define $K(X)$ to be the set of all compact, connected, non-empty subsets of X.

For simplicity, we consider only connected compact subsets. This is essential when considering relative fundamental classes of pairs of type $(M, M - K)$.

Definition 3.2. Let X be a metric space, and let $k \in \mathbb{N}$. Then we write

$$S_k^{\mathrm{lf}}(X) := \bigl\{A \subset \mathrm{map}(\Delta^k, X);\ \forall_{K \in K(X)} \quad |\{\sigma \in A \mid \mathrm{im}(\sigma) \cap K \neq \emptyset\}| < \infty\bigr\}$$
$$S_k^{\mathrm{lf,Lip}}(X) := \bigl\{A \in S_k^{\mathrm{lf}}(X);\ \exists_{L \in \mathbb{R}_{>0}} \quad \forall_{\sigma \in A} \quad \mathrm{Lip}(\sigma) < L\bigr\}.$$

The elements of $S_k^{\mathrm{lf,Lip}}(X)$ are said to be *Lipschitz locally finite*. The subcomplex of $C_*^{\mathrm{lf}}(X)$ of all chains with Lipschitz locally finite support is denoted by $C_*^{\mathrm{lf,Lip}}(M)$, and the corresponding homology – so-called *homology with Lipschitz locally finite support* – is denoted by $H_*^{\mathrm{lf,Lip}}(X)$.

Theorem 3.3. *Let M be a connected Riemannian manifold. Then the homomorphism*

$$H_*^{\mathrm{lf,Lip}}(M) \to H_*^{\mathrm{lf}}(M)$$

induced by the inclusion $C_*^{\mathrm{lf,Lip}}(M) \to C_*^{\mathrm{lf}}(M)$ is an isomorphism.

During the course of the proof of this theorem, we rely on the following notation:

Definition 3.4. Let X be a proper metric space, and let $A \subset X$ be a subspace. Let $L \in \mathbb{R}_{>0}$.

(1) We write $C_*^{\mathrm{lf},<L}(X)$ for the subcomplex of $C_*^{\mathrm{lf,Lip}}(X)$ given by
$$C_*^{\mathrm{lf},<L}(X) := \bigl\{c \in C_*^{\mathrm{lf}}(X);\ \mathrm{Lip}(c) < L\bigr\}.$$
(2) Similarly, we define $C_*^{<L}(X) := \{c \in C_*(X);\ \mathrm{Lip}(c) < L\}$ as well as $C_*^{<L}(X, A) := C_*^{<L}(X)/C_*^{<L}(A)$.
(3) We use the abbreviation
$$C_*^{\mathrm{Lip}}(X) := \operatorname*{colim}_{L \to \infty} C_*^{<L}(X) = \bigl\{c \in C_*(X);\ \mathrm{Lip}(c) < \infty\bigr\}.$$
(4) The corresponding homology groups are denoted by $H_*^{\mathrm{lf},<L}(X)$, $H_*^{<L}(X)$, $H_*^{<L}(X, A)$, and $H_*^{\mathrm{Lip}}(X)$ respectively.

By definition, we can express the chain complex of chains with locally finite Lipschitz support via the colimit

$$C_*^{\mathrm{lf,Lip}}(X) \xleftarrow{\cong} \operatorname*{colim}_{L \to \infty} C_*^{\mathrm{lf},<L}(X).$$

with the obvious inclusions as structure maps. Moreover, if X is connected, the term on the right hand side expands to the inverse limit

$$\forall_{L \in \mathbb{R}_{>0}} \quad C_*^{\mathrm{lf},<L}(X) \xrightarrow{\cong} \varprojlim_{K \in K(X)} C_*^{<L}(X, X - K)$$

with the obvious projections as structure maps.

Proof of Theorem 3.3. We divide the proof into three steps:

(1) For all $L \in \mathbb{R}_{>0}$ and all $K \in K(M)$, the inclusion $C_*^{<L}(M, M-K) \to C_*(M, M-K)$ induces an isomorphism on homology.
(2) For all $L \in \mathbb{R}_{>0}$, the inclusion $C_*^{\mathrm{lf},<L}(M) \to C_*^{\mathrm{lf}}(M)$ induces an isomorphism on homology.
(3) The inclusion $C_*^{\mathrm{lf},\mathrm{Lip}}(M) \to C_*^{\mathrm{lf}}(M)$ induces an isomorphism on homology.

For the first step, let $L \in \mathbb{R}_{>0}$ and $K \in K(M)$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*^{<L}(M-K) & \longrightarrow & C_*^{<L}(M) & \longrightarrow & C_*^{<L}(M, M-K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_*(M-K) & \longrightarrow & C_*(M) & \longrightarrow & C_*(M, M-K) & \longrightarrow & 0 \end{array}$$

of chain complexes. By definition, the rows are exact; hence, there is a corresponding commutative diagram of long exact sequences in homology. In view of the five lemma, it is therefore sufficient to show that the inclusion $C_*^{<L}(U) \to C_*(U)$ induces an isomorphism on the level of homology whenever U is an open subset of M.

By Lemma 3.5 below, the inclusion $C_*^{\mathrm{Lip}}(U) \to C_*(U)$ is a homology isomorphism. Let $\mathrm{sd} : C_*(U) \to C_*(U)$ be the barycentric subdivision operator. The map sd is chain homotopic to the identity via a chain homotopy $h : C_*(U) \to C_{*+1}(U)$ [6, Section IV.17], and the classical construction of sd and h shows that both sd and h restrict to the Lipschitz chain complex $C_*^{\mathrm{Lip}}(U)$. Moreover, for every Lipschitz simplex σ on U there is a $k \in \mathbb{N}$ such that $\mathrm{Lip}(\mathrm{sd}^k \sigma) < L$. Now the same argument as in the classical proof that singular homology is isomorphic to the homology of the chain complex of "small" simplices [6, Section IV.17] shows that the inclusion $C_*^{<L}(U) \to C_*^{\mathrm{Lip}}(U)$ induces an isomorphism on homology. Therefore, $C_*^{<L}(U) \to C_*(U)$ is a homology isomorphism. This proves the first step.

We now come to the proof of the second step. Since the structure maps in the inverse system $(C_*^{<L}(M, M-K))_{K \in K(M)}$ are surjective, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 H_{n+1}^{<L}(M, M-K) & \longrightarrow & H_n^{\mathrm{lf},<L}(M) & \longrightarrow & \lim H_n^{<L}(M, M-K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \lim^1 H_{n+1}(M, M-K) & \longrightarrow & H_n^{\mathrm{lf}}(M) & \longrightarrow & \lim H_n(M, M-K) & \longrightarrow & 0 \end{array}$$

with exact rows [30, Theorem 3.5.8]. By the first step, the outer vertical arrows are isomorphisms. Therefore, the five lemma shows that also the middle vertical arrows is an isomorphism, which proves the second step.

Finally, the third step follows from the second step because homology is compatible with taking filtered colimits. □

Lemma 3.5. *Let M be a Riemannian manifold and let $U \subset M$ be an open subset. Then the inclusion $C_*^{\mathrm{Lip}}(U) \to C_*(U)$ induces an isomorphism on homology.*

Proof. The proof consists of an induction as, for example, in Bredon's proof of the de Rham theorem [6, Section V.9]:

If $U \subset \mathbb{R}^n$ is a bounded convex subset, then one can easily construct a chain contraction for $C_*^{\mathrm{Lip}}(U)$; therefore, the lemma holds for bounded convex subsets in Euclidean spaces.

If U, $V \subset M$ are open subsets such that the lemma holds for both of them as well as for the intersection $U \cap V$, then the lemma also holds for $U \cup V$: The classical construction of barycentric subdivision (and the corresponding chain homotopy to the identity) [6, Section IV.17] restricts to the Lipschitz chain complex and thus Lipschitz homology admits a Mayer-Vietoris sequence.

Proceeding by induction we see that the lemma holds for finite unions of bounded convex subsets of Euclidean space. Then a standard colimit argument shows that the lemma holds for arbitrary open subsets of Euclidean space.

We call an open subset V of M *admissible* if there is a smooth chart $V' \to \mathbb{R}^n$ and a compact set $K \subset M$ such that $V \subset K \subset V'$. In particular, any admissible subset of M is bi-Lipschitz homeomorphic to an open subset of $\mathbb{R}^n$, and hence the lemma holds for admissible subsets of M.

Noting that the intersection of two admissible sets is admissible, the Mayer-Vietoris argument shows that the lemma holds for finite unions of admissible sets. Any open subset of M can be written as a union of admissible sets; hence, a standard colimit argument yields that the lemma holds for arbitrary open subsets of M. □

3.2. **Cohomology with compact supports with a Lipschitz constraint.** The natural cohomological counterpart of locally finite homology is cohomology with compact supports. Similarly, the cohomology theory corresponding to Lipschitz locally finite homology is cohomology with Lipschitz compact supports; here, "corresponding" means in particular that there is an evaluation map linking homology and cohomology (Remark 3.7).

Definition 3.6. Let X be a metric space. A cochain $f \in \hom_{\mathbb{R}}(C_*^{\mathrm{Lip}}(X), \mathbb{R})$ is said to have *Lipschitz compact support* if for all $L \in \mathbb{R}_{>0}$ there exists a compact subset $K \subset X$ such that

$$\forall_{\sigma \in \operatorname{map}(\Delta^k, X)} \quad \bigl(\operatorname{Lip}(\sigma) < L \wedge \operatorname{im}(\sigma) \subset X - K\bigr) \Longrightarrow f(\sigma) = 0.$$

The cochains with Lipschitz compact support form a subcomplex of the cochain complex $\hom_{\mathbb{R}}(C_*^{\mathrm{Lip}}(X), \mathbb{R})$; this subcomplex is denoted by $C^*_{\mathrm{cs,Lip}}(X)$.

The cohomology of $C^*_{\mathrm{cs,Lip}}(X)$, denoted by $H^*_{\mathrm{cs,Lip}}(X)$, is called *cohomology with Lipschitz compact supports*.

Remark 3.7. Let X be a metric space. By construction of the chain complexes $C_*^{\mathrm{lf,Lip}}(X)$ and $C^*_{\mathrm{cs,Lip}}(X)$, the evaluation map

$$\begin{aligned} \langle \cdot\,, \cdot \rangle : C^*_{\mathrm{cs,Lip}}(X) \otimes C_*^{\mathrm{lf,Lip}}(X) &\longrightarrow \mathbb{R} \\ f \otimes \sum_{i \in I} a_i \cdot \sigma_i &\longmapsto \sum_{i \in I} a_i \cdot f(\sigma_i) \end{aligned}$$

is well-defined. Moreover, the same computations as in the case of locally finite homology/cohomology with compact supports show that this evaluation descends to a map $\langle \cdot\,, \cdot \rangle : H^*_{\mathrm{cs,Lip}}(X) \otimes H_*^{\mathrm{lf,Lip}}(X) \longrightarrow \mathbb{R}$ on the level of (co)homology.

Dually to Theorem 3.3, we obtain:

Theorem 3.8. *For all connected Riemannian manifolds, the natural homomorphism $C^*_{\mathrm{cs}}(M) \to C^*_{\mathrm{cs,Lip}}(M)$ given by restriction induces an isomorphism on cohomology.*

Proof. We start by disassembling the cochain complex $C^*_{\mathrm{cs,Lip}}(M)$ into pieces that are accessible by the universal coefficient theorem:

$$\begin{aligned} C^*_{\mathrm{cs,Lip}}(M) &= \varprojlim_{L\to\infty} C^*_{\mathrm{cs},<L}(M), \\ C^*_{\mathrm{cs},<L}(M) &= \operatorname*{colim}_{K\in K(M)} C^*_{<L}(M, M-K). \end{aligned}$$

Here, for all $L \in \mathbb{R}_{>0}$ and all $K \in K(M)$,

$$\begin{aligned} C^*_{<L}(M, M-K) &:= \hom_{\mathbb{R}}\bigl(C^{<L}_*(M, M-K), \mathbb{R}\bigr), \\ C^*_{\mathrm{cs},<L}(M) &:= \bigl\{f \in C^*_{<L}(M, \emptyset);\ i(f) \in C^*_{\mathrm{cs}}(M)\bigr\}, \end{aligned}$$

where $i : C^*_{<L}(M, \emptyset) \to C^*(M)$ is the map sending f to the extension $\overline{f}$ of f with $\overline{f}(\sigma) = 0$ whenever $\operatorname{Lip}\sigma \geq L$.

Let $L \in \mathbb{R}_{>0}$ and $K \in K(M)$. In the proof of Theorem 3.3, we have shown that $C^{<L}_*(M, M-K) \to C_*(M, M-K)$ induces an isomorphism on homology. Therefore, the restriction $C^*(M, M-K) \to C^*_{<L}(M, M-K)$ induces an isomorphism on the level of cohomology by the universal coefficient theorem. Because homology commutes with colimits, it follows that the restriction map $C^*_{\mathrm{cs}}(M) \to C^*_{\mathrm{cs},<L}(M)$ is a cohomology isomorphism.

Notice that the structure maps in the inverse system $(C^*_{\mathrm{cs},<L}(M))_{L\in\mathbb{R}_{>0}}$ are surjective, in particular, they satisfy the Mittag-Leffler condition. Furthermore, for $L < L'$ there is a $k \in \mathbb{N}$ such that the k-fold barycentric subdivision sd^k on $C^{<L'}_*(M)$ lands in $C^{<L}_*(M)$. The classical construction of the barycentric subdivision operator shows that $\mathrm{sd}^k : C^{<L'}_*(M) \to C^{<L}_*(M)$ is a homotopy inverse of the inclusion [6, Section IV.17]. Thus, the restriction map in cohomology $H^*_{\mathrm{cs},<L'}(M) \to H^*_{\mathrm{cs},<L}(M)$ is surjective; in particular, the Mittag-Leffler condition on the level of cohomology is also satisfied. Therefore, the $\lim^1$-term vanishes [30, Proposition 3.5.7], and we obtain [30, Theorem 3.5.8]

$$H^*_{\mathrm{cs,Lip}}(M) \cong \varprojlim_{L\to\infty} H^*_{\mathrm{cs},<L}(M) \cong H^*_{\mathrm{cs}}(M). \qquad \square$$

3.3. **Computing the Lipschitz simplicial volume via cohomology.** Any oriented, connected manifold possesses a (integral) *fundamental class*, which is a distinguished generator of the locally finite homology $H^{\mathrm{lf}}_n(M;\mathbb{Z}) \cong \mathbb{Z}$ with integral coefficients in the top dimension $n = \dim(M)$. The fundamental class in $H^{\mathrm{lf}}_n(M) = H^{\mathrm{lf}}_n(M;\mathbb{R})$ is, by definition, the image of the integral fundamental class under the coefficient change $H^{\mathrm{lf}}_n(M;\mathbb{Z}) \to H^{\mathrm{lf}}_n(M;\mathbb{R})$. Correspondingly, one defines the *cohomological* or *dual fundamental class* as a distinguished generator of the top cohomology with compact supports.

Definition 3.9. Let M be an oriented, connected Riemannian n-manifold (without boundary). The *Lipschitz fundamental class* of M is the homology class $[M]_{\mathrm{Lip}} \in H^{\mathrm{lf,Lip}}_n(M)$ that corresponds to the fundamental class $[M] \in H^{\mathrm{lf}}_n(M)$ via the isomorphism $H^{\mathrm{lf,Lip}}_*(M) \to H^{\mathrm{lf}}_*(M)$ (Theorem 3.3). Analogously, one defines the *Lipschitz dual fundamental class* $[M]^*_{\mathrm{Lip}} \in H^n_{\mathrm{cs,Lip}}(M)$ of M.

Remark 3.10. The proofs of Theorem 3.3 and 3.6 work for any coefficient module. Thus one can equivalently define the Lipschitz fundamental class as the image of the generator of $H_n^{\mathrm{lf,Lip}}(M;\mathbb{Z})$ that corresponds to the integral fundamental class in $H_n^{\mathrm{lf}}(M;\mathbb{Z}) \cong H_n^{\mathrm{lf,Lip}}(M;\mathbb{Z})$ under the change of coefficients $\mathbb{Z} \to \mathbb{R}$. Similar considerations apply to the Lipschitz dual fundamental class.

In the compact case, the simplicial volume can be expressed as the inverse of the semi-norm of the dual fundamental class [16, p. 17]. In the non-compact case, however, one has to be a bit more careful [16, p. 17; 22, Theorem C.2]. Similarly, also the Lipschitz simplicial volume can be computed in terms of certain semi-norms on cohomology (Proposition 3.12).

Definition 3.11. Let M be a topological space, $k \in \mathbb{N}$, and let $A \subset \operatorname{map}(\Delta^k, M)$.

(1) For a locally finite chain $c = \sum_{i\in I} a_i \cdot \sigma_i \in C_k^{\mathrm{lf}}(M)$, let
$$|c|_1^A := \begin{cases} |c|_1 & \text{if } \operatorname{supp}(c) \subset A, \\ \infty & \text{otherwise.}\end{cases},$$
Here, $\operatorname{supp}(c) := \{i \in I;\ a_i \neq 0\}$.

(2) The semi-norms on (Lipschitz) locally finite/relative homology induced by $|\cdot|_1^A$ are denoted by $\|\cdot\|_1^A$.

(3) If M is an oriented, connected n-manifold, then
$$\|M\|^A := \|[M]\|_1^A.$$
If moreover, $K \in K(M)$, then
$$\|M, M-K\|^A := \|[M, M-K]\|_1^A,$$
where $[M, M-K] \in H_n(M, M-K)$ is the relative fundamental class.

(4) For $f \in C^k(M)$ we write
$$\|f\|_\infty^A := \sup_{\sigma\in A} |f(\sigma)| \in [0,\infty].$$

(5) The semi-norms on (relative) cohomology with (Lipschitz) compact supports induced by $\|\cdot\|_\infty^A$ are also denoted by $\|\cdot\|_\infty^A$.

Proposition 3.12 (Duality principle for the Lipschitz simplicial volume)**.** *Let M be an oriented, connected Riemannian n-manifold.*

(1) Then
$$\|M\|_{\mathrm{Lip}} = \inf\bigl\{\|M\|^A;\ A \in S_n^{\mathrm{lf,Lip}}(M)\bigr\}.$$

(2) Moreover, for all $A \in S_n^{\mathrm{lf,Lip}}(M)$, we have
$$\|M\|^A = \frac{1}{\|[M]^*_{\mathrm{Lip}}\|_\infty^A}.$$

Proof. The first part follows directly from the definitions. For the second part let $A \in S_n^{\mathrm{lf,Lip}}(M)$. Then
$$\begin{aligned}\|M\|^A &= \sup_{K\in K(M)} \|M, M-K\|^A \\ &= \sup_{K\in K(M)} \frac{1}{\|[M, M-K]^*_{\mathrm{Lip}}\|_\infty^A} \\ &= \frac{1}{\|[M]^*_{\mathrm{Lip}}\|_\infty^A}.\end{aligned}$$

We now explain these steps in more detail:

- The first equality is shown by constructing an appropriate diagonal sequence out of "small" relative fundamental cycles of the $(M, M-K)$ supported on A [22, Proposition C.3].
- The class $[M, M-K]^*_{\mathrm{Lip}} \in H^n(\hom_{\mathbb{R}}(C^{\mathrm{Lip}}_*(M, M-K), \mathbb{R}))$ is the dual of the relative fundamental class in $H^{\mathrm{Lip}}_n(M, M-K) \cong H_n(M, M-K) \cong \mathbb{R}$.

  Therefore, the second equality is a consequence of the Hahn-Banach theorem – this is exactly the same argument as in the non-Lipschitz case [22, Proposition C.6], but applied to functionals on $C^{\mathrm{Lip}}_*(M)$ instead of $C_*(M)$; this is possible because A is Lipschitz.
- The last equality is equivalent to

  $$\inf_{K \in K(M)} \|[M, M-K]^*_{\mathrm{Lip}}\|^A_\infty = \|[M]^*_{\mathrm{Lip}}\|^A_\infty.$$

  Here the $\geq$-inequality is clear. For the $\leq$-inequality, let $\varepsilon > 0$ and consider $f \in C^n_{\mathrm{cs,Lip}}(M)$ with $\|f\|^A_\infty \leq \|[M]^*_{\mathrm{Lip}}\|^A_\infty + \varepsilon$. By Theorem 3.8, there is a compactly supported cochain g and a $(n-1)$-cochain h with Lipschitz compact support such that $f = g + \delta h$. Since $A \in S^{\mathrm{lf,Lip}}_n(M)$, the chain h' defined by

  $$h'(\sigma) = \begin{cases} h(\tau) & \text{if } \sigma \in \bigcup_{j=0}^n \{\partial_j \sigma;\ \sigma \in A\ \}, \\ 0 & \text{otherwise,} \end{cases}$$

  is compactly supported. Further, $f' := g + \delta h'$ is compactly supported, cohomologous in $C^*_{\mathrm{cs,Lip}}(M)$ to f, and $\|f'\|^A_\infty = \|f\|^A_\infty$. In particular, there is $K \in K(M)$ with $f' \in C^n_{\mathrm{Lip}}(M, M-K)$ and

  $$\|[M, M-K]^*_{\mathrm{Lip}}\|^A_\infty \leq \|f'\|^A_\infty = \|f\|^A_\infty \leq \|[M]^*_{\mathrm{Lip}}\|^A_\infty + \varepsilon.$$

This finishes the proof of the duality principle. □

3.4. **Product structures in the Lipschitz setting.** The definition of product structures in singular (co-)homology is based on the following maps: Let X and Y be topological spaces. Then there exist chain maps $\mathrm{EZ} : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ and $\mathrm{AW} : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$, called the *Eilenberg-Zilber map* and the *Alexander-Whitney map*, respectively, such that $\mathrm{EZ} \circ \mathrm{AW}$ and $\mathrm{AW} \circ \mathrm{EZ}$ both are naturally homotopic to the identity; explicit formulas are, for example, given in Dold's book [10, 12.26 on p. 184].

The map EZ and the composition $C^*(X) \otimes C^*(Y) \to C^*(X \times Y)$, $f \otimes g \mapsto (f \otimes g) \circ \mathrm{AW}$, induce the so-called *cross-products*

$$\begin{aligned} \times : H_m(X) \otimes H_n(Y) &\to H_{m+n}(X \times Y), \\ \times : H^m(X) \otimes H^n(Y) &\to H^{m+n}(X \times Y) \end{aligned} \tag{3.13}$$

in homology and cohomology, respectively.

Next we describe these cross-products more explicitly on the (co)chain level: Let $f \in C^m(X)$ and $g \in C^n(Y)$. Let π_X and π_Y be the projections from $X \times Y$ to X and Y, respectively. For a k-simplex σ, let $\sigma\rfloor_l$ and ${}_{(k-l)}\lfloor\sigma$ the *l-front face* and the *$(k-l)$-back face* of σ, respectively. Then the explicit formula for AW in *loc. cit.* yields

$$(f \times g)(\sigma) = f(\pi_X \circ \sigma\rfloor_m) \cdot g(\pi_Y \circ {}_n\lfloor\sigma). \tag{3.14}$$

For simplices $\sigma : \Delta^m \to X$ and $\varrho : \Delta^n \to Y$, the chain $\mathrm{EZ}(\sigma \otimes \varrho)$ can be described as follows: The product $\Delta^n \times \Delta^m \to X \times Y$ of σ and ϱ is not a simplex but can be chopped into a union of $(m+n)$-simplices (like a square can be chopped into triangles, or a prism into tetrahedra). Then $\mathrm{EZ}(\sigma \otimes \varrho)$ is the sum of these $(m+n)$-simplices.

From this description we see that if $c = \sum_i a_i \sigma_i$ and $d = \sum_j b_j \varrho_j$ are (Lipschitz) locally finite chains in (metric) spaces X and Y, then $\sum_{i,j} a_i b_j (\sigma_i \times \varrho_j)$ is a (Lipschitz) locally finite chain in $X \times Y$. Thus, (3.13) extends to maps

$$\begin{aligned} \times : H_m^{\mathrm{lf}}(X) \otimes H_n^{\mathrm{lf}}(Y) &\to H_{m+n}^{\mathrm{lf}}(X \times Y), \\ \times : H_m^{\mathrm{lf,Lip}}(X) \otimes H_n^{\mathrm{lf,Lip}}(Y) &\to H_{m+n}^{\mathrm{lf,Lip}}(X \times Y). \end{aligned}$$

In general, the cross-product of two cocycles with compact supports has not necessarily compact support. However, the cross-product of two cochains with Lipschitz compact supports again has Lipschitz compact support:

Lemma 3.15. *Let M and N be two complete metric spaces, and let $m, n \in \mathbb{N}$. Then the cross-product on $C^*(M) \otimes C^*(N) \to C^*(M \times N)$ restricts to a cross-product*

$$\times : C^m_{\mathrm{cs,Lip}}(M) \otimes C^n_{\mathrm{cs,Lip}}(N) \to C^{m+n}_{\mathrm{cs,Lip}}(M \times N),$$

which induces a cross-product $H^m_{\mathrm{cs,Lip}}(M) \otimes H^n_{\mathrm{cs,Lip}}(N) \to H^{m+n}_{\mathrm{cs,Lip}}(M \times N)$.

Proof. Let $f \in C^m_{\mathrm{cs,Lip}}(M)$ and $g \in C^n_{\mathrm{cs,Lip}}(N)$. Let $L \in \mathbb{R}_{>0}$. Because f and g are cochains with Lipschitz compact supports, there are compact sets $K_M \subset M$ and $K_N \subset N$ with

$$\forall_{\sigma \in \mathrm{map}(\Delta^m, M)} \ \ \big(\mathrm{Lip}(\sigma) \leq L \ \wedge \ \mathrm{im}(\sigma) \subset M - K_M\big) \Longrightarrow f(\sigma) = 0,$$

and analogously for g and K_N.

We now consider the compact set $K := U_L(K_M) \times U_L(K_N) \subset M \times N$, where $U_L(X)$ denotes the set of all points with distance at most L from X. Because the diameter of the image of a Lipschitz map on a standard simplex is at most as large as $\sqrt{2}$ times the Lipschitz constant of the map in question, we obtain: If $\sigma \in \mathrm{map}(\Delta^{m+n}, M \times N)$ with $\mathrm{Lip}(\sigma) \leq L$ and $\mathrm{im}(\sigma) \subset M \times N - K$, then

$$\mathrm{im}(\pi_M \circ \sigma) \subset M - K_M \qquad \text{or} \qquad \mathrm{im}(\pi_N \circ \sigma) \subset N - K_N.$$

In particular, $f(\pi_M \circ \sigma\rfloor_m) = 0$ or $g(\pi_N \circ {}_n\lfloor\sigma) = 0$. By (3.14), $(f \times g)(\sigma) = 0$. In other words, the cross-product $f \times g$ lies in $C^{m+n}_{\mathrm{cs,Lip}}(M \times N)$. ☐

Definition 3.16. Let M and N be two topological spaces, let $m, n \in \mathbb{N}$, and let $A \subset \mathrm{map}(\Delta^{m+n}, M \times N)$. Then we write

$$\begin{aligned} A_M &:= \big\{\pi_M \circ \sigma\rfloor_m;\ \sigma \in A\big\}, \\ A_N &:= \big\{\pi_N \circ {}_n\lfloor\sigma;\ \sigma \in A\big\}, \end{aligned}$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections. Notice that A_M and A_N depend on m and n, but the context will always make clear which indices are involved; therefore, we suppress m and n in the notation.

The cross-product of cochains with Lipschitz compact support is continuous in the following sense:

Remark 3.17. Let M and N be two topological spaces, and let $m,\, n \in \mathbb{N}$. Then – by the explicit description (3.14) – the cross-product satisfies

$$\|f \times g\|_\infty^A \leq \|f\|_\infty^{A_M} \cdot \|g\|_\infty^{A_N}$$

for all $A \subset \operatorname{map}(\Delta^{m+n}, M \times N)$ and all $f \in C^m_{\mathrm{cs,Lip}}(M)$, $g \in C^n_{\mathrm{cs,Lip}}(N)$.

Notice however, that in general the sets A_M and A_N are *not* locally finite even if A is locally finite. This issue is addressed in Section 3.5.

Lemma 3.18. *Let M and N be oriented, connected, complete Riemannian manifolds. Then*

$$[M \times N]^*_{\mathrm{Lip}} = [M]^*_{\mathrm{Lip}} \times [N]^*_{\mathrm{Lip}} \in H^*_{\mathrm{cs,Lip}}(M \times N).$$

Proof. In view of Remark 3.10, it is enough to show

$$\big\langle [M]^*_{\mathrm{Lip}} \times [N]^*_{\mathrm{Lip}}, [M]_{\mathrm{Lip}} \times [N]_{\mathrm{Lip}} \big\rangle = 1.$$

Let $f \in C^m_{\mathrm{cs,Lip}}(M)$ and $g \in C^n_{\mathrm{cs,Lip}}(N)$ be fundamental cocycles that vanish on degenerate simplices. Such fundamental cocycles always exist; for example, let f be the cocycle $\sigma \mapsto \int_{\Delta^m} \sigma^*\omega$ where $\omega \in \Omega^m(M)$ is a compactly supported differential m-form with $\int_M \omega = 1$. Note that the integral exists by Rademacher's theorem [14]. Let $w = \sum_i a_i \sigma_i \in C^{\mathrm{lf,Lip}}_m(M)$ and $z = \sum_j b_j \varrho_j \in C^{\mathrm{lf,Lip}}_n(N)$ be fundamental cycles of M and N respectively. The Eilenberg-Zilber and Alexander-Whitney maps have the property that $\mathrm{AW} \circ \mathrm{EZ}$ differs from the identity by degenerate chains [13, Theorem 2.1a (2.3)]. Thus, we obtain

$$\begin{aligned}
\langle f \times g, w \times z \rangle &= \sum_{i,j} a_i b_j (f \times g)(\sigma_i \times \varrho_j) \\
&= \sum_{i,j} a_i b_j (f \otimes g)\big(\mathrm{AW} \circ \mathrm{EZ}(\sigma_i \otimes \varrho_j)\big) \\
&= \sum_{i,j} a_i b_j (f \otimes g)\big(\sigma_i \otimes \varrho_j + \text{degenerate simplices}\big) \\
&= \sum_{i,j} a_i b_j f(\sigma_i) g(\varrho_j) = f(w) g(z) = 1.
\end{aligned}$$

□

3.5. **Representing the fundamental class of the product by sparse cycles.** The functor C^{lf}_* is only functorial with respect to *proper* maps. For example, in general, the projection of a locally finite chain on a product of non-compact spaces to one of its factors is not locally finite.

Definition 3.19. Let M and N be two topological spaces, and let $k \in \mathbb{N}$. A locally finite set $A \in S^{\mathrm{lf}}_k(M \times N)$ is called *sparse* if

$$\big\{\pi_M \circ \sigma;\ \sigma \in A\big\} \in S^{\mathrm{lf}}_k(M) \qquad \text{and} \qquad \big\{\pi_N \circ \sigma;\ \sigma \in A\big\} \in S^{\mathrm{lf}}_k(N),$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections.

A locally finite chain $c \in C^{\mathrm{lf}}_*(M \times N)$ is called *sparse* if its support is sparse.

The following proposition is crucial in proving the product inequality for the Lipschitz simplicial volume.

Proposition 3.20. *Let M and N be two oriented, connected, complete Riemannian manifolds (without boundary) with non-positive sectional curvature.*

(1) *For any cycle* $c \in C_*^{\mathrm{lf,Lip}}(M \times N)$ *there is a sparse cycle* $c' \in C_*^{\mathrm{lf,Lip}}(M \times N)$ *satisfying*

$$|c'|_1 \leq |c|_1 \qquad \text{and} \qquad c \sim c' \text{ in } C_*^{\mathrm{lf,Lip}}(M \times N).$$

(2) *In particular, the Lipschitz simplicial volume can be computed via sparse fundamental cycles, i.e.,*

$$\|M \times N\|_{\mathrm{Lip}} = \inf\bigl\{\|M \times N\|^A;\ A \in S^{\mathrm{lf,Lip}}_{\dim M + \dim N}(M \times N),\ A \text{ sparse}\bigr\}.$$

Proof. The second part is a direct consequence of the first part. For the first part, we take advantage of a straightening procedure:

Let $F_M \subset M$ and $F_N \subset N$ be locally finite subsets with $U_1(F_M) = M$ and $U_1(F_N) = N$. Then the corresponding preimages $\widetilde{F}_M := p_M^{-1}(F_M) \subset \widetilde{M}$ and $\widetilde{F}_N := p_N^{-1}(F_N) \subset \widetilde{N}$ satisfy $U_1(\widetilde{F}_M) = \widetilde{M}$ and $U_1(\widetilde{F}_N) = \widetilde{N}$, where $p_M : \widetilde{M} \to M$ and $p_N : \widetilde{N} \to N$ are the Riemannian universal covering maps.

Furthermore, also the product $F := F_M \times F_N \subset M \times N$ is locally finite, and there is a $\pi_1(M) \times \pi_1(N)$-equivariant map $f : \widetilde{M} \times \widetilde{N} \to \widetilde{F}_M \times \widetilde{F}_N =: \widetilde{F}$ such that

$$d_{\widetilde{M}\times\widetilde{N}}\bigl(z, f(z)\bigr) \leq \sqrt{2}$$

holds for all $z \in \widetilde{M} \times \widetilde{N}$.

For $\sigma \in \operatorname{map}(\Delta^k, M \times N)$, we define

$$h_\sigma := (p_M \times p_N) \circ \bigl[\widetilde{\sigma}, [f(\widetilde{\sigma}(v_0)), \dots, f(\widetilde{\sigma}(v_k))]\bigr] : \Delta^k \times [0,1] \to M \times N,$$

where $v_0, \dots, v_k$ are the vertices of the standard simplex Δ^k, and $\widetilde{\sigma}$ is a lift of σ. By Lemma 2.3 and Proposition 2.4 (and Remark 2.5), the map h_σ is Lipschitz, and the Lipschitz constant can be estimated from above in terms of the Lipschitz constant of σ. Moreover, the fact that f is equivariant and covering theory show that

$$h_{\sigma\circ\partial_j} = h_\sigma \circ (\partial_j \times \mathrm{id}_{[0,1]}) \tag{3.21}$$

for all $\sigma \in \operatorname{map}(\Delta^k, M \times N)$ and all $j \in \{0, \dots, k-1\}$, where $\partial_j : \Delta^{k-1} \to \Delta^k$ is the inclusion of the j-th face.

We now consider the map

$$\begin{aligned} H : C_*^{\mathrm{lf,Lip}}(M \times N) &\longrightarrow C_{*+1}^{\mathrm{lf,Lip}}(M \times N) \\ \sum_{i\in I} a_i \cdot \sigma_i &\longmapsto \sum_{i \in I} a_i \cdot \overline{h}_{\sigma_i}, \end{aligned}$$

where $\overline{h}_\sigma$ is the singular chain constructed out of h_σ by subdividing the prism $\Delta^k \times [0,1]$ in the canonical way into a sum of $k+1$ simplices of dimension $k+1$ (compare Lemma 2.13).

The map H is indeed well-defined: As discussed above, for all $c \in C_k^{\mathrm{lf,Lip}}(M \times N)$, all simplices occurring in the (formal) sum $H(c)$ satisfy a uniform Lipschitz condition depending on $\operatorname{Lip}(c)$. Further, it follows from $\operatorname{im}(h_\sigma) \subset U_{\sqrt{2}}(\operatorname{im}(\sigma))$ that H maps locally finite chains to locally finite chains. As next step, we define

$$\begin{aligned} \varphi : C_*^{\mathrm{lf,Lip}}(M \times N) &\longrightarrow C_*^{\mathrm{lf,Lip}}(M \times N) \\ \sum_{i\in I} a_i \cdot \sigma_i &\longmapsto \sum_{i \in I} a_i \cdot h_{\sigma_i}(\,\cdot\,, 1). \end{aligned}$$

In other words, φ is given by replacing each simplex by a straight simplex whose vertices lie in $F_M \times F_N$ and whose vertices are close to the ones of the original simplex. Property (3.21) implies that φ is a chain map and that H is a chain homotopy between the identity and φ (see Lemma 2.13).

By construction, $\|\varphi\| \leq 1$. Therefore, it remains to show that the image of φ contains only sparse chains:

Let $c \in C_k^{\mathrm{lf,Lip}}(M \times N)$. Let $A := \operatorname{supp}(\varphi(c))$. Because the geodesics in $\widetilde{M} \times \widetilde{N}$ are just products of geodesics in $\widetilde{M}$ and $\widetilde{N}$, it follows that the projection $\pi_M : M \times N \to M$ preserves straight simplices. Thus, the set $\{\pi_M \circ \sigma;\ \sigma \in A\}$ consists of straight simplices whose Lipschitz constant is bounded by $\operatorname{Lip}(c)$ and whose vertices lie in F_M. The fact that F_M is locally finite and that there are only finitely many straight simplices with a bounded Lipschitz constant and the same vertices imply that $\{\pi_M \circ \sigma;\ \sigma \in A\}$ is locally finite. Similarly for the projection to N. So the chain $\varphi(c)$ is sparse. □

3.6. **Conclusion of the proof of the product inequality.** Finally, we can put all the pieces collected in the previous sections together to give a proof of the product inequality:

Proof of Theorem 1.7. In the following, we write $m := \dim M$ and $n := \dim N$. In order to prove the product inequality, it suffices to find for each $\varepsilon \in \mathbb{R}_{>0}$ locally finite sets $A_M \in S_m^{\mathrm{lf,Lip}}(M)$ and $A_N \in S_n^{\mathrm{lf,Lip}}(N)$ with

$$\|M\|^{A_M} \cdot \|N\|^{A_N} \leq \|M \times N\|_{\mathrm{Lip}} + \varepsilon.$$

For every $\varepsilon \in \mathbb{R}_{>0}$, Proposition 3.20 provides us with a sparse fundamental cycle $c \in C_{m+n}^{\mathrm{lf,Lip}}(M \times N)$ with support A satisfying $|c|_1 \leq \|M \times N\|_{\mathrm{Lip}} + \varepsilon$. In particular,

$$\|M \times N\|_{\mathrm{Lip}}^{A} \leq \|M \times N\|_{\mathrm{Lip}} + \varepsilon.$$

By sparseness, the sets A_M and A_N associated to A (see Definition 3.16) lie in $S_m^{\mathrm{lf,Lip}}(M)$ and $S_n^{\mathrm{lf,Lip}}(N)$, respectively. The duality principle (Proposition 3.12) yields

$$\|M\|^{A_M} = \frac{1}{\|[M]^*_{\mathrm{Lip}}\|_\infty^{A_M}}, \quad \|N\|^{A_N} = \frac{1}{\|[N]^*_{\mathrm{Lip}}\|_\infty^{A_N}}, \quad \|M \times N\|^{A} = \frac{1}{\|[M \times N]^*_{\mathrm{Lip}}\|_\infty^{A}};$$

the cohomological terms are related as follows

$$\|[M \times N]^*_{\mathrm{Lip}}\|_\infty^{A} \leq \|[M]^*_{\mathrm{Lip}}\|_\infty^{A_M} \cdot \|[N]^*_{\mathrm{Lip}}\|_\infty^{A_N}$$

because $[M \times N]^*_{\mathrm{Lip}} = [M]^*_{\mathrm{Lip}} \times [N]^*_{\mathrm{Lip}}$ (Lemma 3.18) and the cohomological cross-product is compatible with the semi-norms (Remark 3.17).

Therefore, we obtain

$$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leq \|M\|^{A_M} \cdot \|N\|^{A_N} \leq \|M \times N\|^{A} \leq \|M \times N\|_{\mathrm{Lip}} + \varepsilon. \qquad \square$$

4. Proportionality principle for non-compact manifolds

Thurston's proof of the proportionality principle in the compact case is based on "smearing" singular chains to so-called measure chains [28, Chapter 5; 29, p. 6.6–6.10]. We prove the proportionality principle in the non-compact case by combining the smearing technique with a discrete approximation of it; to this end, we replace measure homology by Lipschitz measure homology, a variant that incorporates a Lipschitz constraint (Section 4.2).

Throughout Section 4, we often refer to the following setup:

Setup 4.1. Let M and N be oriented, connected, complete, non-positively curved Riemannian manifolds of finite volume without boundary whose universal covers are isometric. We denote the common universal cover by U. Let $G = \operatorname{Isom}^+(U)$ be its group of orientation-preserving isometries. Then $\Gamma = \pi_1(M)$ and $\Lambda = \pi_1(N)$ are lattices in G by Lemma 4.2 below. Let $\mu_{\Lambda\backslash G}$ denote the normalized Haar measure on $\Lambda\backslash G$. The universal covering maps of M and N are denoted by p_M and p_N, respectively.

The following lemma is well known for locally symmetric spaces and compact manifolds but we were unable to find a reference in the general case.

Lemma 4.2. *Let M be a complete Riemannian manifold of finite volume. Then $\Gamma = \pi_1(M)$ is a lattice in $G = \operatorname{Isom}(\widetilde{M})$.*

Proof. The isometry group G acts smoothly and properly on $\widetilde{M}$. It is easy to see that Γ is a discrete subgroup. Let $x_0 \in \widetilde{M}$, and let $K \subset G$ be the stabilizer of x_0. Let $\nu \to Gx_0$ be the normal bundle of Gx_0, and let $\nu(r)$ denote the sub-bundle of vectors of length at most r. By the slice theorem [11, Chapter 2; 24, Section 2.2], there exists $r > 0$ such that the exponential map $\exp : \nu(r) \to V$ is a diffeomorphism onto a tubular neighborhood V of Gx_0. The map $f : G \times_K \nu_{x_0} \to \nu$, $(g, z) \mapsto Tg(z)$ is a diffeomorphism. Define $g = f^{-1} \circ \exp^{-1}$. We equip G/K with the Riemannian metric that turns the diffeomorphism $G/K \to Gx_0$ into an isometry. Since ν_{x_0} can be equipped with a K-invariant metric (K is compact), it is easy to see that $G \times_K \nu_{x_0}(r)$ carries a G-invariant Riemannian metric such that the projection $G \times_K \nu_{x_0}(r) \to G/K$ is a Riemannian submersion. By compactness, there is $\lambda > 0$ such that $T_z g$ has norm at most λ for all $z \in \exp(\nu_{x_0}(r))$. By G-invariance of the metrics, $T_z g$ has norm at most λ for all $z \in V$, thus, g is λ-Lipschitz, and so is the induced map between the Γ-quotients. We obtain that

$$\operatorname{vol}\bigl(\Gamma\backslash(G \times_K \nu_{x_0}(r))\bigr) \leq \lambda^{\dim(M)} \operatorname{vol}(\Gamma\backslash V) < \infty.$$

Fubini's theorem for Riemannian submersions [26, Theorem 5.6 on p. 66] yields

$$\operatorname{vol}(\Gamma\backslash G/K) \operatorname{vol}(\nu_{x_0}(r)) = \operatorname{vol}\bigl(\Gamma\backslash(G \times_K \nu_{x_0}(r))\bigr) < \infty.$$

Thus $\operatorname{vol}(\Gamma\backslash G/K) < \infty$. Now equip G with a G-equivariant metric such that $G \to G/K$ is a Riemannian submersion. By uniqueness, the corresponding Riemannian measure on G is a Haar measure. Fubini's theorem and $\operatorname{vol}(\Gamma\backslash G/K) < \infty$ show that $\operatorname{vol}(\Gamma\backslash G) < \infty$. □

4.1. **Integrating Lipschitz chains.** Before introducing the smearing operation in Section 4.2, we first discuss integration of Lipschitz chains, which provides a means to detect which class in locally finite homology a given Lipschitz cycle represents.

Let M be an n-dimensional Riemannian manifold, and let $K \subset M$ be a compact, connected subset with non-empty interior. Let $\Omega^*(M, M - K)$ be the kernel of the restriction homomorphism $\Omega^*(M) \to \Omega^*(M - K)$ on differential forms. The corresponding cohomology groups are denoted by $H^*_{\mathrm{dR}}(M, M - K)$. The de Rham map $\Omega^*(M) \to C^*(M)$ restricts to the respective kernels and thus induces a homomorphism, called *relative de Rham map*,

$$\Psi^* : H^*_{\mathrm{dR}}(M, M - K) \to H^*(M, M - K).$$

The relative de Rham map is an isomorphism, which follows from the bijectivity of the absolute de Rham map and an application of the five lemma. Note that integration gives a homomorphism $\int : H^n_{\mathrm{dR}}(M, M-K) \to \mathbb{R}$. Moreover, it is well known that

$$\big\langle \Psi^n[\omega], [M, M-K]\big\rangle = \int_M \omega \tag{4.3}$$

holds for all n-forms ω.

Proposition 4.4. *Let M be a Riemannian n-manifold, and let $c = \sum_{k\in\mathbb{N}} a_k\sigma_k \in C^{\mathrm{lf}}_n(M)$ be a cycle with $|c|_1 < \infty$ and $\mathrm{Lip}(c) < \infty$.*

1. *Then $\langle \mathrm{dvol}_M, \sigma_k\rangle \le \mathrm{Lip}(c)^n \operatorname{vol}(\Delta^n)$ for every $k \in \mathbb{N}$.*
2. *Furthermore, we have the following equivalence:*

$$\sum_{k\in\mathbb{N}} a_k \cdot \langle \mathrm{dvol}_M, \sigma_k\rangle = \operatorname{vol}(M) \iff c \text{ is a fundamental cycle.}$$

Proof. For the first part, it suffices to observe that all Lipschitz simplices σ are almost everywhere differentiable, that $\sigma^*\,\mathrm{dvol}_M$ is measurable (by Rademacher's theorem [14]), and that

$$\big|\langle \mathrm{dvol}_M, \sigma\rangle\big| = \left|\int_{\Delta^n} \sigma^*\,\mathrm{dvol}_M\right| \le \operatorname*{ess-sup}_{x\in\Delta^n} \|T_x\sigma\|^n \operatorname{vol}\big(\Delta^n\big) \le \mathrm{Lip}(\sigma)^n \operatorname{vol}\big(\Delta^n\big)$$

holds. In particular, we see that $\sum_{k\in\mathbb{N}} a_k\langle \mathrm{dvol}_M, \sigma_k\rangle$ converges absolutely.

For the second part, let $s \in \mathbb{R}$ be the number defined by

$$[c] = s\cdot [M] \in H^{\mathrm{lf}}_n(M).$$

In the following, we show that $\sum_{k\in\mathbb{N}} a_k\langle \mathrm{dvol}_M, \sigma_k\rangle = s\cdot\operatorname{vol}(M)$: To this end, we first relate $s\cdot\operatorname{vol}(K)$ for compact K to a finite sum derived from the series on the left hand side, and then use a limit process to compute the value of the whole series.

Let $K \subset M$ be a connected, compact subset with non-empty interior. For $\delta \in \mathbb{R}_{>0}$ let $g_\delta : M \to [0,1]$ be a smooth function supported on the closed δ-neighborhood K_δ of K with $g|_K = 1$. Then $g_\delta \cdot \mathrm{dvol}_M \in \Omega^n(M, M-K_\delta)$ is a cocycle, and

$$s\cdot\operatorname{vol}(K) = \lim_{\delta\to 0} s\cdot \int_M g_\delta\,\mathrm{dvol}_M\,.$$

On the other hand, the map $H_n(j_\delta) : H^{\mathrm{lf}}_n(M) \to H_n(M, M-K_\delta)$ induced by the inclusion $j_\delta : (M,\emptyset) \to (M, M-K_\delta)$ maps the fundamental class of M to the relative fundamental class of $(M, M-K_\delta)$ and $H_n(j_\delta)[c]$ is represented by $\sum_{\operatorname{im}\sigma_k\cap K_\delta\neq\emptyset} a_k\sigma_k$. Therefore, we obtain by (4.3)

$$\begin{aligned}\lim_{\delta\to 0} \sum_{\operatorname{im}\sigma_k\cap K_\delta\neq\emptyset} a_k\cdot\langle g_\delta\cdot\mathrm{dvol}_M, \sigma_k\rangle &= \lim_{\delta\to 0}\big\langle \Psi^n[g_\delta\cdot\mathrm{dvol}_M], s\cdot[M, M-K_\delta]\big\rangle\\ &= \lim_{\delta\to 0} s\cdot\int_M g_\delta\,\mathrm{dvol}_M\\ &= s\cdot\operatorname{vol}(K).\end{aligned}$$

For each $k \in \mathbb{N}$ and $\delta \in \mathbb{R}_{>0}$ we have $|\langle g_\delta \cdot \mathrm{dvol}_M, \sigma_k\rangle| \leq \mathrm{Lip}(c)^n \operatorname{vol}(\Delta^n)$, and hence

$$\Big|\sum_{k\in\mathbb{N}} a_k \langle \mathrm{dvol}_M, \sigma_k\rangle - \sum_{\operatorname{im}\sigma_k\cap K_\delta\neq 0} a_k\langle g_\delta\cdot \mathrm{dvol}_M,\sigma_k\rangle\Big| \leq 2\,\mathrm{Lip}(c)^n \operatorname{vol}(\Delta^n)\cdot \sum_{\operatorname{im}\sigma_k\subset M-K} |a_k|.$$

Because $\sum_{k\in\mathbb{N}} |a_k| < \infty$, there is an exhausting sequence $(K^m)_{m\in\mathbb{N}}$ of compact, connected subsets of M with non-empty interior satisfying

$$\lim_{m\to\infty} \operatorname{vol}(K^m) = \operatorname{vol}(M) \qquad \text{and} \qquad \lim_{m\to\infty} \sum_{\operatorname{im}\sigma_k\subset M-K^m} |a_k| = 0.$$

Thus, the estimates of the previous paragraphs yield

$$\begin{aligned}\sum_{k\in\mathbb{N}} a_k\cdot\langle \mathrm{dvol}_M,\sigma_k\rangle &= \lim_{m\to\infty}\lim_{\delta\to 0}\sum_{\operatorname{im}\sigma_k\cap K^m_\delta\neq\emptyset} a_k\cdot\langle g^m_\delta\cdot \mathrm{dvol}_M,\sigma_k\rangle\\ &= \lim_{m\to\infty} s\cdot\operatorname{vol}(K^m)\\ &= s\cdot\operatorname{vol}(M).\end{aligned}$$

If c is a fundamental cycle, then $s = 1$ and hence the series has value $\operatorname{vol}(M)$. Conversely, if the series evaluates to $\operatorname{vol}(M)$, then $\operatorname{vol}(M)$ must be finite by the first part. Therefore, we can deduce from the computation above that $s = 1$, i.e., c is a fundamental cycle. □

One should be aware that the (locally finite) simplicial volume of a non-compact manifold M might be finite even if $\operatorname{vol}(M) = \infty$, e.g., $\|\mathbb{R}^2\| = 0$ – unlike the Lipschitz simplicial volume as the following direct corollary of Proposition 4.4 shows.

Corollary 4.5. *Let M be a Riemannian manifold. If $\|M\|_{\mathrm{Lip}}$ is finite, then so is $\operatorname{vol}(M)$.*

4.2. **The smearing homomorphism.** Let M and N be smooth manifolds (with or without boundary). The set of smooth maps $M \to N$ equipped with the topology that turns the differential map from this set to $\operatorname{map}(TM, TN)$ into a homeomorphism onto its image is denoted by $C^1(M, N)$. This topology is called *C^1-topology.*

The following defines a variant of Thurston's measure homology [29, p. 6.6f].

Definition 4.6 (Lipschitz measure homology)**.** Let M be a Riemannian manifold.

(a) A signed Borel measure μ on $C^1(\Delta^n, M)$ is said to have *Lipschitz determination* if there is $L > 0$ such that μ is determined on the subset of C^1-simplices whose Lipschitz constant is smaller than L.
(b) Let $\mathscr{C}^{\mathrm{Lip}}_*(M)$ denote the set of signed Borel measures on $C^1(\Delta^n, M)$ that have finite total variation and Lipschitz determination. Then $(\mathscr{C}^{\mathrm{Lip}}_n(M))_{n\geq 0}$ forms a chain complex whose elements are called *Lipschitz measure chains.* The differential is given by the alternating sum of push-forwards induced by face maps [25, p. 539; 31, Corollary 2.9]. The total variation defines a norm on each of these chain groups.
(c) The homology groups of $\mathscr{C}^{\mathrm{Lip}}_*(M)$ are denoted by $\mathscr{H}^{\mathrm{Lip}}_*(M)$. They are equipped with the quotient semi-norm.

The Lipschitz determination condition ensures that the function $\sigma \mapsto \int \sigma^* \operatorname{dvol}_M$ is bounded on the supports of the measure chains in question. Therefore, Lipschitz measure chains can be evaluated against the volume form:

Remark 4.7. Let M be a Riemannian n-manifold and let $\mu \in \mathscr{C}_n^{\mathrm{Lip}}(M)$. Then the function

$$I : C^1(\Delta^n, M) \to \mathbb{R},\ \sigma \mapsto \langle \operatorname{dvol}_M, \sigma \rangle = \int_{\Delta^n} \sigma^* \operatorname{dvol}_M$$

is well defined, measurable, and μ-almost everywhere bounded, thus μ-integrable. We denote the integral $\int I d\mu$ by $\langle \operatorname{dvol}_M, \mu \rangle$.

Definition 4.8. For a Riemannian manifold M, we define the following subcomplex of $C_*^{\mathrm{lf,Lip}}(M)$ (see Definition 3.2)

$$C_*^{\ell^1,\mathrm{Lip}}(M) = \Big\{ \sum_{i\in\mathbb{N}} a_i \sigma_i \in C_*^{\mathrm{lf,Lip}}(M);\ \sigma_i \text{ smooth for all } i \in \mathbb{N}, \text{ and} \sum_{i\in\mathbb{N}} |a_i| < \infty \Big\}.$$

A cycle in $C_{\dim M}^{\ell^1,\mathrm{Lip}}(M)$ is called a *fundamental cycle* if it is a locally finite fundamental cycle in $C_{\dim M}^{\mathrm{lf}}(M)$.

From now on, we refer to the setting in Setup 4.1. Thurston's smearing technique is a cunning way of averaging the simplices over the isometry group of the universal cover:

Proposition 4.9. *Let $\sigma : \Delta^i \to M$ be a smooth simplex, and let $\widetilde{\sigma} : \Delta^i \to U$ be a lift of σ to U. The push-forward of $\mu_{\Lambda\backslash G}$ under the map*

$$\mathrm{smear}_{\widetilde{\sigma}} : \Lambda\backslash G \to C^1(\Delta^i, N),\ \Lambda g \mapsto p_N \circ g\widetilde{\sigma}$$

does not depend on the choice of the lift of σ and is denoted by μ_σ. Further there is a well-defined chain map

$$\mathrm{smear}_* : C_*^{\ell^1,\mathrm{Lip}}(M) \longrightarrow \mathscr{C}_*^{\mathrm{Lip}}(N),\ \sum_\sigma a_\sigma \sigma \mapsto \sum_\sigma a_\sigma \mu_\sigma.$$

Proof. One uses the right G-invariance of $\mu_{\Lambda\backslash G}$ for showing that smear_* is independent of the choice of the lifts and compatible with the boundary. The computations are similar to the ones in the classical case [28, Section 5.4]. □

In the proof of the proportionality principle (Theorem 1.5), it is essential to be able to determine the map induced by smearing in the top homology. We achieve this by evaluating with respect to the volume form.

Lemma 4.10. *For every fundamental cycle $c \in C_n^{\ell^1,\mathrm{Lip}}(M)$ we have*

$$\big\langle \operatorname{dvol}_N, \mathrm{smear}_n(c) \big\rangle = \int_{C^1(\Delta^n,N)} \int_{\Delta^n} \sigma^* \operatorname{dvol}_N \, d\,\mathrm{smear}_n(c)(\sigma) = \mathrm{vol}(M).$$

Remark 4.11. There exists a fundamental cycle in $C_n^{\ell^1,\mathrm{Lip}}(M)$ if and only if $\|M\|_{\mathrm{Lip}} < \infty$. Equivalently, the Lipschitz simplicial volume can be computed by smooth cycles:

$$\|M\|_{\mathrm{Lip}} = \inf\big\{ |c|_1;\ c \in C_n^{\mathrm{lf}}(M) \text{ smooth fundamental cycle, } \mathrm{Lip}(c) < \infty \big\}. \tag{4.12}$$

This can be shown without curvature conditions using relative approximation theorems for Lipschitz maps by smooth ones but in the case of non-positively curved manifolds the straightening technique gives a quick proof of (4.12):

If $c = \sum_{i \in I} a_i \sigma_i \in C_*^{\mathrm{lf}}(M)$ satisfies $\mathrm{Lip}(c) < \infty$, then Proposition 2.4 and Remark 2.5 show that also the straightened chain $c' = \sum_{i \in I} a_i \cdot s_M(\sigma_i)$ is both Lipschitz and locally finite. Moreover, it is smooth by 2.4. Thus, straightening chains gives rise to a chain map $C_*^{\mathrm{lf,Lip}}(M) \to C_*^{\mathrm{lf,Lip}}(M)$. The same arguments as in Proposition 2.12 and Lemma 2.13 also apply in the locally finite case, which implies that this chain map is homotopic to the identity. Hence $[c'] = [c]$, which, combined with $|c'|_1 \leq |c|_1$, shows (4.12).

Proof of Lemma 4.10. In view of Remark 4.7, the double integral in the lemma is well-defined. Because the universal covering maps p_M and p_N are locally isometric, we obtain (where we write $c = \sum_\sigma a_\sigma \sigma$)

$$\begin{aligned}
\langle \mathrm{dvol}_N, \mathrm{smear}_n(c) \rangle &= \sum_\sigma a_\sigma \langle \mathrm{dvol}_N, \mu_{\widetilde{\sigma}} \rangle \\
&= \sum_\sigma a_\sigma \int_{C^1(\Delta^n, N)} \langle \mathrm{dvol}_N, \varrho \rangle \; d\mu_{\widetilde{\sigma}}(\varrho) \\
&= \sum_\sigma a_\sigma \int_{\Lambda \backslash G} \langle \mathrm{dvol}_N, p_N \circ g\widetilde{\sigma} \rangle \; d\mu_{\Lambda \backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Lambda \backslash G} \langle \mathrm{dvol}_U, g\widetilde{\sigma} \rangle \; d\mu_{\Lambda \backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Lambda \backslash G} \langle \mathrm{dvol}_U, \widetilde{\sigma} \rangle \; d\mu_{\Lambda \backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Lambda \backslash G} \langle \mathrm{dvol}_M, \sigma \rangle \; d\mu_{\Lambda \backslash G}(g).
\end{aligned}$$

By Proposition 4.4, the last expression equals $\mathrm{vol}(M)$. □

4.3. **Proof of Theorem 1.5.** In order to prove the proportionality principle (Theorem 1.5), we proceed in the following steps:

(1) First we construct a Λ-equivariant partition of U into Borel sets of small diameter and a corresponding Λ-equivariant 1-net.
(2) Using the 1-net and a straightening procedure, we develop a discrete version of the smearing map – i.e., a mechanism turning fundamental cycles on M into cycles on N. This has some similarity with the construction by Benedetti and Petronio [1, p. 114f].
(3) By comparing the discrete smearing with the original smearing, integration enables us to identify which class the smeared cycle represents.
(4) In the final step, we compute the ℓ^1-norm of the smeared cycle, thereby proving the theorem.

Proof of Theorem 1.5. Like in the previous paragraphs, we refer to the notation established in Setup 4.1.

4.3.1. *Construction of a suitable Λ-equivariant partition of U into Borel sets.* By locally subdividing a triangulation of N, it is possible to construct a locally finite (and hence countable) set $T \subset N$ and a partition $(F_x)_{x \in T}$ of N into Borel sets with the following properties: For each $x \in T$ we have $x \in F_x$, the diameter of F_x is at most $1/2$ (thus, T is a 1-net in N), and the universal cover p_N is trivial over F_x.

Let $\widetilde{T} \subset U$ be a lift of T to $U = \widetilde{N}$. In view of the triviality condition, we find a corresponding Λ-equivariant partition $\widetilde{F} := (\widetilde{F}_x)_{x \in \Lambda \cdot \widetilde{T}}$ of U into Borel sets of diameter at most $1/2$. Note that $\Lambda \cdot \widetilde{T}$ is locally finite since Λ acts properly on $\widetilde{N}$.

4.3.2. *Discrete version of the smearing map.* In order to construct the discrete version of the smearing map, we first define a version str of the geodesic straightening that turns simplices in U into geodesic simplices with vertices in $\Lambda \cdot \widetilde{T}$: For an i-simplex $\varrho : \Delta^i \to U$ we define the geodesic simplex

$$\operatorname{str}_i(\varrho) := [x_0, \dots, x_i],$$

where $x_0, \dots, x_i \in \Lambda \cdot \widetilde{T}$ are the elements uniquely determined by the requirement that for all $j \in \{0, \dots, i\}$ the j-th vertex of ϱ lies in $\widetilde{F}_{x_j}$. By Proposition 2.4, the simplex $\operatorname{str}_i(\varrho)$ is smooth. Because the partition $\widetilde{F}$ is Λ-equivariant, so is str_i. Using the fact that all elements of $\widetilde{F}$ are Borel and that $\Lambda \cdot \widetilde{T}$ is countable, it is not difficult to see that the map $\operatorname{str}_i : C^1(\Delta^i, U) \to C^1(\Delta^i, N)$ is Borel with respect to the C^1-topology. Moreover, for all $k \in \{0, \dots, i\}$

$$\operatorname{str}_{i-1}(\partial_k \varrho) = \partial_k \operatorname{str}_i(\varrho). \tag{4.13}$$

For $i \in \mathbb{N}$ we write

$$\begin{aligned} S_i := \big\{ &p_N \circ \sigma;\ \sigma : \Delta^i \to U \text{ geodesic simplex with vertices in } \Lambda \cdot \widetilde{T} \big\} \\ &\subset C^1(\Delta^i, N), \end{aligned}$$

and for every simplex $\sigma : \Delta^i \to U$ we define a map

$$f_\sigma : G \to S_i,\ g \mapsto p_N \circ \operatorname{str}_i(g\sigma);$$

The map f_σ is Borel because str_i is Borel and the action of G is C^1-continuous (the compact-open topology on G coincides with the C^1-topology [28, Theorem 5.12]). Furthermore, f_σ induces a well-defined Borel map $f_\sigma : \Lambda \backslash G \to S_i$, which we denote by the same symbol.

We now consider the following discrete approximation of the smearing map defined in Proposition 4.9

$$\varphi_* : C_*^{\ell^1, \mathrm{Lip}}(M) \to C_*^{\ell^1, \mathrm{Lip}}(N)$$

$$\varphi_i\Big(\sum_{k \in \mathbb{N}} a_k \sigma_k\Big) := \sum_{\varrho \in S_i} \Big(\sum_{k \in \mathbb{N}} a_k \cdot \mu_{\Lambda \backslash G}\big(f_{\widetilde{\sigma}_k}^{-1}(\varrho)\big)\Big) \cdot \varrho \tag{4.14}$$

where each $\widetilde{\sigma}_k$ is a lift of σ_k to U. First we show that φ_* is well-defined: The number $\mu_{\Lambda \backslash G}(f_{\widetilde{\sigma}}^{-1}(\varrho))$ does not depend on the choice of the lift $\widetilde{\sigma}$ of the simplex σ because $\mu_{\Lambda \backslash G}$ is invariant under right multiplication of G. If $L = \operatorname{Lip}(\sigma)$, any lift $\widetilde{\sigma}$ has diameter at most $\sqrt{2}L$. Hence, each pair of vertices of $\operatorname{str}_i(g\widetilde{\sigma})$ has distance at most $1 + \sqrt{2}L$. In view of Proposition 2.4 and Remark 2.5, $\operatorname{str}_i(g\widetilde{\sigma})$, and thus $f_{\widetilde{\sigma}}(g)$, are smooth and have a Lipschitz constant depending only on L. Hence there is a uniform bound on the Lipschitz constants of simplices appearing in the right hand sum of (4.14). This also implies that (4.14) defines a locally finite chain because both $\Lambda \cdot \widetilde{T}$ and T are locally finite. Therefore, φ_i is a well-defined homomorphism for every $i \in \mathbb{N}$.

Next we prove that φ_* is a chain homomorphism: From (4.13) we obtain

$$\bigcup_{\varrho \text{ with } \partial_k \varrho = \xi} \big\{ \Lambda g \in \Lambda \backslash G;\ p_N \circ \operatorname{str}_i(g\widetilde{\sigma}) = \varrho \big\} = \big\{ \Lambda g \in \Lambda \backslash G;\ p_N \circ \operatorname{str}_{i-1}(g \partial_k \widetilde{\sigma}) = \xi \big\}$$

for all $\sigma \in \operatorname{map}(\Delta^i, N)$, $k \in \{0, \dots, i\}$, and all $\xi \in \operatorname{map}(\Delta^{i-1}, N)$. Because the left hand side is a disjoint, at most countable, union this implies that

$$\sum_{\varrho \text{ with } \partial_k \varrho = \xi} \mu_{\Lambda\backslash G}\bigl(f_{\widetilde{\sigma}}^{-1}(\varrho)\bigr) = \mu_{\Lambda\backslash G}\bigl(f_{\partial_k \widetilde{\sigma}}^{-1}(\xi)\bigr).$$

Therefore, we deduce

$$\begin{aligned}
\partial_k \varphi_i(\sigma) &= \sum_{\varrho \in S_i} \mu_{\Lambda\backslash G}\bigl(f_{\widetilde{\sigma}}^{-1}(\varrho)\bigr) \cdot \partial_k \varrho \\
&= \sum_{\xi \in S_{i-1}} \sum_{\varrho \text{ with } \partial_k \varrho = \xi} \mu_{\Lambda\backslash G}\bigl(f_{\widetilde{\sigma}}^{-1}(\varrho)\bigr) \cdot \xi \\
&= \sum_{\xi \in S_{i-1}} \mu_{\Lambda\backslash G}\bigl(f_{\partial_k \widetilde{\sigma}}^{-1}(\xi)\bigr) \cdot \xi \\
&= \varphi_{i-1}(\partial_k \sigma),
\end{aligned}$$

which shows that φ_* is a chain map.

4.3.3. *Comparison with the original smearing map.* Let

$$j_* : C_*^{\ell^1,\mathrm{Lip}}(N) \to \mathscr{C}_*^{\mathrm{Lip}}(N)$$

be the chain map that is the obvious extension of the map given by mapping a simplex σ to the atomic measure concentrated in $\{\sigma\}$. Next we show that there is a chain homotopy between the smearing map smear_* given in Proposition 4.9 and the composition $j_* \circ \varphi_*$: For any smooth simplex $\sigma : \Delta^i \to U$ and $g \in G$ the geodesic homotopy from $\mathrm{str}_i(g\sigma)$ to $g\sigma$ followed by p_N defines a map $h_\sigma(g) : \Delta^i \times I \to N$. By Proposition 2.1, Proposition 2.4, and Remark 2.5, $h_\sigma(g)$ is smooth and its Lipschitz constant is bounded from above in terms of the Lipschitz constant of σ. Moreover, Proposition 2.1 shows that the map $h_\sigma : G \to C^1(\Delta^i, N)$ is Borel with respect to the C^1-topology. Because str_* is Λ-equivariant, we obtain a well-defined Borel map

$$h_\sigma : \Lambda\backslash G \to C^1(\Delta^i \times I, N)$$

satisfying

$$\begin{aligned}
h_\sigma(\Lambda g)|_{\Delta^i \times \{0\}} &= f_\sigma(g), \\
h_\sigma(\Lambda g)|_{\Delta^i \times \{1\}} &= p_N \circ g\sigma.
\end{aligned} \tag{4.15}$$

It is also clear that for each face map $\partial_k : \Delta^{i-1} \to \Delta^i$ and every simplex $\sigma : \Delta^i \to U$ we have

$$h_{\sigma \circ \partial_k}(\Lambda g) = h_\sigma(\Lambda g) \circ \bigl(\partial_k \times \mathrm{id}_I\bigr).$$

Retaining the notation of Lemma 2.13 and Remark 2.14, for every $\sigma : \Delta^i \to U$ and every $k \in \{0, \dots, i\}$ let $\nu_{\sigma,k}$ be the push-forward of $\mu_{\Lambda\backslash G}$ under the map

$$\Lambda\backslash G \to C^1(\Delta^{i+1}, N),\ \Lambda g \mapsto h_\sigma(g) \circ G_{i,k}.$$

If σ is a simplex in M and $\widetilde{\sigma}$ a lift to U, then $\nu_{\widetilde{\sigma},k}$ does not depend on the choice of the lift and will be also denoted by $\nu_{\sigma,k}$. We now define the homomorphism

$$H_* : C_*^{\ell^1,\mathrm{Lip}}(M) \to \mathscr{C}_{*+1}^{\mathrm{Lip}}(N),\ H_i(\sigma) := \sum_{k=0} \nu_{\sigma,k}.$$

Lemma 2.13 and (4.15) yield [31, Theorem 2.1 (1)]

$$\partial H_i(\sigma) + H_{i-1}\partial\sigma = j_i(\varphi_i(\sigma)) - \mathrm{smear}_i(\sigma)$$

for every i-simplex σ in M. Thus H_* is the desired chain homotopy $j_* \circ \varphi_* \simeq \operatorname{smear}_*$.

The evaluation with dvol_N (cf. Remark 4.7) is compatible with j_*, that is,

$$\big\langle \operatorname{dvol}_N, j_*(c)\big\rangle = \langle \operatorname{dvol}_N, c\rangle$$

for every $c \in C_*^{\ell^1,\mathrm{Lip}}(N)$.

Let $c \in C_n^{\ell^1,\mathrm{Lip}}(M)$ be a fundamental cycle. Because evaluation with dvol_N is well-defined on homology classes and by Lemma 4.10, we obtain that

$$\begin{aligned}\big\langle \operatorname{dvol}_N, \varphi_n(c)\big\rangle &= \big\langle \operatorname{dvol}_N, j_n(\varphi_n(c))\big\rangle \\ &= \big\langle \operatorname{dvol}_N, \operatorname{smear}_n(c)\big\rangle \\ &= \operatorname{vol}(M).\end{aligned}$$

Now Proposition 4.4 lets us determine the homology class of $\varphi_n(c)$ as

$$[\varphi_n(c)] = \frac{\operatorname{vol}(M)}{\operatorname{vol}(N)} \cdot [N]. \tag{4.16}$$

4.3.4. *The norm estimate and conclusion of proof.* By symmetry we only have to show that

$$\frac{\|M\|_{\mathrm{Lip}}}{\operatorname{vol}(M)} \geq \frac{\|N\|_{\mathrm{Lip}}}{\operatorname{vol}(N)},$$

and in addition we can assume $\|M\|_{\mathrm{Lip}} < \infty$. By Remark 4.11, we can compute the Lipschitz simplicial volume $\|M\|_{\mathrm{Lip}}$ by fundamental cycles lying in the chain complex $C_*^{\ell^1,\mathrm{Lip}}(M)$. Let $c = \sum_{k\in\mathbb{N}} a_k\sigma_k \in C_n^{\ell^1,\mathrm{Lip}}(M)$ be a fundamental cycle of M. Because of (4.16) it suffices to show that

$$|\varphi_n(c)|_1 \leq |c|_1,$$

which is a consequence the following computation:

$$\begin{aligned}|\varphi_n(c)|_1 &\leq \sum_{\varrho\in S_n}\sum_{k\in\mathbb{N}} |a_k| \cdot \mu\big(f_{\widetilde{\sigma_k}}^{-1}(\varrho)\big) \\ &= \sum_{k\in\mathbb{N}}\sum_{\varrho\in S_n} |a_k| \cdot \mu\big(f_{\widetilde{\sigma_k}}^{-1}(\varrho)\big) \\ &= \sum_{k\in\mathbb{N}} |a_k| \\ &= |c|_1.\end{aligned}$$

This finishes the proof of the proportionality principle. ☐

5. Vanishing results for the locally finite simplicial volume

In this section, we give a proof of the vanishing theorem (Theorem 1.1); the proof is based on the fact that locally symmetric spaces of higher $\mathbb{Q}$-rank admit "amenable" coverings of sufficiently small multiplicity and Gromov's vanishing finiteness theorem.

As a first step, we recall Gromov's definition of amenable subsets and sequences of subsets that are amenable at infinity [16, p. 58] and his vanishing-finiteness theorem:

Definition 5.1. Let X be a topological space.

(1) A subset $U \subset X$ is called *amenable in* X if for every basepoint $x \in U$ the subgroup $\operatorname{im}\big(\pi_1(U,x) \to \pi_1(X,x)\big)$ is amenable.

(2) A sequence $(U_i)_{i\in\mathbb{N}}$ of subsets of X is called *amenable at infinity* if there is an increasing sequence of compact subsets $(K_i)_{i\in\mathbb{N}}$ of X with $U_i \subset X - K_i$, $X = \bigcup_{i\in\mathbb{N}} K_i$, and such that U_i is amenable in $X - K_i$ for sufficiently large $i \in \mathbb{N}$.

Theorem 5.2 (Vanishing-finiteness theorem for simplicial volume [16, Corollary (A) on p. 58])**.** *Let M be a manifold without boundary of dimension n. Let $(U_i)_{i\in\mathbb{N}}$ be a locally finite covering of M by open, relatively compact subsets such that each point of M is contained in at most n such subsets. If every U_i is amenable in M and $(U_i)_{i\in\mathbb{N}}$ is amenable at infinity, then $\|M\| = 0$.*

As a next step, we provide a construction of locally finite coverings with small multiplicity by relatively compact, open, amenable subsets; notice however that such a covering is not necessarily amenable at infinity.

Theorem 5.3. *Let M be a manifold and $\Gamma = \pi_1(M)$. Assume that Γ admits a finite model for its classifying space $B\Gamma$ of dimension k. Then there is a locally finite covering of M by relatively compact, amenable, open subsets such that every point of M is contained in at most $k+2$ such subsets.*

Proof. Since $B\Gamma$ is k-dimensional and compact, every open covering of $B\Gamma$ has a finite refinement with multiplicity at most $k+1$ [17, Theorem V 1 on p. 54]. Starting with a covering of $B\Gamma$ by open, contractible sets, let $(V_j)_{j\in J}$ be a finite refinement of multiplicity at most $k+1$.

We pull this covering back to M via the classifying map $\varphi : M \to B\Gamma$: For $j \in J$ let

$$U_j := \varphi^{-1}(V_j).$$

By construction, $(U_j)_{j\in J}$ is an open covering of M with multiplicity at most $k+1$. However, the sets U_j may not be relatively compact.

To achieve a nice covering of M by relatively compact sets, we combine the covering $(U_j)_{j\in J}$ with another covering of M of small multiplicity consisting of relatively compact sets, which is constructed as follows: For every $j \in J$ we choose a covering $\mathcal{R}_j$ of $\mathbb{R}$ by bounded, open intervals such that each $\mathcal{R}_j$ has multiplicity 2 and for $i \neq j$ the cover $\mathcal{R}_i \sqcup \mathcal{R}_j$ (disjoint union) has multiplicity at most 3. This is possible because J is finite.

Let $f : M \to \mathbb{R}$ be a proper function. We show now that the combined covering

$$\mathcal{U} := \bigl(U_j \cap f^{-1}(W)\bigr)_{j\in J,\ W\in\mathcal{R}_j}$$

of M has the desired properties: In the following, by definition, we say that the *J-index* of $U_j \cap f^{-1}(W)$ is j.

Because f is proper and the elements of the $\mathcal{R}_j$ are bounded, each set in $\mathcal{U}$ is relatively compact.

Since $\varphi : \pi_1(M) \to \pi_1(B\Gamma)$ is an isomorphism, the inclusion $U_j \cap f^{-1}(W) \hookrightarrow M$ is trivial on the level of π_1 if and only if its composition with φ is so. But the composition with φ factors over the inclusion $V_j \hookrightarrow B\Gamma$, which is trivial in π_1. In particular, each element of $\mathcal{U}$ is an amenable subset of M.

It remains to verify that $\mathcal{U}$ has multiplicity at most $k+2$: Suppose there is a subset $\mathcal{U}_0 \subset \mathcal{U}$ of $k+3$ sets whose intersection is non-empty. Because the elements of $\mathcal{U}_0$ have at most $k+1$ different J-indices, and the multiplicity of each of the $\mathcal{R}_j$ is at most 2, there must be $i \neq j \in J$ such that there are at least two elements in $\mathcal{U}_0$

having J-index i, and at least two with J-index j. But this contradicts the fact that $\mathcal{R}_i \sqcup \mathcal{R}_j$ has multiplicity at most 3. So the multiplicity of $\mathcal{U}$ is at most $k+2$. □

In order to obtain a suitable amenable covering that is amenable at infinity, we impose additional constraints on the fundamental group of the boundary; one should compare this also with Gromov's remark on subpolyhedra [16, p. 59].

Corollary 5.4. *Let M be the interior of a compact, n-dimensional manifold W with boundary ∂W. Assume that $B\pi_1(M)$ admits a finite model of dimension at most $n-2$ and that at least one of the following conditions is satisfied*

(1) *The fundamental group $\pi_1(\partial W; x)$ is amenable for all $x \in \partial W$.*
(2) *For all $x \in \partial W$ the inclusion induces an injection $\pi_1(\partial W; x) \to \pi_1(W; x)$.*

Then $\|M\| = 0$.

Proof. By Theorem 5.3 we obtain a covering $(U_i)_{i\in\mathbb{N}}$ of M by open, relatively compact, amenable subsets in M which has multiplicity $\leq n$. Let $(V_i)_{i\in\mathbb{N}}$ be a decreasing sequence of open neighborhoods in W of the boundary ∂W with $\bigcap_{i\in\mathbb{N}} V_i = \partial W$ and $\bigcup_{i\in\mathbb{N}} V_i = W$. By choosing collar neighborhoods of ∂W we can assume that ∂W is a deformation retract of V_i for all large $i \in \mathbb{N}$. Because $(U_i)_{i\in\mathbb{N}}$ is locally finite, we additionally can assume that $U_i \subset V_i$ for all $i \in \mathbb{N}$.

If $\pi_1(\partial W; x)$ is amenable for every basepoint x then U_i is obviously an amenable subset of V_i for all large $i \in \mathbb{N}$. If the inclusion maps $\partial W \to W$ are π_1-injective then so are the inclusion maps $V_i \cap M \to M$ for all large $i \in \mathbb{N}$, and the amenability of the subset $U_i \subset V_i \cap M$ follows from the one of $U_i \subset M$.

In either case we can now apply Gromov's vanishing-finiteness theorem 5.2. □

Example 5.5 (Products of open manifolds with non-zero simplicial volume)**.** We consider the open manifold $M := W^\circ \times \mathbb{R}$, where $(W, \partial W)$ is the surface with boundary obtained by removing a finite number of pairwise disjoint, open discs from an oriented, closed, connected surface of genus at least 1.

Then the fundamental group $\pi_1(M) \cong \pi_1(W)$ is a finitely generated free group and thus admits a finite model of dimension $1 = \dim M - 2$.

However, we can view M as the interior of the compact manifold $W \times [0,1]$ whose boundary is nothing but an oriented, closed, connected surface of genus at least 2; in particular, this boundary has non-zero simplicial volume, which forces the simplicial volume of M to be infinite [16, p. 17; 22, Corollary 6.2].

In fact, tracking down the construction of an open covering in the proof of Theorem 5.3 shows that this particular covering is amenable but not amenable at infinity.

In particular, the finiteness hypothesis in the corollary is not sufficient for the vanishing of the simplicial volume. The following cohomological criterion helps to check whether the finiteness hypothesis in the corollary is satisfied.

Lemma 5.6. *Let Γ be a group that has a finite model for its classifying space $B\Gamma$.*

(1) *If $\operatorname{cd}\Gamma \neq 2$, then there is a finite model of $B\Gamma$ whose dimension equals the integral cohomological dimension $\operatorname{cd}\Gamma$ of Γ.*
(2) *If $\operatorname{cd}\Gamma = 2$, then there is a finite model of $B\Gamma$ of dimension at most 3.*

Proof. Because there is a finite model for $B\Gamma$, the group Γ is finitely presented and of type FL. Therefore, a classic result of Eilenberg and Ganea shows that there is a finite model of $B\Gamma$ of dimension $\max\{\operatorname{cd}\Gamma, 3\}$ [7, Theorem VIII.7.1].

If $\operatorname{cd}\Gamma = 0$, then Γ is the trivial group and hence the one-point space is a model for $B\Gamma$. If $\operatorname{cd}\Gamma = 1$, then Γ is free by a theorem of Stallings and Swan [27]; because Γ is finitely presented, Γ is a finitely generated free group. In particular, we can take a finite wedge of circles as a finite, one-dimensional model for $B\Gamma$. $\square$

Using the techniques established in this section, we prove the vanishing theorem for the locally finite simplicial volume of non-compact locally symmetric spaces of $\mathbb{Q}$-rank ≥ 3 (Theorem 1.1):

Proof of Theorem 1.1. The locally symmetric space $M = \Gamma\backslash X$ is a model of $B\Gamma$ because X is non-positively curved [12, Sections 2.1 and 2.2], thus contractible. Moreover, M is homotopy equivalent to the Borel-Serre compactification W of M [5], which thus is a finite model of $B\Gamma$. For $\operatorname{rk}_{\mathbb{Q}}\Gamma \geq 3$ the inclusion $\partial W \to W$ is a π_1-isomorphism [3, Proposition 2.3]. Furthermore, we have [5, Corollary 11.4.3]

$$\operatorname{cd}\Gamma = \dim X - \operatorname{rk}_{\mathbb{Q}}\Gamma.$$

Therefore, Lemma 5.6 shows that there is a finite model for $B\Gamma$ of dimension at most $\max\{\dim X - \operatorname{rk}_{\mathbb{Q}}\Gamma, 3\} \leq \dim X - 2$.

Thus, Corollary 5.4 yields the vanishing of $\|M\|$. $\square$

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