FUNDAMENTAL CLASSES OF NEGATIVELY CURVED MANIFOLDS CANNOT BE REPRESENTED BY PRODUCTS OF MANIFOLDS

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Abstract. Not every singular homology class is the push-forward of the fundamental class of some manifold. In the same spirit, one can study the following problem: Which singular homology classes are the push-forward of the fundamental class of a given type of manifolds? In the present article, we show that the fundamental classes of negatively curved manifolds cannot be represented by a non-trivial product of manifolds. This observation sheds some light on the functorial semi-norm on singular homology given by products of compact surfaces.

1. Introduction

As manifolds form a particularly accessible class of topological spaces, it is natural to ask which classes in singular homology with $\mathbf{Z}$-coefficients are the push-forward of the fundamental class of some manifold; indeed, not every singular homology class is such a push-forward [10; Théorème 3]. In the same spirit, one can study the following more restrictive problem: Which singular homology classes are the push-forward of the fundamental class of a given type of manifolds? An interesting type of manifolds in the context of functorial semi-norms on singular homology [4; 5.34 on p. 302] is products of compact surfaces.

Definition (1.1). Let M be an oriented, closed, connected manifold. We say that the fundamental class of M can be *represented by a product of manifolds*, if there exist $d \in \mathbf{Z} \setminus \{0\}$, $r \in \mathbf{N}_{>1}$, and oriented, closed, connected manifolds $S_1, \dots, S_r$ of non-zero dimension admitting a continuous map $f\colon S_1 \times \cdots \times S_r \longrightarrow M$ such that

$$H_*(f; \mathbf{Z})\bigl([S_1 \times \cdots \times S_r]\bigr) = d \cdot [M]. \qquad \diamond$$

Gromov suspected that most interesting homology classes – such as fundamental classes of irreducible locally symmetric spaces – cannot be represented by products of manifolds [4; 5.36 on p. 303f]; in the present article, we confirm this anticipation, at least in the case of manifolds of negative curvature:

Theorem (1.2). *Let M be an oriented, closed, connected, smooth manifold that admits a Riemannian metric with negative sectional curvature. Then the fundamental class of M cannot be represented by a product of manifolds, let alone by a product of surfaces.*

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In particular, we obtain that the functorial semi-norm given by products of compact surfaces [4; 5.36 on p. 303f] is infinite when evaluated on the fundamental class of an oriented, closed, connected Riemannian manifold with negative sectional curvature.

In order to prove Theorem (1.2), we assume that the fundamental class of M can be represented by a product of manifolds and lead this assumption to a contradiction by studying the corresponding scene on the level of fundamental groups. Roughly speaking, the source of this contradiction lies in the tension between $\pi_1(M)$ being a complicated group (because M is negatively curved) and $\pi_1(M)$ being too commutative (the images of the fundamental groups of the factors commute in $\pi_1(M)$).

In fact, we present two directions of generalisations of Theorem (1.2) in Sections 1.1 and 1.2 below, and establish Theorem (1.2) as an instance of these more general statements.

1.1. **Irrepresentability – aspherical manifolds.** The first generalisation applies to a larger class of aspherical manifolds and provides an obstruction to representability by a product of manifolds in terms of group theory; in order to formulate the statement, we introduce the following conventions:

Definition (1.3).

– Let C be a class of groups. We say that a group G *has centralisers in* C, if for every element $g \in G$ of infinite order the centraliser $C_G(g)$ lies in C.
– A class of groups is a *class with products*, if it is closed under taking subgroups, quotients, and finite products. ◇

Theorem (1.4). *Let M be an aspherical, oriented, closed, connected manifold whose fundamental class can be represented by a product of manifolds. If the fundamental group of M has centralisers in a class C of groups with products, then the fundamental group of M contains a subgroup of finite index lying in C (i.e., $\pi_1(M)$ is virtually in C).*

1.2. **Irrepresentability – manifolds with positive simplicial volume.** For the second generalisation, we trade the asphericity condition for a suitable hypothesis on the simplicial volume, an invariant unaware of higher homotopy.

Theorem (1.5). *Let M be an oriented, closed, connected manifold with non-zero simplicial volume whose fundamental group is large and has amenable centralisers. Then the fundamental class of M cannot be represented by a product of manifolds.*

We now explain the occurring terminology in more detail:

Definition (1.6). A group is *large*, if it contains an element of infinite order. ◇

Recall that a group G is called *amenable* if there is a left-invariant mean on the set $B(G, \mathbf{R})$ of bounded functions from G to $\mathbf{R}$, i.e., if there is a G-invariant, linear map $m\colon B(G, \mathbf{R}) \longrightarrow \mathbf{R}$ with $\inf_{g \in G} f(g) \leq m(f) \leq \sup_{g \in G} f(g)$ for all $f \in B(G, \mathbf{R})$.

Every finite, every Abelian, and every solvable group is amenable. The class of amenable groups is closed with respect to taking subgroups, quotients, and extensions; in particular, the class of amenable groups is a class with products in the sense of Definition (1.3). The group $\mathbf{Z} * \mathbf{Z}$ is not amenable. A thorough account of amenable groups is given in Paterson's book [9].

For a topological space X and a class $\alpha \in H_k(X)$ in singular homology with $\mathbf{R}$-coefficients, the *ℓ^1-semi-norm of α* is defined by

$$\|\alpha\|_1 := \inf\bigl\{\|c\|_1 \bigm| c \in C_k(X) \text{ is an } \mathbf{R}\text{-cycle representing } \alpha\bigr\};$$

here, for a singular chain $c = \sum_{j=0}^r a_j \cdot \sigma_j \in C_k(X)$, we write $\|c\|_1 := \sum_{j=0}^r |a_j|$.

The simplicial volume is an example of a topological invariant defined in terms of the ℓ^1-semi-norm [2; p. 8]: The *simplicial volume* of an oriented, closed, connected manifold M is defined by

$$\|M\| := \|[M]_{\mathbf{R}}\|_1 = \inf\bigl\{\|c\|_1 \bigm| c \in C_*(M) \text{ is an } \mathbf{R}\text{-fundamental cycle of } M\bigr\},$$

where $[M]_{\mathbf{R}} \in H_*(M)$ denotes the image of the fundamental class $[M] \in H_*(M;\mathbf{Z})$ of M under the change of coefficients homomorphism $H_*(M;\mathbf{Z}) \longrightarrow H_*(M)$.

The simplicial volume of spheres and tori is zero; on the other hand, the simplicial volume of oriented, closed, hyperbolic surfaces S equals $2 \cdot |\chi(S)|$ [2; p. 8f], and more generally the simplicial volume of closed Riemannian manifolds of negative curvature is non-zero [6, 11].

The simplicial volume of oriented, closed, connected, locally symmetric spaces of non-compact type is non-zero [8]. Therefore, if it were known that the fundamental groups of oriented, closed, connected, irreducible locally symmetric spaces of non-compact type have amenable centralisers, one could apply Theorem (1.5) also in this case.

As an indication of the applicability of Theorems (1.4) and Theorem (1.5), we mention the class of word-hyperbolic groups (cf. Section 2.1).

Organisation. This article is organised as follows: In Section 2, we derive Theorem (1.2) from the more general statements in Theorem (1.4) and Theorem (1.5). Theorems (1.4) and (1.5) in turn are proved in Section 3.

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2. Deriving the negatively curved case

To show that fundamental classes of negatively curved manifolds cannot be represented by products of manifolds (Theorem (1.2)), it suffices to establish this statement as a special case of Theorem (1.5):

2.1. **Word-hyperbolic groups.**

The notion of word-hyperbolicity is a group-theoretic analogue of negative curvature introduced by Gromov [3]. For example, fundamental groups of oriented, closed, connected Riemannian manifolds of negative curvature are word-hyperbolic by the Švarc-Milnor lemma [5; Section V.D].

Remark (2.1).

- Word-hyperbolic groups have amenable centralisers: Let G be a word-hyperbolic group. If $g \in G$ has infinite order, then the subgroup of G generated by g has finite index in the centraliser $C_G(g)$ [1; Corollary 3.10 on p. 462]. Therefore, the group $C_G(g)$ contains a normal, infinite cyclic subgroup of finite index and hence is amenable [9].
- All infinite word-hyperbolic groups are large in the sense of Definition (1.6) above [1; Proposition 2.22 on p. 458]. ◇

2.2. **Proof of Theorem (1.2).** Using the facts on word-hyperbolic groups listed above, we can show that fundamental classes of negatively curved Riemannian manifolds cannot be represented by products of manifolds:

Proof (of Theorem (1.2)). Because M is an oriented, closed, connected, smooth manifold admitting a Riemannian metric of negative sectional curvature, Thurston's straightening technique shows that M has non-zero simplicial volume [6, 11].

By the Cartan-Hadamard theorem, M is aspherical; in particular, $\pi_1(M)$ is torsion-free and we may assume that $\pi_1(M)$ is non-trivial; hence, $\pi_1(M)$ is large. Moreover, the fundamental group of M is word-hyperbolic by the Švarc-Milnor Lemma. Therefore, Remark (2.1) shows that $\pi_1(M)$ has amenable centralisers.

Applying Theorem (1.5) to M yields that the fundamental class of M cannot be represented by a product of manifolds. □

3. Proof of Theorems (1.4) and (1.5)

In this section, we prove Theorems (1.4) and (1.5). After introducing the relevant notation, we collect in Section 3.2 a number of observations on the obstructions to representability by products on the level of fundamental groups. In Section 3.3, we conclude the proofs of Theorem (1.4) and (1.5).

3.1. **Notation.** For the remainder of this section, we assume that M is an oriented, closed, connected manifold whose fundamental class of M can be represented by a product of manifolds and whose fundamental group has centralisers in a class C with products. Combining multiple factors we may assume that the fundamental class of M can be represented by a product of two manifolds, i.e., there are $d \in \mathbf{Z} \setminus \{0\}$, and oriented, closed, connected manifolds S_1, S_2 of non-zero dimension admitting a continuous map $f \colon S_1 \times S_2 \longrightarrow M$ such that

$$H_n(f; \mathbf{Z})([S_1 \times S_2]) = d \cdot [M],$$

where $n := \dim M$; in particular, $\dim S_1 + \dim S_2 = n$ and $\deg f = d$. Notice that $n = \dim M > 0$ because the fundamental class of M can be represented by a product of manifolds.

In the course of the proofs, we use the following abbreviations:

- We choose base-points $s_1 \in S_1$ and $s_2 \in S_2$, and set $m := f(s_1, s_2) \in M$. In the following, all fundamental groups are taken with respect to these base-points or base-points deduced from these in an obvious way.
- We write

$$f_1 := f|_{S_1 \times \{s_2\}} \colon S_1 \times \{s_2\} \longrightarrow M,$$
$$f_2 := f|_{\{s_1\} \times S_2} \colon \{s_1\} \times S_2 \longrightarrow M.$$

- Correspondingly, on the level of fundamental groups, we define

$$H_1 := \operatorname{im} \pi_1(f_1) \subset \pi_1(M, m),$$
$$H_2 := \operatorname{im} \pi_1(f_2) \subset \pi_1(M, m),$$

and

$$H_1' := \pi_1(S_1 \times \{s_2\}, (s_1, s_2)) \subset \pi_1(S_1 \times S_2, (s_1, s_2)),$$
$$H_2' := \pi_1(\{s_1\} \times S_2, (s_1, s_2)) \subset \pi_1(S_1 \times S_2, (s_1, s_2)).$$

– Finally, $G := \operatorname{im} \pi_1(f) \subset \pi_1(M, m)$.

3.2. **The scene on the level of fundamental groups.** As indicated in the introduction, we now analyse the corresponding scene on the level of fundamental groups:

(1) *The group G has centralisers in C and G has finite index in $\pi_1(M)$.*

Proof. Centralisers in G are subgroups of centralisers in $\pi_1(M)$, and hence are subgroups of groups in C. Because C is closed with respect to taking subgroups, the group G has centralisers in C.

Furthermore, G has finite index in $\pi_1(M)$: Let $p\colon \overline{M} \longrightarrow M$ be the covering associated with the subgroup $G \subset \pi_1(M)$; in particular, $\overline{M}$ is a manifold with $\dim \overline{M} = \dim M$. By definition of G, covering theory provides us with a lift $\overline{f}\colon S_1 \times S_2 \longrightarrow \overline{M}$ of f, i.e., $p \circ \overline{f} = f$. Because $\deg f \neq 0$, applying top homology shows that $\overline{M}$ must be compact and $\deg f = \deg p \cdot \deg \overline{f}$. Therefore, the index

$$[\pi_1(M) : G] = |\deg p|$$

must be finite. □

(2) *The set $H_1 \cup H_2$ generates G, and H_1 and H_2 commute with each other.*

Proof. The inclusions $H_1' \hookrightarrow \pi_1(S_1 \times S_2)$ and $H_2' \hookrightarrow \pi_1(S_1 \times S_2)$ induce an isomorphism $H_1' \times H_2' \cong \pi_1(S_1 \times S_2)$. In particular, $\pi_1(S_1 \times S_2)$ is generated by $H_1' \cup H_2'$. Therefore

$$H_1 \cup H_2 = \pi_1(f)(H_1' \cup H_2')$$

generates $\operatorname{im} \pi_1(f)$, which coincides – by definition – with G.

Because H_1' and H_2' commute with each other in $\pi_1(S_1 \times S_2) \cong H_1' \times H_2'$, it follows that also their images H_1 and H_2 under the homomorphism $\pi_1(f)$ commute. □

(3) *The group G is a quotient of $H_1 \times H_2$.*

Proof. Because H_1 and H_2 commute, there is a homomorphism $H_1 \times H_2 \longrightarrow G$, which is surjective by part 2. Hence, G is a quotient of $H_1 \times H_2$. □

(4) *If H_1 is large (in the sense of Definition (1.6)), then H_2 lies in C, and vice versa .*

Proof. If H_1 is large, then there is an element $g \in H_1$ of infinite order, and hence part 1 implies that $C_G(g) \in C$. On the other hand, $H_2 \subset C_G(g)$ by part 2. □

(5) *If $\pi_1(M)$ is large, then at least one of the groups H_1 and H_2 is large.*

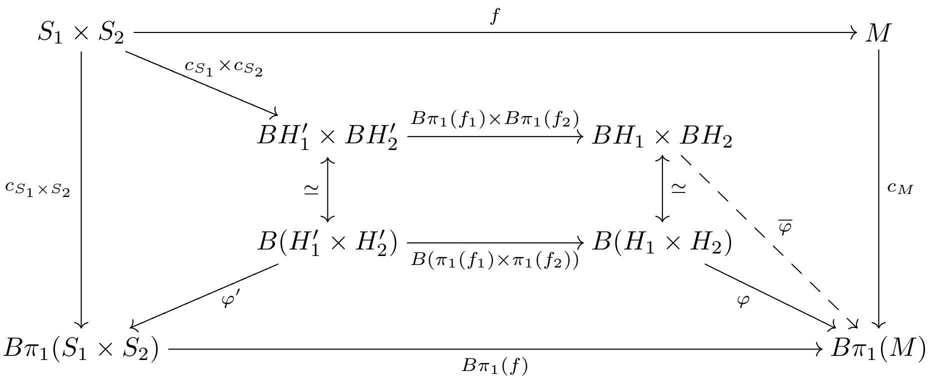

Figure (3.1): Proof of part 6

Proof. In view of part 3 it suffices to show that G is large: If $\pi_1(M)$ is large, we find an element $g \in \pi_1(M)$ of infinite order. Because G has finite index in $\pi_1(M)$, there are $m, n \in \mathbf{N}_{>1}$ with $g^m \cdot G = g^n \cdot G$ and $m \neq n$. Hence, $g^{m-n} \in G$. On the other hand, $m - n \neq 0$ implies that g^{m-n} has infinite order. I.e., G is large. □

(6) *The diagram in Figure (3.1) is commutative. In particular, there are homology classes $\alpha_1 \in H_{\dim S_1}(BH_1)$ and $\alpha_2 \in H_{\dim S_2}(BH_2)$ satisfying*

$$H_n(c_M)(d \cdot [M]_{\mathbf{R}}) = H_n(\overline{\varphi})(\alpha_1 \times \alpha_2).$$

Proof. We first explain the notation occurring in Figure (3.1): For a path-connected space X, we write $c_X \colon X \longrightarrow B\pi_1(X)$ for the classifying map. Recall that every homomorphism $\psi \colon K' \longrightarrow K$ of groups yields a continuous map $B\psi \colon BK' \longrightarrow BK$ that induces the given homomorphism ψ on the level of fundamental groups; moreover, $B\psi$ is characterised uniquely up to homotopy by this property.

The vertical homotopy equivalences in the centre of the diagram are induced by the projections/inclusions on the level of groups. The map φ' is induced by the canonical isomorphism $H_1' \times H_2' \longrightarrow \pi_1(S_1 \times S_2)$ given by the inclusions. Finally, for φ we observe that the inclusions $H_1 \longrightarrow \pi_1(M)$ and $H_2 \longrightarrow \pi_1(M)$ give rise to a homomorphism $H_1 \times H_2 \longrightarrow \pi_1(M)$, because H_1 and H_2 commute in $G \subset \pi_1(M)$; the map φ is the continuous map induced by this homomorphism.

It is a routine matter to verify that the diagram in Figure (3.1) is commutative up to homotopy. Hence, the corresponding diagram on the level of singular homology with $\mathbf{R}$-coefficients is commutative.

In the following, we abbreviate the composition of φ with the homotopy equivalence $BH_1 \times BH_2 \longrightarrow B(H_1 \times H_2)$ by $\overline{\varphi} \colon BH_1 \times BH_2 \longrightarrow B\pi_1(M)$.

Using the naturality of the homological cross-product, we obtain in singular homology with **R**-coefficients the relation

$$\begin{aligned} H_n(c_M)(d \cdot [M]_{\mathbf{R}}) &= H_n(c_M) \circ H_n(f)([S_1 \times S_2]_{\mathbf{R}}) \\ &= H_n(c_M) \circ H_n(f)([S_1]_{\mathbf{R}} \times [S_2]_{\mathbf{R}}) \\ &= H_n(\overline{\varphi}) \circ H_n\big(B\pi_1(f_1) \times B\pi_1(f_2) \circ c_{S_1} \times c_{S_2}\big)([S_1 \times S_2]_{\mathbf{R}}) \\ &= H_n(\overline{\varphi})(\alpha_1 \times \alpha_2), \end{aligned}$$

where we put $\alpha_j := H_{\dim S_j}\big(B\pi_1(f_j) \circ c_{S_j}\big)([S_j]_{\mathbf{R}})$ for $j \in \{1, 2\}$. □

3.3. **Completing the proofs.** Using the facts on the groups H_1 and H_2 established in the previous section, we can finish the proofs of Theorem (1.4) and (1.5):

Proof (of Theorem (1.4)). By hypothesis, the manifold M is aspherical; in particular, $\pi_1(M)$ is torsion-free and non-trivial (because $\dim M > 0$) and hence any non-trivial subgroup of $\pi_1(M)$ is large.

If H_1 were trivial, then $\alpha_1 = 0$ in part 6, which would show that $H_n(c_M)([M]_{\mathbf{R}}) = 0$. However, M is aspherical and therefore, c_M is a homotopy equivalence. Analogously, H_2 cannot be trivial; i.e., H_1 and H_2 are large.

Therefore, we obtain from part 4 that H_1 and H_2 lie in C. Using part 3 and the fact that C is closed with respect to taking finite products, subgroups, and quotients, we deduce that G is in C. Furthermore, G has finite index in $\pi_1(M)$ by part 1. □

Proof (of Theorem (1.5)). By hypothesis, $\pi_1(M)$ is large and $\|M\| > 0$. *Assume* that the fundamental class of M can be represented by a product of manifolds; thus, the observations of Section 3.2 apply.

By part 5, we may assume that H_1 is large and hence that H_2 is amenable (part 4).

We now apply the ℓ^1-semi-norm to the equation of part 6. At this point, we rely on the following classic results on the ℓ^1-semi-norm:

- The homomorphism $H_n(c_M)\colon H_n(M) \longrightarrow H_n(B\pi_1(M))$ is isometric with respect to the ℓ^1-semi-norm by the mapping theorem in bounded cohomology [2, 7; p. 40/18, Theorem 4.3].
- The ℓ^1-semi-norm on connected, countable CW-complexes with amenable fundamental group is zero in non-zero degree [2, 7; p. 40/18, Theorem 4.3].
- The standard triangulation of products of simplices shows that the ℓ^1-semi-norm is compatible with the homological cross-product in the following sense: For all spaces X, Y and all $\alpha \in H_k(X)$, $\beta \in H_\ell(Y)$, we have

$$\|\alpha \times \beta\|_1 \leq 2^{k+\ell} \cdot \|\alpha\|_1 \cdot \|\beta\|_1 .$$

- The ℓ^1-semi-norm is a functorial semi-norm, i.e., the map $H_n(\overline{\varphi})$ does not increase the ℓ^1-semi-norm.

Since H_2 is countable, there exists a countable CW-model of BH_2. Because H_2 is amenable, the properties of $\|\cdot\|_1$ listed above and part 6 imply that

$$\begin{aligned}\|M\| &= \frac{1}{|d|} \cdot \big\|H_n\,(c_M)(d \cdot [M]_{\mathbf{R}})\big\|_1 = \frac{1}{|d|} \cdot \big\|H_n\,(\overline{\varphi})(\alpha_1 \times \alpha_2)\big\|_1 \\ &\leq \frac{2^n}{|d|} \cdot \|\alpha_1\|_1 \cdot \|\alpha_2\|_1 \\ &= 0,\end{aligned}$$

which contradicts the hypothesis that the simplicial volume $\|M\|$ of M is non-zero. □

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