JOURNAL OF FUNCTIONAL ANALYSIS 45, 297–340 (1982)

Dimension Functions and Traces on C^*-Algebras

BRUCE BLACKADAR* AND DAVID HANDELMAN[†,‡]

*Department of Mathematics, University of Nevada, Reno, Nevada 89557, and
[†]Department of Mathematics, University of Ottawa,
Ottawa, Ontario K1N 9B4, Canada

Communicated by A. Connes

Received June 1981

Contents.

INTRODUCTION

One of the most important ideas in the study of operator algebras has been the notion of finiteness. There have been a number of proposed definitions of finiteness; the weakest of these (for unital algebras) is Murray–von Neumann finiteness ($xx^* = 1$ implies $x^*x = 1$), and the strongest is the existence of a separating family of tracial states. Classical results of Murray and von Neumann show that these notions coincide for W^*-algebras, but the relationship is unclear even for AW^*-algebras, and thus more generally.

A primary goal of the study of finiteness in operator algebras is the gathering of information about the order structure and comparability within the algebra (cf. [6, 15]). Related to this, is the K-theory of C^*-algebras. The Grothendieck group (K_0) of a stably finite C^*-algebra admits a natural ordering which has become very important, particularly in the study of AF algebras by means of dimension groups [9, 10].

The groundwork for a theory unifying and clarifying many aspects of finiteness in C^*-algebras was laid by Cuntz in [4]. He defined (for simple C^*-algebras) a partially ordered abelian group K_0^* similar to K_0, and

[‡] Supported in part by an operating grant from the Natural Sciences and Engineering Research Council.

0022-1236/82/030297–44$02.00/0
Copyright © 1982 by Academic Press, Inc.
All rights of reproduction in any form reserved.

showed that its states could be identified with "dimension functions" (similar to those on W^*- or AW^*-algebras) on the algebra.

Our aim in this paper is to develop the general theory of K_0^* and dimension functions to a considerably greater extent than [4], to clarify the relation between dimension functions and traces, and to apply the theory to obtain new results about the internal structure of C^*-algebras and their K_0 and K_0^* groups.

One new concept (actually only new terminology) considered, is that of quasitrace; this is a complex-valued function on a C^*-algebra having all the usual properties of a tracial state, but with linearity assumed only on commutative C^*-subalgebras.

The principal results of this article are:

1. Any lower semicontinuous subadditive rank function (I.1.2) on a C^*-algebra extends to matrix rings, and thus to a lower semicontinuous dimension function(Theorem II.3.1); without "subadditive," the result fails.

2. If A is any C^*-algebra and D is a lower semicontinuous dimension function, then there is a $*$-homomorphism Φ from A to a finite AW^*-algebra M, and a lower semicontinuous dimension function $\bar{D}$ on M, such that $D = \bar{D} \cdot \Phi$.

3. If A is a C^*-algebra, there is a natural bijection between the lower semicontinuous dimension functions and the 2-quasitraces on A (a 2-quasitrace is a quasitrace which extends to $M_2 A$, the ring of 2×2 matrices with entries from A) (II.2.2).

4. For unital A, the set of 2-quasitraces on A has the structure of a simplex (II.4.4).

5. If $\mathscr{C}$ denotes the class of C^*-algebras generated by type I and W^* algebras, and closed under the formation of ideals, quotients, direct limits, extensions, and matrix rings, then every quasitrace on a member of $\mathscr{C}$ is a trace (hence every stably finite unital C^*-algebra in $\mathscr{C}$ possesses a trace) (II.4.9, II.4.11).

6. For simple AF algebras (a larger class of C^*-algebras is considered), the module isomorphism classes of closed right ideals are described, as are the corresponding K_0^* groups (III.2.11, III.2.12, III.3.4).

7. If A is a stably finite C^*-algebra which is "rich in projections," then the state space of $K_0(A)$ is the simplex $QT(A)$ (4 above) (III.1.3).

The paper is divided, like Gaul, into three parts. Part I is a study of dimension functions, Part II deals with quasitraces, and Part III discusses the structure of K_0^* with applications to K_0. Each is divided into several sections.

I. DIMENSION FUNCTIONS

Roughly speaking, a rank or dimension function on a ring, is a real-valued function whose values measure the size of the "support projections" of the elements.

The study of such functions goes back at least to Murray and von Neumann, who used them (defined only on projections) in their classification of factors. They have since become an important tool in the study of von Neumann regular rings [12]. Cuntz [4] gave definitions appropriate for their study on general C^*-algebras.

Section 1 develops some of the elementary properties of rank and dimension functions; Section 2 examines dimension functions on commutative C^*-algebras; and Section 3 discusses the possibility of extending subadditive rank functions to enveloping regular rings. Section 4 concerns the problem of extending them to related AW^*-algebras.

Some of the methods used in Sections 2, 3, and 4 require knowledge of basic facts about regular rings, which may be unfamiliar to the reader. A good general reference is [12]; for the relationship with AW^*-algebras [1; Chap. 8] is the usual source.

Caveat lector! There is an unfortunate nonuniformity of terminology in the literature concerning rank and dimension functions (e.g., [14]), so the reader is warned to observe the proper definitions. We shall follow [4], but our definitions will be carefully stated, so as to avoid confusion.

I.1. *General Theory*

DEFINITION I.1.1(a). A pre-C^*-algebra A is called a *local C^*-algebra* if every positive element of A is contained in a (complete) C^*-subalgebra of A; that is, A admits a functional calculus on its positive elements. The ordering on A is that induced from its completion.

An algebraic direct limit of C^*-algebras is an important example.

I.1.1(b). If A is a ring let $M_n A$ denote the ring of $n \times n$ matrices over A, and φ_n: $M_n A \to M_{n+1} A$ the upper left corner embeddings. Define $M_\infty A$ to be the (algebraic) inductive limit of $(M_n A, \varphi_n)$. We shall think of $M_n A$ as a subring of $M_\infty A$.

I.1.1(c). If $\varepsilon > 0$, let f_ε be the continuous function from $\mathbb{R}$ to $\mathbb{R}$ which is zero on $(-\infty, \varepsilon/2]$, linear on $[\varepsilon/2, \varepsilon]$, and one on $[\varepsilon, \infty)$.

DEFINITION I.1.2. Let A be a local C^*-algebra. A *rank* (*dimension*) *function* on A is a mapping $D: A \to [0, 1]$ ($D: M_\infty A \to [0, \infty)$) such that:

(i) $\sup\{D(a) \mid a \in A\} = 1$ (normalization).

(ii) If $a \perp b$ (i.e., $ab = ab^* = a^*b = a^*b^* = 0$), then $D(a+b) = D(a) + D(b)$.

(iii) For all a, $D(a) = D(aa^*) = D(a^*a) = D(a^*)$.

(iv) If $0 \leqslant a \leqslant b$, then $D(a) \leqslant D(b)$.

(v) If $a \lesssim b$ (i.e., there exist x_n, y_n with $\{x_n b y_n\}$ converging to a in norm [4]), then $D(a) \leqslant D(b)$.

A rank function which satisfies (vi) below is *subadditive*:

(vi) For all a, b, $D(a+b) \leqslant D(a) + D(b)$.

A rank function satisfying (vi′), is called *weakly subadditive*:

(vi′) For all positive commuting a, b in A, $D(a+b) \leqslant D(a) + D(b)$.

There are many equivalent formulations of these definitions; the next few propositions explore some variations and consequences.

PROPOSITION I.1.3. *Let D be a function on a local C^*-algebra A satisfying* (iii), (iv), *and* (v). *Then D satisfies*:

(vii) *For all a in A, λ in $\mathbb{C}$-$\{0\}$, $D(\lambda a) = D(a)$;*

(viii) *For a, b in A, $D(ab) \leqslant \min\{D(a), D(b)\}$.*

Proof. That (vii) holds follows easily from (v). To prove (viii), observe that $D(ab) = D(abb^*a^*) \leqslant D(\|b\|^2\, aa^*) = D(aa^*) = D(a)$; similarly for $D(ab) \leqslant D(b)$. ∎

PROPOSITION I.1.4. *Let D be a function on a C^*-algebra A such that both* (viii) *and* (ix) *below hold*:

(ix) *For all positive a in A, $D(a) = D(a^2)$. Then D satisfies* (iii) *and* (iv).

Proof. There exists u in A such that $a = u(a^*a)^{1/4}$ [18; 1.4.5]; thus $D(a) \leqslant D((a^*a)^{1/4}) = D(a^*a) \leqslant D(a)$. Further, $D(a^*a) = D((a^*a)(a^*a)) = D(a^*(aa^*)a) \leqslant D(aa^*)$, etc., yielding (iii).

If $0 \leqslant a \leqslant b$, by [18; 1.4.5], there exists w in A with $a = wb^{1/4}w^*$; hence $D(a) \leqslant D(b^{1/4}) = D(b)$. ∎

PROPOSITION I.1.5. *Let D be a function on a local C^*-algebra A such that* (iii), (iv), (vii), *and* (x) *below hold*:

(x) $D(a) = \sup\{D(f_\delta(a)) \mid \delta > 0\}$ *for all positive a in A. Then D is lower semicontinuous.*

Proof. Let $\{x_n\}$ converge to x; we may assume that all of x_n and x are positive by (iii). Then $f_\delta(x_n)$ converges to $f_\delta(x)$ for all δ greater than zero.

Fix $\varepsilon > 0$, and choose $\delta > 0$ so that $D(f_\delta(x)) \geqslant D(x) - \varepsilon$. Set $y_n = f_{\delta/2}(x_n) f_\delta(x)$. Since $f_{\delta/2}(x)$ is a unit for $C^*(f_\delta(x))$ and $\{f_{\delta/2}(x_n)\} \to f_{\delta/2}(x)$, applying [2; Lemma 4.1] (with $a = f_{\delta/2}(x_n)$ and $z = f_\delta(x)$), we obtain that for n sufficiently large, $y_n^* y_n \geqslant \lambda f_\delta(x)^2$ for some $\lambda > 0$. Further, $y_n y_n^* \leqslant f_{\delta/2}(x_n)^2$. So, for n sufficiently large,

$$\begin{aligned} D(x) - \varepsilon &\leqslant D(f_\delta(x)) = D(f_\delta(x)^2) \leqslant D(y_n^* y_n) = D(y_n y_n^*) \\ &\leqslant D(f_{\delta/2}(x_n)^2) \leqslant D(x_n). \quad \blacksquare \end{aligned}$$

PROPOSITION I.1.6. *Let D be a function on a C^*-algebra A satisfying* (i), (ii), (viii), (ix), (x). *Then D is a lower semicontinuous rank function.*

Proof. Lower semicontinuity follows from I.1.5, and it along with (viii) jointly imply (v). ∎

We now consider subadditivity in rank functions.

PROPOSITION I.1.7. *Let D be a rank function on a local C^*-algebra, A. If D extends to a rank function on M_2A, then D is subadditive. In particular, dimension functions are subadditive.*

Proof. [4; 3.1]. We observe that

$$\left\{ \begin{bmatrix} f_{1/n}(aa^*) & f_{1/n}(bb^*) \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} f_{1/n}(a^*a) & 0 \\ f_{1/n}(b^*b & 0 \end{bmatrix} \right\}$$

converges to $\left[\begin{smallmatrix} a+b & 0 \\ 0 & 0 \end{smallmatrix}\right]$; thus

$$\begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix} \precsim \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

and so

$$D \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix} \leqslant D \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = D(a) + D(b). \quad \blacksquare$$

A partial converse will be proved subsequently (II.3.1).

EXAMPLE I.1.8. Even a lower semicontinuous rank function on a commutative C^*-algebra need not be weakly subadditive.

On $C([0, 1])$, define a function D via

$$\begin{aligned} D(f) &= \inf\{x \mid f(x) = 0\} && \text{if } f(x) = 0 \text{ for some } x; \\ &= 1 && \text{if } f(x) \neq 0 \text{ for all } x \text{ in } [0, 1]. \end{aligned}$$

Properties (i), (iii), and (iv) are obvious. If $fg = 0$, and f does not vanish on $[0, x_0)$ but $f(x_0) = 0$, then g is identically zero on $[0, x_0]$; thus $D(f+g) = x_0 = D(f) = D(f) + D(g)$, since $D(g) = 0$. So D satisfies (ii).

Suppose $\{f_n\}$ is a sequence of elements of $C([0, 1])$ converging to f, and that $D(f_n) = x_n$. Set $x_0 = \liminf x_n$. Passing to a subsequence, we may assume x_n converges to x_0. Thus $\{f_n(x_n)\}$ converges to $f(x_0)$, so that $f(x_0) = 0$. Hence $D(f) \leqslant x_0$, so D is lower semicontinuous; by I.1.6, D is a rank function.

However, if f is a nonnegative function vanishing only at $\frac{1}{4}$, and g is nonnegative, vanishing only at $\frac{1}{2}$, then $D(f) = \frac{1}{4}$, $D(g) = \frac{1}{2}$ and $D(g+f) = 1$. So D is not weakly subadditive.

More generally, if μ is a probability measure on a connected locally connected compact Hausdorff space X, and x_0 is a point of X, let $D(f)$ be the measure of the connected component of $\operatorname{coz} f = \{x \mid f(x) \neq 0\}$ containing x_0. Then D is a lower semicontinuous rank function on $C(X)$ which is not generally weakly subadditive. ∎

If, however, A has sufficiently many projections, we can show that every lower semicontinuous rank function is subadditive. The specific condition needed here (and also in Part III) is the following.

DEFINITION I.1.9 [19]. A local C^*-algebra A has property (HP) if every singly generated closed hereditary $*$-subalgebra of A has an approximate identity consisting of an increasing sequence of projections.

This can be restated in terms of closed right ideals which contain a dense singly generated submodule. All AF, all W^*-algebras have property (HP).

This definition is slightly different from that of [19]; however, if A is a C^*-algebra, the definitions are equivalent.

PROPOSITION I.1.10. *Let A be a C^*-algebra such that every singly generated hereditary C^*-subalgebra has an approximate identity consisting of projections (not assumed to be increasing). Then every hereditary $*$-subalgebra of A has an approximate identity of projections, and every countably generated hereditary C^*-subalgebra of A has an approximate identity consisting of an increasing sequence of projections.*

Proof. If B is an hereditary $*$-subalgebra of A and b belongs to B, then $(f_{1/n}(b^*b) A f_{1/n}(b^*b))^- \subseteq b^*Ab \subseteq B$, and $\bigcup_n f_{1/n}(b^*b) A f_{1/n}(b^*b)$ is dense in b^*Ab. Let $P(b, n)$ be a set of projections forming an approximate identity for $(f_{1/n}(b^*b) A f_{1/n}(b^*b))^-$. Then $P = \bigcup_{b,n} P(b, n)$ is a set of projections in B which constitutes an approximate identity for B. If B is countably generated, then we may choose the set P to be countable, say, $P = \{p_1, p_2, \ldots\}$.

Set $q_{11} = p_1$. Suppose a finite set of projections $\{q_{ij} \mid i \leqslant j \leqslant k\}$ have been chosen. Since $\{p_n q_{kk} p_n\}$ converges to q_{kk}, for sufficiently large n there is a projection $q_{k,k+1} \leqslant p_n$ with $\|q_{kk} - q_{k,k+1}\| < 2^{-k}$, and a partial isometry u_k so that $u_k^* u_k = q_{kk}$, $u_k u_k^* = q_{k,k+1}$, and $\|q_{kk} - u_k\| < 2^{-k}$. For $1 \leqslant i < k$, set $q_{i,k+1} = u_k q_{ik} u_k^*$, and define $q_{k+1,k+1} = p_n$. We hereby obtain a set of projections $\{q_{ij} \mid i \leqslant j\}$ with the properties

$$\|q_{ij} - q_{i,j+1}\| \leqslant 2^{-j} \text{ for } i \leqslant j \qquad \text{and} \qquad q_{ij} \leqslant q_{i+1,j} \text{ for } i < j.$$

Let $q_k = \lim_j q_{kj}$. Then $q_k \leqslant q_{k+1}$, and $\|q_k - q_{kk}\| \leqslant 2^{-k+1}$. Since the set $\{q_{kk}\}$ is cofinal in $\{p_n\}$, $\{q_k\}$ is an increasing approximate identity for B. ∎

If A is a C^*-algebra without 1, then $\tilde{A}$ will denote its unitification.

LEMMA I.1.11. *Let A be a C^*-algebra, and suppose x, y belong to A. Then*:

$$x^*y + y^*x \leqslant x^*x + y^*y$$

$$(x+y)^*(x+y) \leqslant 2(x^*x + y^*y).$$

If z, w lie in A, with w positive, then

$$w \leqslant 2(z^*wz + (1-z)^* w(1-z)).$$

(*The last computation is done in $\tilde{A}$ formally, but both sides belong to A even if it has no identity.*)

Proof. Clearly:

$$0 \leqslant (x-y)^*(x-y) = x^*x + y^*y - (x^*y + y^*x);$$

$$(x+y)^*(x+y) = x^*x + y^*y + (x^*y + y^*x).$$

Finally, $w = (z^* + (1-z)^*)\ w(z + (1-z)) = z^*wz + (1-z)^* w(1-z) + z^*w(1-z) + (1-z)^* wz$. Apply the first inequality with $x = w^{1/2}z$, $y = w^{1/2}(1-z)$, to obtain $z^*w(1-z) + (1-z)^* wz \leqslant z^*wz + (1-z)^* w(1-z)$. ∎

PROPOSITION I.1.12. *Let D be a lower semicontinuous rank function on a local C^*-algebra A that satisfies* (HP). *Then D is subadditive.*

Proof. Let a, b belong to A. Suppose $D(a+b) > D(a) + D(b)$. Since $D(a^*a) + D(b^*b) = D(a) + D(b) < D(a+b) = D((a+b)^*(a+b)) \leqslant D(2(a^*a + b^*b)) = D(a^*a + b^*b)$, we may assume that a, b are both positive. Choose an approximate identity $\{p_n\}$ of increasing projections for $(aAa)^-$. Since $a_n = p_n a p_n \to a$, for n sufficiently large, $D(a_n + b) > D(a) + D(b)$. By I.1.11,

$$a_n + b \leqslant 2(p_n(a_n + b)p_n + (1-p_n)(a_n+b)(1-p_n));$$

thus, as $p_n \perp (1-p_n)$,

$$D(p_n(a_n+b)\,p_n + (1-p_n)(a_n+b)(1-p_n))$$
$$= D(p_n(a_n+b)\,p_n) + D((1-p_n)(a+b)(1-p_n)).$$

Clearly, $D(p_n(a_n+b)\,p_n) \leqslant D(p_n) \leqslant D(a)$; further, $(1-p_n)(a_n+b)(1-p_n) = (1-p_n)\,b(1-p_n)$ (as $a_n \perp (1-p_n)$); hence,

$$D((1-p_n)(a_n+b)(1-p_n)) = D((1-p_n)) \leqslant D(b),$$

a contradiction. ∎

It is also possibly true that a rank function on a simple C^*-algebra must be subadditive. Here is a partial result in this direction.

PROPOSITION I.1.13. *Let D be a rank function on a simple C^*-algebra A. Then there is a (full) hereditary C^*-subalgebra of A on which D is subadditive.*

Proposition I.1.13 folows immediately from I.1.7 and the next lemma.

LEMMA I.1.14. *Let A be a simple C^*-algebra which is not 1-dimensional. Then there are nonzero (hence full) hereditary C^*-algebras $B \subset C$ of A such that $C \simeq M_2 B$.*

Proof. The hypothesis ensures that A contains two nonzero orthogonal positive elements a and b. By [3, 1.8], there exists nonzero y in A with $z = y^*y$ in $(aAa)^-$ and $w = yy^*$ in $(bAb)^-$. Thus z is orthogonal to w. Set $B = (zAz)^-$ and $C = ((z+w)\,A(z+w))^-$. ∎

Finally, we examine the behaviour of subadditive rank functions under quotients.

LEMMA I.1.15. *Let D be a function on a local C^*-algebra A such that* (vi) *and* (vii) *hold. If a, b are elements of A with $D(b) = 0$, then $D(a+b) = D(a)$.*

Proof. We observe that $D(a+b) \leqslant D(a) + D(b) = D(a)$, and $D(a) = D((a+b) - b) \leqslant D(a+b) + D(-b) = D(a+b)$. ∎

If D is a subadditive rank function on A, then $\ker D = \{a \in A \mid D(a) = 0\}$ is a two-sided *-ideal of A. By Lemma I.1.15, D induces a well-defined function $\bar{D}$ on $\bar{A} = A/\ker D$. Clearly, $\bar{D}$ satisfies (i), (iii), (vi), (vii), (viii), and (ix), but it is not clear that (ii) holds. If D can be extended to $M_2 A$, however, it is orthogonally additive.

PROPOSITION I.1.16. *Let A be a local C^*-algebra, and suppose that D is* *a rank function on A that extends to a function on M_2A satisfying* (ii), (iii), (vi), *and* (vii). *Then* $\bar{D}$ *satisfies* (ii).

Proof. Let $\pi: A \to \bar{A}$ denote the quotient map, and select orthogonal x, y in $\bar{A}$. Choose a, b in A with $\pi(a) = x$, $\pi(b) = y$. Then $ab, ab^*, \ldots$ are all in $\ker D$. Set $u = \left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right] \in M_2A$; then

$$u^*u = \begin{bmatrix} a^*a + b^*b & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$uu^* = \begin{bmatrix} aa^* & ab^* \\ ba^* & bb^* \end{bmatrix} = \begin{bmatrix} aa^* & 0 \\ 0 & bb^* \end{bmatrix} + \begin{bmatrix} 0 & ab^* \\ ba^* & 0 \end{bmatrix};$$

in addition,

$$\begin{bmatrix} 0 & ab^* \\ ab^* & 0 \end{bmatrix}$$

lies in $\ker D$. Thus,

$$\begin{aligned} \bar{D}(x + y) &= D(a + b) = D((a + b)^* (a + b)) = D(a^*a + b^*b) = D(u^*u) \\ &= D(uu^*) = D \begin{bmatrix} aa^* & 0 \\ 0 & bb^* \end{bmatrix} = D(aa^*) + D(bb^*) = D(a) + D(b) \\ &= \bar{D}(x) + \bar{D}(y). \quad \blacksquare \end{aligned}$$

Even if D is a dimension function, it is not clear that $\bar{D}$ possesses any analogue of property (v); indeed, this does not make sense (although sense could be made of it) unless $\bar{A}$ is a local C^*-algebra, that is, unless $\ker D$ is closed. It seems that $\bar{D}$ is not of much interest unless $\ker D$ is closed (but see the proof of [14; 2.4]), so we shall restrict our attention (for the moment) to lower semicontinuous D.

THEOREM I.1.17. *Let D be a lower semicontinuous dimension function on a C^*-algebra A. Then* $\ker D$ *is closed, and $\bar{D}$ is a lower semicontinuous dimension function on $\bar{A} = A/$*$\ker D$.

Proof. By I.1.6, it suffices to show that $\bar{D}$ satisfies (x). Select positive $\bar{a}$ in $\bar{A}$. If a is positive in A with $\pi(a) = \bar{a}$, then $\pi(f_\epsilon(a)) = f_\epsilon(\bar{a})$ for all ε, and $\bar{D}(\bar{a}) = D(a) = \sup_\epsilon D(f_\epsilon(a)) = \sup_\epsilon \bar{D}(f_\epsilon(\bar{a}))$. $\blacksquare$

COROLLARY I.1.18. *Let D be a lower semicontinuous dimension function on a C^*-algebra A, and suppose that J is a closed two-sided ideal of A*

contained in $\ker D$. *Then D induces a lower semicontinuous dimension function on* A/J.

Remarks I.1.19(a). A dimension function on A can be thought of as a coherent family of subadditive rank functions defined on matrix algebras over A. It is not known whether every subadditive rank function D on A extends to a dimension function; this is true when A is commutative (I.2.2) or if D is lower semicontinuous (II.3.1).

(b) The normalization condition (i) can be relaxed to simply require that $\{D(a) \mid a \in A\}$ be bounded. This is automatic if A is complete (i.e., if A is a C^*-algebra, not just a local C^*-algebra)—for, let $\{a_n\}$ be positive elements of A such that $D(a_n) \geqslant n$, $\|a_n\| < 2^{-n}$; then $\sum a_n$ would not admit a (finite) value under D. If A is not complete, however, a function satisfying (ii)–(v) need not be bounded. A theory of unbounded dimension functions and quasitraces may be developed, but is beyond the scope of this article. If A is unital, (i) and (iv) imply $D(1) = 1$.

(c) A convex combination of rank, subadditive rank, or dimension functions is a function of the same kind; if A has a unit, then each type of function is preserved under pointwise limits. Hence if A is unital, the set of rank functions (subadditive rank functions, weakly subadditive rank functions, dimension functions) is a compact convex set in the topology of pointwise convergence.

(d) It is plausible that every weakly subadditive rank function is subadditive. A proof should be possible along the lines of I.1.5 and I.1.12. If this were true, the hypotheses of II.2.2 etc., could be weakened to eliminate the irritating assumption of extendability to matrix algebras.

(e) It should be pointed out that, by [14; 2.4], every stably finite unital C^*-algebra possesses a lower semicontinuous dimension function, so the results of these sections apply to a large class of C^*-algebras.

1.2. *Dimension Functions on Commutative C^*-Algebras*

In this section, we characterize subadditive rank functions on commutative C^*-algebras, and show that all such extend to dimension functions. The completion of commutative C^*-algebras in the rank metric is also described.

Let X be a locally compact Hausdorff space, and let $\mathscr{F}$ be the set of finitely additive probability measures on X with σ-compact support, defined on the algebra of subsets of X generated by the σ-compact open sets.

PROPOSITION I.2.1. *There is a natural one to one correspondence between* $\mathscr{F}$ *and the set of subadditive rank functions on* $C_0(X)$. *A subadditive rank function is lower semicontinuous if and only if its associated measure is countably additive.*

Proof. Most of the proof is routine and is left to the reader; we outline the correspondence. If $\mu \in \mathscr{F}$, define $D_\mu(f) = \mu(\operatorname{coz} f)$, where $\operatorname{coz} f = \{x \in X \mid f(x) \neq 0\}$. Conversely, if D is a subadditive rank function and U is a σ-compact open subset of X, let f be an element of $C_0(X)$ with $U = \operatorname{coz} f$. Set $\mu(U) = D(f)$. If g is any function with $U = \operatorname{coz} g$, then $D(g) = D(f)$ by property (v), so μ is well-defined.

COROLLARY I.2.2. *Each subadditive rank function on $C_0(X)$ extends uniquely to a dimension function.*

Proof. The extension is given as follows. If D is a subadditive rank function, let μ be the corresponding measure. Then for f in $M_n(C_0(X)) \simeq C_0(X, M_n\mathbb{C})$, set

$$D(f) = \sum_{k=1}^{n} k \cdot \mu\{x \mid \operatorname{rank} f(x) = k\}.$$

It is routine (although tedious) to verify that D is a dimension function, and the only one extending the subadditve rank function. ∎

From Theorem I.2.1, we obtain the "known" result that the lower semicontinuous dimension functions on $C_0(X)$ are exactly those induced by traces (states) on $C_0(X)$ [3, 4]. The function $D_\mu \mapsto \mu$ thus gives an affine bijection between the sets of lower semicontinuous dimension functions and traces of $C_0(X)$; this is continuous when both are equipped with the topology of pointwise convergence. The inverse function is not, however, continuous, except in trivial cases. This situation will be studied in the non-commutative case, in parts II and III.

EXAMPLE I.2.3. Using I.2.1, it is easy to give an example of a dimension function which is not lower semicontinuous. Let X be the one-point compactification of $\mathbb{N}$, and ω an ultrafilter on X. For f in $C(X)$, set $D(f) = 1$ if $\operatorname{coz} f \in \omega$, $D(f) = 0$ otherwise. These are precisely the extremal dimension functions on $C(X)$; they are lower semicontinuous if and only if the corresponding ultrafilter is principal.

The spirit of Example I.2.3 pervades the proof of the following.

THEOREM I.2.4. *If X is compact, then the set of lower semicontinuous dimension functions is dense in $DF(C(X))$, the space of dimension functions on $C(X)$.*

Proof. Each finitely additive measure in $\mathscr{F}$ extends to a finitely additive probability measure on all subsets of X; these correspond naturally to the countably additive probability measures on βX_d, the Stone–Čech compactification of X with the discrete topology. There is thus an affine map

Ω (given by restriction) from $M_1(\beta X_d)$, the state space of $C(\beta X_d)$, onto $DF(C(X))$. Weak -* convergence in this state space implies setwise convergence of the corresponding measures on X (if $U \subseteq X$, consider convergence on the extension of χ_U to βX_d), which is equivalent to pointwise convergence in $DF(C(X))$; thus Ω is continuous. By the Krein–Milman theorem, it suffices to show that the lower semicontinuous extremal dimension functions are dense in the space of extreme points of $DF(C(X))$. The extremal dimension functions correspond to the $\{0, 1\}$-valued measures, which are the images of the point masses in $M_1(\beta X_d)$. The lower semicontinuous ones are the point masses from points in X, the set of which is dense in βX_d. ∎

We conjecture that this result holds for all C^*-algebras.

We now examine the completion of $C_0(X)$ with respect to a lower semicontinuous dimension function. Let μ be a countably additive measure in $\mathscr{F}$, and D_μ the corresponding dimension function. Define a (pseudo-)metric on $C_0(X)$, via $d_\mu(f, g) = D_\mu(f-g)$; then d_μ is readily checked to be a pseudometric, and addition and multiplication are uniformly continuous with respect to d_μ. Thus the completion R has a natural structure as a commutative*-algebra over $\mathbb{C}$, and there is a *-homomorphism from $C_0(X)$ to R, with kernel ker D.

A ring R with involution *, is said to be **-regular*, if for all r in R, there exists $p = p^2 = p^*$ in R such that $rR = pR$; equivalently, R is (von Neumann) regular and $xx^* = 0$ implies $x = 0$. For any ring with involution $(R, *)$ that contains at least the rationals, and satisfies

$$\sum x_i x_i^* = 0 \qquad \text{implies all } x_i = 0, \tag{A}$$

we may define a subset of R,

$$R_b = \{r \in R \mid \text{there exist } s_i \text{ in } R,\ n \text{ in } N, \text{ so that } rr^* + \sum x_i x_i^* = n \cdot 1\}.$$

Then R_b is a subring with involution, called the bounded subring of R. For details, see [1; Sect. 54] or [23].

Let μ be a countably additive measure on X, and define

$$M(X, \mu) = \frac{\{f\colon X \to \mathbb{C} \cup \{\pm\infty\} \mid f \text{ is measurable and is finite a.e.}\}}{\{f\colon X \to \mathbb{C} \cup \{\pm\infty\} \mid f \text{ is measurable and zero a.e.}\}}.$$

As will be seen below, $M(X, \mu)$ is a *-regular ring, and its bounded subring is $L^\infty(X, \mu)$. The kernel of the map $C_0(X) \to M(X, \mu)$ is precisely ker D_μ.

PROPOSITION I.2.5. *The completion of $C_0(X)$ with respect to d_μ is naturally isomorphic to $M(X, \mu)$. Further, $M(X, \mu)$ is a *-regular ring satisfying* (A) *above, and $L^\infty(X, \mu)$ is its bounded subring.*

Proof. Define $B=\{(f_i)\mid f_i\in C_0(X),\ (f_i)$ Cauchy re $d_\mu\}\subseteq\prod_{\mathbb{N}} C_0(X)$ (the latter term is the full cartesian product), and define $N=\{(f_i)\in B\mid \lim D_\mu(f_i)=0\}$. If (f_i) belongs to B, then the sequence converges in measure; we thus define a measurable function $f(x)=\lim f_i(x)$ (observe that for almost all x, the sequence $\{f_i(x)\}$ is ultimately stationary). Then f belongs to $M(X,\mu)$, and modulo N, is independent of the approximating sequence. Now D_μ extends in the obvious way to $M(X,\mu)$, as does d_μ to a metric. It is routinely verified that $(f_i)\mapsto f$ is a *-algebra homomorphism with kernel N, and the quotient map, $R=B/N\to M(X,\mu)$, is an isometry with respect to d_μ. However, every finite a.e. measurable function can be approximated in measure by functions in $C_0(X)$, so the map is onto.

Given an element f in $M(X,\mu)$, define g via

$$\begin{aligned} g(x) &= 0 && \text{if } f(x)=0 \\ &= 1/f(x) && \text{if } f(x)\neq 0. \end{aligned}$$

Then g lies in $M(X,\mu)$, and $fgf=f$, $gfg=g$. Hence $M(X,\mu)$ is regular, and since $\sum f_i f_i^*=0$ implies all f_i are zero (routine), $M(X,\mu)$ is *-regular.

To be bounded with respect to the *-order in $M(X,\mu)$ requires the existence of g_i in $M(X,\mu)$ and n in $\mathbb{N}$ so that $ff^*+\sum g_i g_i^*=n$. Then $\mu\{x\mid |f(x)|^2>n\}=0$, so f lies in $L^\infty(X,\mu)$. Conversely, if $|f|$ is essentially bounded, with essential supremum less than n, then $ff^*+(\sqrt{n^2-ff^*})^2=n$ in $M(X,\mu)$; thus f lies in the bounded subring. ∎

I.3. *Completions and Extensions of Subadditive Rank Functions*

In this section, it is shown that a subadditive lower semicontinuous rank function on a local C^*-algebra is induced by a *-homomorphism into a *-regular ring which is complete at a (regular ring) rank function. One eventual consequence is that each such function extends uniquely to subadditive rank functions on the rings of matrices, and thus to a dimension function on A.

LEMMA I.3.1. *Let R be any ring, with elements x_i, y_i $(i=1,2)$. Let $N\colon R\to[0,1]$ satisfy*

$$N(rs)\leqslant \min\{N(r),N(s)\}$$

and

$$N(r+s)\leqslant N(r)+N(s) \text{ all } r,\ s \text{ in } R.$$

Assume that for $i=1,2$, the following hold:

$$x_i y_i = y_i x_i, \qquad x_i y_i x_i = x_i, \qquad y_i x_i y_i = y_i.$$

Then $N(y_1-y_2)\leqslant 9N(x_1-x_2)$.

Proof. Set $x_i y_i = e_i$; then $e_i^2 = e_i$. Since

$$y_1 - y_2 = y_1(e_1 - e_2) + (e_1 - e_2) y_2 - y_1(x_1 - x_2) y_2,$$

we obtain

$$N(y_1 - y_2) \leqslant 2N(e_1 - e_2) + N(x_1 - x_2). \tag{1}$$

But $e_1 - e_2 = y_1(x_1 - x_2) - (x_1 - x_2) y_2 + y_1 x_2 - x_1 y_2$, so

$$N(e_1 - e_2) \leqslant 2N(x_1 - x_2) + N(y_1 x_2 - x_1 y_2). \tag{2}$$

Finally

$$\begin{aligned} y_1 x_2 - x_1 y_2 &= -y_1 x_1^2 y_2 + y_1 x_2^2 y_2 = -y_1(x_1^2 - x_2^2) y_2 \\ &= -y_1(x_1(x_1 - x_2) + (x_1 - x_2) x_2) y_2; \end{aligned}$$

thus

$$N(y_1 x_2 - x_1 y_2) \leqslant 2N(x_1 - x_2). \tag{3}$$

Combining (1), (2), and (3), the esired estimate results. ∎

It follows that in a *-regular ring, if $'$ denotes "relative inverse," and N is a (regular ring) pseudo-rank function, then $N(r' - s') \leqslant 19N(r - s)$. K. R. Goodearl has reduced 19 to 5.

Let A be a local C^*-algebra, D a subadditive lower semicontinuous rank function on A; then D induces a pseudo-metric $d(x, y) = D(x - y)$ on A; the completion R of A has a natural structure as a complex *-algebra as in Section 2, and D extends to a function $\bar{D}$ on R. The kernel of the natural *-homomorphism $\rho: A \to R$, is ker D, a closed *-ideal.

THEOREM I.3.2. *The following properties hold for R and $\bar{D}$:*

(i) *R is *-regular, * satisfies* (A);

(ii) *$\bar{D}$ is a regular ring rank function* [12] *on R;*

(iii) *R is self-injective on either side.*

Proof. Define $B = \{(a_i) \in \prod_{\mathbb{N}} A \mid (a_i) \text{ is } d\text{-Cauchy}\}$, and $N = \{(a_i) \in B \mid \lim D(a_i) = 0\}$. Then $R = B/N$, and $\bar{D}((a_i)) = \lim D(a_i)$ is well-defined. It follows easily that $\bar{D}$ satisfies

$$\bar{D}(r + s) \leqslant \bar{D}(r) + \bar{D}(s)$$

and

$$\bar{D}(rs) \leqslant \bar{D}(r), \bar{D}(s) \text{ for all } r, s \text{ in } R.$$

The proof proceeds in a series of lemmas.

LEMMA I.3.3. *If $\sum_1^n r_i r_i^* = 0$, then each $r_i = 0$.*

Proof. Let $\varepsilon > 0$. Choose a_i in A, with $\bar{D}(\rho(a_i) - r_i) < \varepsilon/2n$. Then $D(\sum a_i a_i^*) \leqslant \bar{D}(\sum r_i r_i^*) + \varepsilon = \varepsilon$. Since for each i, $a_i a_i^* \leqslant \sum a_i a_i^*$, $D(a_i) = D(a_i a_i^*) \leqslant \varepsilon$; thus $\bar{D}(r_i) \leqslant 2\varepsilon$. As ε is arbitrary, $\bar{D}(r_i) = 0$, and thus $r_i = 0$. ∎

LEMMA I.3.4. *Let r and s be orthogonal positive elements of R. Then there are sequences (c_n), (d_n) of elements of A^+ such that $\rho(c_n) \to r$, $\rho(d_n) \to s$, and $d_n \perp c_n$. In particular, $\bar{D}(r + s) = \bar{D}(r) + \bar{D}(s)$.*

Proof. Find a_n in A^+ such that $\rho(a_n) \to r$. Then there is a sequence of real positive numbers ε_n, for which $\rho(c_n) \to r$, where $c_n = a_n f_{\epsilon_n}(a_n)$. Choose b_n in A^+ so that $\rho(b_n) \to s$. Since $\rho(f_{\epsilon_n/2}(a))s \to 0$, by passing to subsequences we may assume that $\rho(f_{\epsilon_n/2}(a_n)\, b_n) \to 0$. Define

$$d_n = b_n - f_{\epsilon_n/2}(a_n)\, b_n - b_n f_{\epsilon_n/2}(a_n) + f_{\epsilon_n/2}(a_n)\, b_n f_{\epsilon_n/2}(a_n).$$

Then d_n is positive, $\rho(d_n)$ converges to s, and $d_n \perp c_n$ (formally, $d_n = (1 - f_{\epsilon_n/2}(a_n))$ in $\tilde{A}$). ∎

LEMMA I.3.5. *R has an identity.*

Proof. Choose $a_n \geqslant 0$ in A, with $D(a_n) \geqslant 1 - 1/2n$; then there is an $\varepsilon_n > 0$, with $D(f_{\epsilon_n}(a_n)) > 1 - 1/n$. Set $b_n = f_{\epsilon_n}(a_n)$, $c_n = f_{\epsilon_n/2}(a_n)$. Then $b_n c_n = b_n$. As $(c_n - c_n^2) \perp b_n$, $D(c_n - c_n^2) < 1/n$. Set $x = c_m - c_n$. Then

$$\begin{aligned} D(x) &= D((c_m - c_n)x) = D(x - c_n x - x + c_m x) \leqslant D(x - c_n x) + D(x - c_m x) \\ &= D((x - c_n x)(x - x c_n)) + D((x - c_m x)(x - x c_m)) \\ &= D(x^2 - c_n x^2 - x^2 c_n + c_n x^2 c_n) + D(x^2 - c_m x^2 - x^2 c_m + c_m x^2 c_m) \\ &< 1/n + 1/m; \end{aligned}$$

the last inequality follows since $(x^2 - c_i x^2 - x^2 c_i + c_i x^2 c_i) \perp b_i$, for $i = n, m$. Thus $\{\rho(c_n)\}$ is a Cauchy sequence whose limit is a projection p in R, with $\bar{D}(p) = 1$. For y in R, $(y - py)(y - py)^* = yy^* - pyy^* - yy^*p + pyy^*p$; this is orthogonal to p, so $\bar{D}(y - py) = 0$ and thus $y = py$. Similarly $yp = y$, and p is thus an identity of R. ∎

LEMMA I.3.6. *The ring R is *-regular.*

Proof. In view of Lemmas I.3.3 and I.3.5, we need only show that the equation $bxb = b$ can be solved for any element b of R. If $b^*byb^*b = b^*b$ for some y, then $(b - byb^*b)^*\ (b - byb^*b) = b^*b - b^*by^*b^*b - b^*byb^*b + b^*by^*b^*byb^*b = 0$; thus $b = byb^*b$, and so $x = yb^*$ would be a solution of

$bxb = b$. It therefore suffices to show that $czc = c$ has a solution for every $b^*b = c$ in R.

To this end, select a sequence of elements of A^+, (a_n), so that $\rho(a_n) \to c$. Each a_n sits inside a commutative C^*-subalgebra of A, and the closure in R is a $*$-regular ring by I.2.5. Thus each $\rho(a_n)$ has a relative inverse $\rho(a_n)'$ (see the function g defined in the course of the proof of I.2.5) in R. By Lemma I.3.1, the sequence $(\rho(a_n)')$ is a Cauchy sequence in R, and its limit z satisfies $czc = c$. ∎

LEMMA I.3.7. *Let E be a function on satisfying* (ii), (iii), (vi), *and* (viii), *that is uniformly continuous with respect to D. Then E extends to a regular ring pseudo-rank function on R; if $E = D$, the appropriate extension is $\bar{D}$, and this is a regular ring rank function.*

Proof. Obviously E extends to R by defining $\bar{E}((a_i)) = \lim E(a_i)$ if (a_i) is a Cauchy sequence with respect to D. Since $E(ab) \leqslant E(a), E(b)$ for all a, b in A, it follows that $\bar{E}$ is also submultiplicative. It remains to show that if e, f are idempotents such that $0 = ef = fe$ (these are called "orthogonal" in the theory of regular rings; to avoid confusion with the notion of orthogonal used in this paper, we shall refrain from using "orthogonal" in the sense of regular rings), then $\bar{E}(e+f) = \bar{E}(e) + \bar{E}(f)$.

By $*$-regularity, there exist projections p, q in R so that $Rp = Re$, and $qR = fR$. As $eR \simeq pR$ and $Rq \simeq Rf$ (as R-modules) and R is von Neumann regular, it follows that there exist elements x, y, z, w in R such that

$$ex = x = xp, \quad py = y = ye, \quad fz = z = zq, \quad qw = w = wf,$$

and

$$xy = e, \quad yx = p; \qquad wz = q, \quad zw = f.$$

As $e = xpy$ and $p = yex$, $\bar{E}(e) = \bar{E}(p)$; similarly, $\bar{E}(f) = \bar{E}(q)$. Also,

$$e + f = (x+z)(p+q)(y+w) \qquad \text{and} \qquad p + q = (y+w)(e+f)(x+z);$$

thus, $\bar{E}(e+f) = \bar{E}(p+q)$. As $ef = 0$, $p \perp q$; by the second sentence of I.3.4, $\bar{E}(p+q) = \bar{E}(p) + \bar{E}(q)$. Combining all of the equalities deduced above, we obtain $\bar{E}(e+f) = \bar{E}(p+q) = \bar{E}(p) + \bar{E}(q) = \bar{E}(e) + \bar{E}(f)$. Thus $\bar{E}$ is a pseudo-rank function.

If $E = D$, $\bar{E} = \bar{D}$; then $\bar{D}(p) = 0$ implies $p = 0$ and thus $\bar{D}$ is a rank function. ∎

Since R is complete at the $\bar{D}$-rank metric, it follows from [12; 19.7] that R is right and left self-injective. This completes the proof of I.3.2. ∎

COROLLARY I.3.8. *Let D be a lower semicontinuous subadditive rank function on a local C^* algebra A, and suppose that E satisfies the hypothesis of* I.3.7. *Then for every n, E extends uniquely to a function on M_nA also called E, satisfying* (ii), (iii), (iv), (vi), (vii), (viii) *such that* $\sup\{E(a) \mid a \in M_nA\} = n$; *further, the completion of M_nA in the (extended) D-metric is M_nR.*

Proof. Since R, the D-completion, is $*$-regular satisfying (A), M_nR is also $*$-regular. Now $\bar{E}$, $\bar{D}$ (on R) extend uniquely to regular ring pseudo-rank functions on $S = M_nR$ [12; 16.10]; the restriction to M_nA is the desired extension. Property (ii) is proved by observing that for elements a, b in M_nA which are orthogonal, then in the $*$-regular ring S, their projections obtained via $Sa = Sp$, $Sb = Sq$, are orthogonal; as $Sa + Sb = Sp \oplus Sq = S(p+q)$, and as it is easily verified that $S(a+b) = Sa + Sb$, we obtain that $\bar{E}(a) + \bar{E}(b) = \bar{E}(p) + \bar{E}(q) = \bar{E}(p+q) = \bar{E}(a+b)$.

In a $*$-regular ring, $aS = aa^*S \simeq a^*S$, and (iii) follows.

If $0 \leqslant c \leqslant d$ in S (i.e., $c = \sum c_i c_i^*$, $d = d^*$, and $d - c = \sum x_i x_i^*$), then the right annihilator of d is contained in that of c; it follows that if $Sc = Sp$, $Sd = Sq$ (p, q projections), then $p \leqslant q$, so that $Sc \subseteq Sd$; thus $\bar{E}(c) \leqslant \bar{E}(d)$. Hence (iv) holds.

Properties (vi), (vii), (viii) follows from the corresponding properties of regular ring rank functions [12; Chap. 16].

Let D_1, E_1 be extensions of D, E to M_nA that satisfy (ii), (iii), (vi), and (viii). For $a = (a_{ij})$ in M_nA, let X^{ij} be the matrix whose only nonzero entry is a_{ij} in the ij position; then $D_1(X^{ij}) = D(a_{ij})$, and $D_1(a) \leqslant \sum D_1(X^{ij}) = \sum D(a_{ij}) \leqslant n^2 D(a)$. Thus D_1 is uniformly continuous with respect to D, so by I.3.6, D_1 yields a regular ring rank function on M_nA; by the uniqueness of the extension to matrix rings [12; 16.10], $D_1 = D$, and one similarly proves that E_1 is uniformly continuous with respect to E and thus to D, so E_1 induces a pseudo-rank function, and the uniqueness result implies $E_1 = E$. ∎

Property (v) and lower semicontinuity of the extensions to matrix rings also hold (II.3.1), but more work is required.

I.4. *Mapping to AW^*-Algebras*

In this section, we show that a lower semicontinuous subadditive rank function on a local C^*-algebra is induced by a homomorphism into a finite AW^*-algebra. This is obtained via an ultraproduct construction similar to that of [14], and is closely related to the regular ring considered in I.3.

THEOREM I.4.1. *Let A be a local C^*-algebra, D a lower semicontinuous subadditive rank function on A. Then there is a finite AW^*-algebra M, a* *faithful lower semicontinuous subadditive rank function $\tilde{D}$ on M, and a homomorphism $\sigma: A \to M$ such that $D = \tilde{D} \circ \sigma$.*

Let $l^\infty(A)$ be the local C^*-algebra of bounded sequences of elements of A, and ω a non-principal ultrafilter on $\mathbb{N}$. Define D^ω on $l^\infty(A)$ via $D^\omega((a_n)) = \lim_\omega D(a_n)$. Let J be the (closed) *-ideal of $l^\infty(A)$ consisting of sequences converging to 0 in norm along ω, and set $A_\omega = l^\infty(A)/J$. Then A_ω is a C^*-algebra (any quotient of $l^\infty(A)$ by a closed *-ideal containing $c_0(A)$ is automatically complete). Let π be the quotient map from $l^\infty(A)$ onto A_ω, and define D_ω on A_ω by $D_\omega(x) = \inf\{D^\omega((a_n)) \mid \pi((a_n)) = x\}$, (the infimum being taken over all sequences representing x).

LEMMA I.4.2. *Let x be an element of A_ω. Then there is a sequence (a_n) in $l^\infty(A)$ with $\pi((a_n)) = x$ and $D_\omega(x) = D^\omega((a_n))$.*

Proof. For each positive integer k, there is a sequence (a_{kn}) in $l^\infty(A)$ with $\pi((a_{kn})) = x$ and $D_\omega(x) \geqslant D^\omega((a_{kn})) - 1/k$. For each $k > 1$, the set $S_k = \{n \mid \|a_{kn} - a_{1n}\| < 1/k,\ |D(a_{kn}) - D_\omega(x)| < 2/k\}$ belongs to ω. Set $\tilde{S}_1 = \mathbb{N}$, $\tilde{S}_k = S_1 \cap \cdots \cap S_k$ for $k > 1$; then $\tilde{S}_k$ belongs to ω for all k, and $\tilde{S}_1 \supseteq \tilde{S}_2 \supseteq \cdots$. If $n \in \tilde{S}_k / \tilde{S}_{k+1}$, set $a_n = a_{kn}$, and (a_n) is the desired sequence. ∎

LEMMA I.4.3. *If x, y lie in A, then $D_\omega(x+y) \leqslant D_\omega(x) + D_\omega(y)$, and $D_\omega(xy) \leqslant \min\{D_\omega(x), D_\omega(y)\}$.*

Proof. Let (a_n) and (b_n) represent x and y as in I.4.2. Then

$$\begin{aligned} D_\omega(x+y) &\leqslant \lim_\omega D(a_n + b_n) \leqslant \lim_\omega (D(a_n) + D(b_n)) \\ &= \lim_\omega D(a_n) + \lim_\omega D(b_n) \\ &= D_\omega(x) + D_\omega(y); \end{aligned}$$

submultiplicativity is proved in a similar fashion. ∎

LEMMA I.4.4. *If x is a positive element of A_ω, then $D_\omega(x) = D_\omega(x^2)$.*

Proof. By I.4.3, $D_\omega(x^2) \leqslant D_\omega(x)$; conversely, let (a_n) represent x^2 as in I.4.2. Then $\pi((a_n^* a_n)^{1/4}) = x$, so $D_\omega(x) \leqslant \lim_\omega D((a_n^* a_n)^{1/4}) = \lim_\omega D((a_n)) = D_\omega(x^2)$. ∎

LEMMA I.4.5. *The function D is lower semicontinuous.*

Proof. (Similar to I.4.2). Let $\{x_k\}$ converge in norm to x in A_ω; represent each x_k by (a_{kn}) in $l^\infty(A)$, and x by (a_n), as in I.4.2. By passing to a subsequence of (x_k), it suffices to show for any $r > 0$, that if $D_\omega(x_k) \leqslant r$ for all k, then $D_\omega(x) \leqslant r$. For each integer k, there is an integer $m(k)$ such that the set S_k,

$$S_k = \{n \in \mathbb{N} \mid \|a_{m(k)n} - a_n\| < 1/k, D(a_{m(k)n}) < r + 1/k\}$$

belongs to ω. Set $\tilde{S}_1 = \mathbb{N}$, $\tilde{S}_k = S_1 \cap \cdots \cap S_k$, and define $b_n = a_{m(k)n}$ if n belongs to $\tilde{S}_k / \tilde{S}_{k+1}$. Then $\pi((b_n)) = x$, and $\lim_\omega D(b_n) \leqslant r$. ∎

There remains the difficult step of showing that D_ω is orthogonally additive.

LEMMA I.4.6. *If x, y are orthogonal elements of A, then*

$$D_\omega(x+y) = D_\omega(x) + D_\omega(y).$$

Proof. We may assume that $x, y \geqslant 0$, since $|x+y| = |x| + |y|$. We may extend D to M_2A by I.3.8, and this induces a function $D_\omega^{(2)}$ on M_2A, which extends D_ω. Because the completion of M_2A at D is M_2R (where R is the D-completion of A), we may apply the results of I.4.2–I.4.5 to $D_\omega^{(2)}$ (observe in particular, that the proof of I.4.5 adjusted for $D_\omega^{(2)}$ does not require lower semicontinuity of the extended D on M_2A). Thus $D_\omega^{(2)}$ satisfies (iii), (vi), (vii), (viii), and is lower semicontinuous.

Let

$$h = \begin{pmatrix} x^{1/2} & 0 \\ y^{1/2} & 0 \end{pmatrix} \in M_2A_\omega;$$

then $h^*h = \left(\begin{smallmatrix} x+y & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $h^*h = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, so $D_\omega(x+y) = D_\omega^{(2)}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$. Let

$$(u_n) = \begin{pmatrix} r_n & s_n \\ s_n^* & t_n \end{pmatrix}$$

be a sequence in $(M_2A)+$ representing $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ as in I.4.2. Then $\pi((r_n)) = x$, $\pi((t_n)) = y$, and $\lim_\omega \|s_l\| = 0$.

Fix $\varepsilon > 0$; then for any $\delta > 0$,

$$\left\{ n \;\middle|\; \left\| f_\varepsilon(u_n) - \begin{pmatrix} f_\varepsilon(r_n) & 0 \\ 0 & f_\varepsilon(t_n) \end{pmatrix} \right\| < \delta \right\}$$

belongs to ω. For such an n in this set, $a = f_\varepsilon(u_n)^{1/2}$ satisfies for sufficiently small δ,

$$\|aw - w\| < (1/3)\|w\| \qquad \text{for all } w \text{ in } C^*(z),$$

where

$$z = \begin{pmatrix} f_{2\varepsilon}(r_n) & 0 \\ 0 & f_{2\varepsilon}(t_n) \end{pmatrix};$$

this follows since

$$\begin{pmatrix} f_\epsilon(r_n) & 0 \\ 0 & f_\epsilon(t_n) \end{pmatrix}$$

is a unit for $C^*(z)$. By [2; 4.1], $za^2z \leqslant (1/3)\, z^2$; thus

$$D(z) = D(z^2) \leqslant D(za^2z) = D(az^2a) \leqslant D(a^2) = D(f_\epsilon(u_n)) \leqslant D(u_n).$$

As

$$\pi\left(\begin{pmatrix} f_{2\epsilon}(r_n) & 0 \\ 0 & f_{2\epsilon}(t_n) \end{pmatrix}\right) = \begin{pmatrix} f_{2\epsilon}(x) & 0 \\ 0 & f_{2\epsilon}(y) \end{pmatrix},$$

we obtain

$$\begin{aligned} D_\omega(x) + D_\omega(y) &= \sup_\epsilon\, (D_\omega(f_{2\epsilon}(x)) + D_\omega(f_{2\epsilon}(y))) \\ &= \sup_\epsilon\, (\lim_\omega D(f_{2\epsilon}(r_n)) + \lim_\omega D(f_{2\epsilon}(t_n))) \\ &= \sup_\epsilon \left(\lim_\omega D\begin{pmatrix} f_{2\epsilon}(r_n) & 0 \\ 0 & f_{2\epsilon}(t_n) \end{pmatrix}\right) \\ &\leqslant \lim_\omega D(u_n) = D_\omega^{(2)}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = D_\omega(x+y); \end{aligned}$$

as $D_\omega(x+y) \leqslant D_\omega(x) + D_\omega(y)$, orthogonal additivity is established. ∎

So D_ω is a lower semicontinuous subadditive rank function on A_ω. There is a natural map $\delta\colon A \to l^\infty(A)$, the diagonal map, and the composition $\theta = \pi\delta$ yields a *-homomorphism from A to A_ω. It follows from the lower semicontinuity of D that $D = D_\omega \theta$. ∎

LEMMA I.4.7. *The C^*-algebra A_ω is $\aleph_0$-injective.*

Proof. Follows immediately from [14; 2.1, 2.2]. ∎

Thus if $M = A_\omega/\ker D_\omega$, then M is a finite AW^*-algebra by [14; 2.3], and D_ω induces a faithful lower semicontinuous subadditive rank function $\tilde{D}$ on M by I.3.8, I.1.16 and its preceding comments. This completes the proof of Theorem I.4.1. ∎

COROLLARY I.4.8. *Let D be a lower semicontinuous subadditive rank function on a local C^*-algebra A. Then D extends uniquely to a lower semicontinuous subadditive rank function on the norm completion of $\tilde{A}$.*

Further consequences will be obtained in Section II.3.

We now explain the relationship between the construction in this section and that of I.3; we also outline an alternate proof of I.4.1 and other results to be proved later.

We may assume A is a unital C^*-algebra.

Suppose T is a *-regular ring; then the bounded subring (see I.2) T_b admits a C^*-like pseudo-norm (if T is at least an algebra over the rational numbers),

$$\|t\|^2 = \inf\{q \in \mathbb{Q} \mid tt^* \leqslant q\},$$

and the Jacobson radical of T_b is the set of elements of pseudo-norm zero (if T is at least an algebra over $\mathbb{R}$). Details may be found in [23].

If R is the ring constructed in I.3, form the complete direct product ΠR of countably many copies of R. Then ΠR is *-regular, and the natural homomorphism $\rho^\infty: \Pi A \to \Pi R$ maps $l^\infty(A)$ into $(\Pi R)_b$.

Set $\bar{Y} = \{(r_n) \in \Pi R \mid \lim_\omega \bar{D}(r_n) = 0\}$, and $Y = (\rho^\infty)^{-1}\, \bar{Y}$. From the density of $\rho(A)$ in R, it follows that the natural map $\Pi A/Y \to \Pi R/\bar{Y}$ is an isomorphism. Set $S = (\Pi R/\bar{Y})_b = (\Pi A/Y)_b$; there is a natural map $\alpha: l^\infty(A) \to S$.

LEMMA I.4.9. *The mapping α is surjective.*

Proof. Let $x = x^*$ be an element of S; we may assume $0 \leqslant x \leqslant 1$. Lift x to $a = (a_n)$ in ΠA, with $a = a^*$; then a, $1 - a$ belong to $(\Pi A)^+ + Y$. There exist commutative C^*-subalgebras C_n of A, with a_n in C_n. We may thus write $a_n = b_n - c_n$, where b_n, c_n are positive orthogonal elements of C_n. Pre- and post-multiplying by $c = (c_n) \in \Pi A$, yields $c(1-a)c = -c^3$. The left side is positive modulo Y; as c is as well and the quotient ring $\Pi R/\bar{Y}$ has its induced involution positive definite,[1] we conclude c^3 belongs to Y, and thus c belongs to Y. Set $b = (b_n) \in \pi A$; then $b - (1-a) \in Y$, and $b \geqslant 0$. Apply a similar process to $1 - b$; write $1 - b_n = d_n - e_n$ with $d_n, e_n \geqslant 0$, $d_n e_n = 0$. As above, (e_n) belongs to Y, and if $d = (d_n)$, then $a - d \in Y$ and $0 \leqslant d \leqslant 1$. ∎

Standard techniques in regular rings reveal that the function on ΠR defined by $\bar{D}^\omega((r_n)) = \lim_\omega \bar{D}(r_n)$ is a regular ring pseudo-rank function. The kernel is exactly $\bar{Y}$, and it is an automatic consequence of the properties of regular rings that $\bar{D}^\omega$ induces a regular ring rank function $\bar{D}_\omega$ on $\Pi R/\bar{Y}$ [12; Sect. 16], and thus to S by restriction.

The kernel of the map from $l^\infty(A)$ to A_ω is the set of elements of $l^\infty(A)$ which are sent into the radical J_0 of S by α. Hence α induces a surjective homomorphism $\beta: A_\omega \to B = S/J_0$.

Furthermore, if $\tilde{D}$ is defined on B via $\tilde{D}(b) = \inf\{\bar{D}_\omega(s) \mid \pi(s) = b\}$, then

[1] Since ΠR has all its matrix rings *-regular, the same holds for all quotients of ΠR; this translates to condition (A), the positive definiteness of the induced involution, on all quotients.

$$
\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[d, "\rho"'] & & \ell^{\infty}(A) \arrow[rr, "\pi"] \arrow[d, "\rho^{\infty}"] \arrow[drr, "\alpha"] \arrow[ddl, "\iota"'] & & A_{\omega} \arrow[r, "\pi"] \arrow[dr, "\beta"] & M \arrow[d, "\gamma"] \\
R_b \arrow[rr, "\delta"] \arrow[dd, "\iota"'] & & (\pi R)_b \arrow[rr] \arrow[dd, "\iota"] & & S=(\pi R/Y)_b \arrow[r, "\pi"] \arrow[dd, "\iota"] & B=S/J_0 \\
& \pi A \arrow[rr, "\pi"] \arrow[dr, "\rho^{\infty}"'] & & \pi A/Y \arrow[dr, "\simeq"] & & \\
R \arrow[rr, "\delta"] & & \pi R \arrow[rr, "\pi"] & & \pi R/Y &
\end{tikzcd}
$$

TABLE 1. Each iota (ι) is an inclusion, δ is an embedding, and every π is a quotient map.

$D_{\omega} = \tilde{D}\beta$, and $\ker \beta D_{\omega}$. Thus β induces an isomorphism $\gamma: M \to B$ that respects $\tilde{D}$.

Table 1 summarizes the constructions made earlier.

The second, third lines can be used to give an alternate proof of I.4.1, as well as of II.3.1; namely, complete A to obtain R, form $(\Pi R)_b$, use I.4.9 to obtain the ontoness of α; in fact the same proof shows that the natural map from $l^{\infty}(A)$ to any quotient of $(\Pi R)_b$ by a *-ideal containing $c_0(R_b)$ is onto. In particular, any such quotient that is semiprimitive (older terminology: semisimple) is automatically a C^*-algebra, and $\aleph_0$-injective. Using standard results about ultrafilters on regular rings and corresponding rank functions, we can obtain $\bar{Y}$ as the kernel of the pseudo-rank function induced on ΠR by $\bar{D}$ and ω. Then B will inherit a rank function from its regular ring, which coincides with the original rank function on A. Further, we observe that every normal element of B is contained inside a subalgebra of B of the form $L^{\infty}(X)$; since the regular ring rank function is countably additive on projections of the AW^*-algebra B, it can be shown to be lower semicontinuous on commutative subalgebras; hence it is lower semicontinuous. The same can be applied to matrix rings to show that the extended functions (on $M_n B$, and by restriction on $M_n A$) are also lower semicontinuous.

II. QUASITRACES

Quasitraces have been studied for a long time (cf. [17; Chap. IV]), except by name. Here we develop the basic properties of quasitraces and show the correspondence between 2-quasitraces and lower semicontinuous dimension functions; some results on dimension functions follow.

Section II.1 concerns quasitraces on AW^*-algebras, and II.2 deals with general quasitraces. In II.3, we obtain results on dimension functions. Finally in II.4, we show that the set of 2-quasitraces on a unital C^*-algebra constitutes a simplex, and consider the relationships between traces an quasitraces.

II.1. *Quasitraces on AW^*-Algebras*

DEFINITION II.1.1. A quasitrace on a pre-C^*-algebra A is a function $\tau: A \to \mathbb{C}$ such that:

(i) $0 \leqslant \tau(x^*x) = \tau(xx^*)$ for all x in A;

(ii) τ is linear on commutative $*$-subalgebras of A;

(iii) If $x = a + ib$ with a, b self-adjoint, then

$$\tau(x) = \tau(a) + i\tau(b).$$

If τ extends to a quasitrace on $M_2 A$, then τ is called a *2-quasitrace.*

A linear quasitrace is a trace.

Whether every (2-) quasitrace on a C^*-algebra is linear, is a well-known open question (asked by Kaplansky). By Corollary II.2.4 below, this is equivalent to the problem of whether the canonical quasitrace on a II_1 AW^*-factor is linear.

We first require a result about AW^*-algebras which may already be known; if M is an AW^*-algebra, a a normal element, and S a Borel subset of $\sigma(a)$, we write $E_S(a)$ for the corresponding spectral projection.

If p, q are projections, then we write $p \precsim q$ if there is a partial isometry (in the relevant C^*-algebra) x such that $xx^* = p$ and $x^*x \leqslant q$.

We are indebted to L. Zsido for the proof of the following lemma.

LEMMA II.1.2. *Let M be an AW^*-algebra, containing elements a, b such that $0 \leqslant a \leqslant b$, and suppose λ, μ are real numbers with $0 \leqslant \lambda < \mu$. Set $p = E_{(\mu,\infty)}(a)$, $q = E_{(\lambda,\infty)}(b)$. Then*

$$p \precsim q.$$

Proof. We observe that $\mu p \leqslant a \leqslant b$ and $\lambda(1-q) \geqslant b(1-q)$. Now $\|pqp - p\| = \|p(1-q)p\| = \|(1-q)\,p(1-q)\| \leqslant (1/\mu)\,\|(1-q)\,b(1-q)\| = (1/\mu)\,\|(1-q)b\| \leqslant \lambda/\mu < 1$. Hence pqp is invertible in pMp, and it easily follows that $p \precsim q$. ∎

COROLLARY II.1.3. *Under the same hypotheses as* II.1.2,

$$E_{(\lambda,\infty)}(a) \precsim E_{(\lambda,\infty)}(b) \qquad \textit{and} \qquad E_{(\lambda,\infty)}(a) \precsim E_{[\lambda,\infty)}(b).$$

Proof. This follows from II.1.2 and the continuity of $\precsim$ under monotone limits. ∎

COROLLARY II.1.4. *A quasitrace on an AW^*-algebra is oder-preserving on self-adjoint elements.*

Proof. Since a quasitrace is linear on commutative C^* subalgebras, it can be calculated on any self-adjoint element by integrating over its spectral projections. If $a \leqslant b$, it follows from II.3 that

$$\tau(E_{(\lambda,\infty)}(a)) \leqslant \tau(E_{(\lambda,\infty)}(b)) \qquad \text{for all } \lambda,$$

and a routine argument then shows that $\tau(a) \leqslant \tau(b)$. ∎

Remark II.1.5. It is not clear that a faithful quasitrace is strictly monotone even if M is a factor. This is closely related to the question of whether a II_1 AW^*-algebra factor can contain a non-normal hyponormal element. It is possible that strict monotonicity implies linearity for quasitraces.

The proof of the next proposition is due to S. Berberian, who has also obtained Corollary II.1.4 (unpublished).

PROPOSITION II.1.6. *A quasitrace on an AW^*-algebra is norm-continuous; if a, b are self-adjoint, then*

$$|\tau(a) - \tau(b)| \leqslant \tau(1)\,\|a - b\|.$$

In general,

$$|\tau(x) - \tau(y)| \leqslant 2^{1/2}\tau(1)\,\|x - y\|.$$

Proof. Let $\lambda = \|a - b\|$; then $-\lambda \cdot 1 \leqslant a - b \leqslant \lambda \cdot 1$, and $b - \lambda \cdot 1 \leqslant a \leqslant b + \lambda \cdot 1$; $\tau(b) - \lambda\tau(1) = \tau(b - \lambda \cdot 1) \leqslant \tau(a) \leqslant \tau(b + \lambda \cdot 1) = \tau(b) + \lambda\tau(1)$, since b commutes with 1. The second statement follows by applying the first to the real and imaginary parts. ∎

It follows immediately from II.1.1(i) and [1; Sect. 17, Theorem 1], that if τ is a quasitrace on an AW^* algebra M, then $\tau(x) = \tau(px)$ for all x in M, where p is the largest finite central projection of M. Thus it suffices to study quasitraces on finite AW^*-algebras.

Let M be a finite AW^*-algebra. Then there is a unique centre-valued dimension function D on M, defined in [1; Chap. 6]. This can be extended to a "centre-valued quasitrace" T on finite linear combinations of orthogonal projections. The function T is order-preserving; hence by an argument similar to that of the proof of II.1.6, T can be extended by continuity to all of M. The following properties hold for T:

(i) $T(x + y) = T(x) + T(y)$ if x, y are commuting normal elements.

(ii) $T(\lambda x) = \lambda T(x)$ for all x in M, $\lambda \in \mathbb{C}$.

(iii) $T(z) = z$ if z is central.

(iv) $0 \leqslant T(x^*x) = T(xx^*)$ for all x in M.

(v) $a \leqslant b$ implies $T(a) \leqslant T(b)$.

(vi) T is norm-continuous.

THEOREM II.1.7. *Every quasitrace on M is uniquely expressible in the form $\varphi \cdot T$, for some positive linear functional φ on $Z(M)$.*

Proof. Let τ be a quasitrace on M, and set $\varphi = \tau / Z(M)$. Set $\tau_1 = \varphi \cdot T$; then τ_1 is a quasitrace on M, as τ is linear on $Z(M)$. Observe that $\tau(e) = \tau_1(e)$ for every simple projection e in M, and hence $\tau = \tau_1$ on linear combinations of orthogonal simple projections.

Write $Z(M) \simeq C(X)$ for a compact Hausdorff X. Let $\varepsilon > 0$ be fixed. For any projection p in M, let $\hat{p}$ be the continuous function on X corresponding to $T(p)$. Since X is totally disconnected, we may find continuous functions f and g on X, each taking on only a finite number of values, all rational (if M has a type I summand, we also require that f and g lie in the set of values T takes on projections), with $f \leqslant \hat{p} \leqslant g$ and $\varphi(g - f) < \varepsilon$. There are thus finite orthogonal families $\{q_i\}$, $\{r_j\}$ of simple projections in M such that $\sum q_i \leqslant p \leqslant \sum r_j$ and $\tau_1(\sum r_j - \sum q_i) < \varepsilon$. But since $\tau(\sum q_i) = \tau_1(\sum q_i)$, and similarly with the r_j, it follows that both $\tau(p)$ and $\tau_1(p)$ are between $\tau_1(\sum q_i)$ and $\tau_1(\sum r_j)$; hence $|\tau(p) - \tau_1(p)| < \varepsilon$. Since ε is arbitrary, $\tau = \tau_1$ on projections, hence by continuity, $\tau = \tau_1$. ∎

COROLLARY II.1.8. *The set of quasitraces on an AW^*-algebra forms a complete lattice.*

COROLLARY II.1.9. *Let M be an AW^*-algebra, τ_0 a finite trace on M. If τ is a quasitrace on M with $\tau \leqslant \tau_0$, then τ is a trace.*

Proof. We may assume M is finite. Then there is a largest central projection p such that T is linear on pM, and $\operatorname{supp}(\tau) \leqslant \operatorname{supp}(\tau_0) \leqslant p$ in $Z(M)$. ∎

COROLLARY II.1.10. *Every quasitrace on an AW^*-algebra is a 2-quasitrace.*

COROLLARY II.1.11. *Let τ be a quasitrace on an AW^*-algebra M, and suppose a, b are positive elements of M. Then*

$$\tau(a + b) \leqslant 2(\tau(a) + \tau(b)).$$

Proof. In $M_2 M$, set

$$x = \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix};$$

then $x^*x = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}$, and

$$xx^* = \begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix}.$$

Let z be

$$\begin{pmatrix} a^{1/2} & 0 \\ -b^{1/2} & 0 \end{pmatrix};$$

then $\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} - xx^* = zz^*$; thus $\tau(a+b) = \tau(x^*x) = \tau(xx^*) \leqslant \tau\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} = 2(\tau(a) + \tau(b))$. ∎

II.2. *Quasitraces on General C^*-Algebras*

In this section, we show that there is a natural bijection between the set of 2-quasitraces and the set of lower semicontinuous dimension functions on a C^*-algebra. This was partially described in [4]. Using this, we show that every 2-quasitrace on a C^*-algebra extends to the enveloping AW^*-algebra described in I.4. Consequences include the fact that 2-quasitraces are almost linear on almost commuting elements.

LEMMA II.2.1. *Let A be a C^*-algebra, and a, b positive elements of A with $a \leqslant b$. For any $\varepsilon > 0$, there exists x in A such that $x^*x = f_\varepsilon(a)$ and $f_{\varepsilon/6}(b)\, xx^* = xx^*$.*

Proof. Represent A on a Hilbert space. Choose λ and μ with $\varepsilon/3 < \lambda < \mu < \varepsilon/2$, and $p = E_{(\mu,\infty)}(a)$, $q = E_{(\lambda,\infty)}(b)$. Set $c = f_{\varepsilon/3}(b) f_\varepsilon(a)$. Inasmuch as $f_\varepsilon(a) \leqslant p$, $f_{\varepsilon/3}(b) \geqslant q$, and the right projection of qp is p (notation: $RP(qp) = p$) (II.1.2), it follows that $RP(c) = RP(f_\varepsilon(a))$. If $c = u\,|c|$ is the polar decomposition of c, then the element $x = u(f_\varepsilon(a))^{1/2}$ belongs to A. Now $x^*x = f_\varepsilon(a)$ and $RP(xx^*) \leqslant RP(f_{\varepsilon/3}(b))$, so $f_{\varepsilon/6}(b)$ is a unit for xx^*. ∎

THEOREM II.2.2. *There is a natural affine bijection between the sets of quasitraces and weakly subadditive (unnormalized) lower semicontinuous rank functions, on any C^*-algebra A. The 2-quasitraces correspond to the subadditive (lower semicontinuous) rank functions.*

Proof. If D is a weakly subadditive lower semicontinuouus rank function on A, define a quasitrace τ_D as follows. If B is a commutative C^*-subalgebra of A, say, $B \simeq C_0(X)$, hen D induces a countably additive finite measure on X, which defines a positive linear functional on B; call it τ_D. This defines τ_D unambiguously on normal elements.

In general, if x is an element of A, write $x = a + ib$ with a and b self-adjoint; define $\tau_D(x) = \tau_D(a) + i\tau_D(b)$. Then τ_D satisfies II.1.1(ii) and (iii). To show $\tau_D(x^*x) = \tau_D(xx^*)$, it suffices to show that for all non-negative continuous functions f on $\sigma(x^*x)\cup\{0\}$ vanishing at 0, $D(f(x^*x))= D(f(xx^*))$. Represent A on a Hilbert space, and let $x=u|x|$ be the polar decomposition. If $y=u(f(x^*x))^{1/2}$, then y belongs to A, $y^*y=f(x^*x)$, and $yy^*=f(xx^*)$. Thus τ_D is a quasitrace.

Conversely, let τ be a quasitrace on A. For a in A, define $D_\tau(a)= \sup_\varepsilon \tau(f_\varepsilon(|a|))$. Being a positive linear functional on $C^*(|a|)$, τ is bounded, and so $D_\tau(a)$ is finite. We shall show that D_τ is a weakly subadditive lower semicontinuous rank function.

(I) D_τ yields a subadditive lower semicontinuous rank function on commutative C^*-subalgebras of A, by the results of I.2.

(II) If a, b are orthogonal elements, then $|a|\perp|b|$, so $|a|$ and $|b|$ commute. Hence, $D_\tau(a+b)=D_\tau(|a+b|)=D_\tau(|a|+|b|)=D_\tau(a)+D_\tau(b)$ by I.

(III) If $0\leqslant a\leqslant b$ and $\varepsilon>0$, let x be as in Lemma II.2.1; since $f_{\varepsilon/6}(b)$ is a unit for xx^*, they commute; and since $\|xx^*\|\leqslant 1$, $xx^*\leqslant f_{\varepsilon/6}(b)$. Thus $\tau(f_\varepsilon(a))=\tau(xx^*)\leqslant\tau(f_{\varepsilon/6}(b))$, so $D_\tau(a)\leqslant D_\tau(b)$.

(IV) By I, we have that $D_\tau(a)=D_\tau(a^2)=D_\tau(\lambda a)$ for $a\geqslant 0$ and λ in $\mathbb{C}-\{0\}$. Hence $D_\tau(x)=D_\tau(x^*x)=D_\tau(xx^*)=D_\tau(x^*)$ for all x.

(V) For a, b in A, we have that $D_\tau(ab)=D_\tau(b^*a^*ab)\leqslant D_\tau(b^*b)= D_\tau(b)$ $(b^*a^*ab\leqslant\|b\|^2\,b^*b)$; similarly $D_\tau(ab)\leqslant D_\tau(a)$.

(VI) By (I) and I.1.5, D_τ is lower semicontinuous.

Thus D_τ is a weakly subadditive lower semicontinuous rank function. It is readily checked that the two assignments described above are mutual inverses, and are affine.

If τ is a 2-quasitrace, then D_τ is subadditive by I.1.7. Conversely, if D_τ is subadditive, it can be extended to an enveloping AW^*-algebra M as in Section I.4, and so τ can be extended to a quasitrace on M. Thus τ is a 2-quasitrace by II.1.10.

This completes the proof of the theorem. ∎

COROLLARY II.2.3. *Every quasitrace on a C^*-algebra is bounded; in particular,* $\|\tau\|=\sup\{\tau(a)\mid 0\leqslant a,\ \|a\|\leqslant 1\}<\infty$.

Proof. By I.19(b), D_τ is bounded, and if $0\leqslant a$ and $\|a\|<1$, then $\tau(a)\leqslant D_\tau(a)$. ∎

It is easily seen that $\|\tau\|$ is also equal to sup $\{D_\tau(a)\mid a\in A\}$.

COROLLARY II.2.4. *Let τ, τ_0 be 2-quasitraces on a C^*-algebra with $\tau\leqslant\tau_0$, and let M be the AW^*-algebra constructed for D_{τ_0} in Section* I.4. *Then there is a 2-quasitrace $\bar\tau$ on M such that $\tau=\bar\tau\circ\theta$.*

Proof. This follows from II.2.2 and I.1.18. ∎

COROLLARY II.2.5. *If τ is a 2-quasitrace on a C^*-algebra A, then*

(i) *τ extends uniquely to a 2-quasitrace on $\tilde{A}$ so that $\|\tau\| = \tau(1)$;*

(ii) *τ is order-preserving;*

(iii) *if a, b are self-adjoint elements of A, then $|\tau(a) - \tau(b)| \leqslant \|\tau\| \, \|a - b\|$; in general, $|\tau(x) - \tau(y)| \leqslant 2^{1/2} \|\tau\| \, \|x - y\|$; in particular, τ is norm-continuous;*

(iv) *if a, b are positive elements of A, then $\tau(a + b) \leqslant 2(\tau(a) + \tau(b))$.*

It is clear that the 2-quasitrace τ_ω on A_ω determined by D_ω (I.4) is given by the formula $\tau_\omega(\pi(a_n)) = \lim_\omega \tau(a_n)$ (this is well-defined by the continuity of τ), so we have the following corollary.

COROLLARY II.2.6. *Let τ be a 2-quasitrace on a C^*-algebra A. Then for every $\varepsilon > 0$, there is a $\delta > 0$, there is a $\delta > 0$ such that whenever a, b are self-adjoint elements inside the unit ball of A,*

$$\|ab - ba\| < \delta \qquad \text{implies} \quad \|\tau(a + b) - \tau(a) - \tau(b)\| < \varepsilon.$$

Proof. Suppose not; then for some $\varepsilon > 0$, there exist self-adjoint elements a_n, b_n in the unit ball of A so that $\|a_n b_n - b_n - a_n\| < 1/n$ and $|\tau(a_n + b_n) - \tau(a_n) - \tau(b_n)| \geqslant \varepsilon$. Set $x = \pi((a_n))$, $y = \pi((b_n))$ in A. Then x and y commute and are self-adjoint; thus $\tau_\omega(x + y) = \tau_\omega(x) + \tau_\omega(y)$. However, $\tau_\omega(x + y) = \lim_\omega \tau(a_n + b_n)$, $\tau_\omega(x) = \lim_\omega \tau(a_n)$, $\tau_\omega(y) = \lim_\omega \tau(b_n)$, a contradiction. ∎

II.3. *Results about Dimension Functions*

In this section, we obtain some results about extendability of rank functions to dimension functions; these are consequences of the work in I.2.

THEOREM II.3.1 (Compare I.3.8). *Let D be a subadditive lower semicontinuous rank function on a C^*-algebra. Then D extends uniquely to a lower semicontinuous dimension function on A.*

Proof. By I.4.1, D extends to its enveloping AW^*-algebra M, and $D = D_\tau$ for a unique quasitrace τ on M. By II.1.7, τ extends uniquely to a quasitrace on any matrix algebra over M, hence by II.2.2 and the uniqueness of the extension obtained in I.3.8, we conclude that the extension of D to matrix rings is lower semicontinuous. ∎

Recall that a hereditary C^*-subalgebra B of a C^*-algebra A is *full* if the closed two-sided ideal, $(ABA)^-$, generated by B, equals A; B is *completely full* if the two-sided ideal ABA generated (algebraically) by B is A.

THEOREM II.3.2. *Let B be a hereditary C^*-subalgebra of A, and D a subadditive lower semicontinuous rank function on B. Then D extends uniquely to a function on ABA satisfying* I.1.2(ii)–(x). *If B is full, then D extends to a lower semicontinuous dimension function on A if and only if the extension to ABA is bounded. In particular, if B is completely full, D automatically extends to a lower semicontinuous dimension function on A.*

Proof. Let $a = \sum_i^n c_i x_i d_i$ be a typical element of ABA, with c_i, d_i in A, and x_i in B. Define three elements of $M_n A$,

$$c = \begin{bmatrix} c_1 c_2 \cdots c_n \\ \\ \bigcirc \end{bmatrix}; \qquad x = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}; \qquad d = \begin{bmatrix} d_1 & \\ \vdots & \bigcirc \\ d_n & \end{bmatrix}.$$

Considering A as the upper left hand corner subalgebra of $M_n A$, we have $a = cxd$, and thus $a^*a = d^*x^*c^*cxd$. Set $y = (x^*c^*cx)^{1/2}$; as $M_n B$ is a hereditary subalgebra of $M_n A$ (write $B = L \cap L^*$ for a left ideal L of A; then $M_n B = (M_n L) \cap (M_n L)^*$, so $M_n B$ is hereditary) and x belongs to it, so does y. We also observe that $a^*a = d^*y^*yd$; set $w = yd$. We may define (as in [3; 1.4]) an isomorphism Φ from $A_a = (M_n A)_w$ onto $(M_n A)_{w^*} \subseteq M_n(B)$; Φ extends to an isomorphism from $(a^*Aa)^-$ onto a C^*-subalgebra of $M_n B$. We can use this isomorphism to transfer the extension of D to $M_n B$, to $(a^*Aa)^-$, and therefore to a, once we show that the function constructed in this way is well-defined.

For this, we need only show that if u and v are elements of $M_n A$ such that $u^*u = v^*v = a^*a$, and uu^*, vv^* belong to $M_n B$, then $D(uu^*) = D(vv^*)$. However, if $z = vu^*$, then $z_k = f_{1/k}(vv^*)\, z f_{1/k}(uu^*)$ converges to z, and each z_k belongs to $M_n B$, so that z does also. Now, $z^*z = uv^*vu^* = (uu^*)^2$, and $zz^* = (vv^*)^2$, so

$$D(uu^*) = D((uu^*)^2) = D(z^*z) = D(zz^*) = D((vv^*)^2) = D(vv^*).$$

The extended function is a rank function, since any countable collection of elements of A is contained in a singly generated hereditary C^*-subalgebra.

If B is full, then ABA contains the Pedersen ideal of A, which is a local C^*-algebra; thus if the extension of D to ABA is bounded, it extends to A by Corollary I.4.8. ∎

The hypothesis that B be full can be eliminated by II.4.7.

COROLLARY II.3.3. *Let D_1 and D_2 be lower semicontinuous subadditive rank functions on a C^*-algebra A. If D_1 and D_2 agree on a full hereditary C^*-subalgebra of A, then $D_2 = D_1$.*

A further extension theorem will be obtained (II.4.7).

II.4. *Results about Quasitraces*

We begin with some corollaries of Theorems II.3.1 and II.3.2.

PROPOSITION II.4.1. *Let τ be a 2-quasitrace on a C^*-algebra A. Then τ extends to a quasitrace on $M_n A$ for all n.*

PROPOSITION II.4.2. *Let B be a full hereditary C^*-subalgebra of a C^*-algebra A, and let τ be a 2-quasitrace on B. Then τ extends uniquely to a 2-quasitrace τ on ABA, and τ extends to a 2-quasitrace on A if and only if it is bounded on ABA.*

COROLLARY II.4.3. *Let τ be a quasitrace on a simple C^*-algebra A. Then there is a 2-quasitrace on A which agrees with τ on a hereditary C^*-subalgebra of A.*

Proof. This follows immediately from I.1.13 and II.3.2. ∎

So if A is a simple C^*-algebra, there is an affine retraction from the set of quasitraces on A onto the set of 2-quasitraces. This should be true in general; in fact, it seems likely that every quasitrace is a 2-quasitrace (see remark I.1.19(d)).

Denote the set of normalized 2-quasitraces on a C^*-algebra A by $QT(A)$, and the set of (normalized) dimension functions by $DF(A)$. If A is unital, then $QT(A)$ and $DF(A)$ are compact convex sets.

THEOREM II.4.4. *If A is a unital C^*-algebra, then $QT(A)$ is a simplex.*

Proof. The proof is quite similar to Thoma's proof for traces [22]. We outline the argument. It suffices to show that the set of 2-quasitraces is a (complete) lattice. If τ_1 and τ_2 are 2-quasitraces on A, then both extend to the enveloping AW^*-algebra M for $D_{\tau_1+\tau_2}$ by II.2.4; call the extensions α_1, α_2. Then α_1 and α_2 have a greatest lower bound on M (II.1.8); this infimum restricted to A is clearly the infimum for τ_1 and τ_2. Thus the set of 2-quasitraces is a lattice. Now if $\{\tau_i\}$ is any collection of 2-quasitraces, let τ_0 be one of the τ_i, and set $\sigma_i = \inf(\tau_i, \tau_0)$. All of the σ_i extend to M_{τ_0}, and the set of extensions has an infimum by II.1.8, which (when restricted to A) is the infimum of $\{\tau_i\}$. ∎

It is reasonable to conjecture that $DF(A)$ is also a simplex in general, although it can fail to be metrizable when A is separable (example I.2.3).

PROPOSITION II.4.5. *Let A be a C^*-algebra. The set* $\mathrm{T}(A)$ *of normalized traces of A is a closed face in $QT(A)$.*

Proof. If τ is a trace on A, and $\tau = \lambda\tau_1 + (1-\lambda)\tau_2$ with λ unequal to

0, 1, where τ_1, τ_2 belong to $QT(A)$, set $\alpha = \min(\lambda, 1-\lambda)$. Then $\tau_1, \tau_2 \leqslant \tau_0 = \alpha^{-1}\tau$; they extend to M_{τ_0}, and are therefore linear by II.1.9. ∎

PROPOSITION II.4.6. *The set, denoted* LSCDF(A), *of lower semi-continuous dimension functions on a C^*-algebra A is a face in $DF(A)$.*

Proof. This follows immediately from I.1.5. ∎

Summarizing, if A is a unital C^*-algebra, there is a continuous affine bijection from a face, LSCDF(A), of the compact convex set $DF(A)$ onto the simplex $QT(A)$. The inverse is not in general continuous (I.2); in fact, LSCDF(A) is frequently dense in $DF(A)$, and is possibly always dense in it. If A is stably finite, then LSCDF(A) is at least nonempty [14], so that so is $QT(A)$.

If A is not unital, then by considering the unitification $\tilde{A}$, II.4.4 yields that the set of quasitraces of bound at most 1, together with zero, forms one of Effros' simplex spaces; and analogues of II.4.5 and II.4.6 follow.

THEOREM II.4.7. *Let A be a C^*-algebra, I a closed two-sided ideal, and τ an element of $QT(I)$. Then τ extends uniquely to an element of $QT(A)$. Hence if D belongs to* LSCDF(I), *then D extends uniquely to an element of* LSCDF(A).

Proof. By II.2.2, it suffices to prove existence of extensions for dimension functions, and uniqueness for quasitraces. Let $\{c_n\}$ be a sequence in I as in the proof of I.3.5. If a is an element of A, we have $(c_m - c_n)a^*a(c_m - c_n) \leqslant \|a\|^2 (c_m - c_n)^2$, so $D(ac_m - ac_n) \leqslant D(c_m - c_n) \to 0$ (as $m, n \to \infty$). Thus (ac_n) is a Cauchy sequence in the completion R of I in the D-metric. Call the limit $\hat{a}$. It is easily verified that the map $a \longmapsto \hat{a}$ is a $*$-homomorphism of A into R_b. Via the maps described at the end of I.4, we obtain a $*$-homomorphism of A into the AW^*-algebra M extending the natural map of I into M, so the rank function $\tilde{D}$ yields the desired extension.

To prove uniqueness, we may assume A possesses an identity. Let F be the closed face consisting of elements of $QT(A)$ which vanish on I. Let τ_1 and τ_2 be elements of $QT(A)$ with $\tau_1 = \tau_2$ on I. Then $\tau_1 - \tau_2 = \lambda f_1 - \mu f_2$ with f_1, f_2 in F and λ, μ positive. Evaluation at 1 yields $\lambda = \mu$. Decompose $\tau_2 = \alpha f_3 + (1-\alpha)\tau_3$ with $f_3 \in F$, τ_3 in the complementary face. Since

$$\sup\{\tau_2(x) \mid x \in I^+, \|x\| \leqslant 1\} = 1,$$

we obtain $\alpha = 0$, so τ_2 is in the complementary face to F, and it follows that $\tau_1 = \tau_2$. ∎

PROPOSITION II.4.8. *Let $\{A_i\}$ be a directed system of unital C^*-algebras.*

and $A = \varinjlim A_i$ *the unital* C^**-direct limit. Then* $QT(A)$ *is affinely homeomorphic to* $\varprojlim QT(A_i)$.

Proof. This is routine, except that II.2.2 and II.4.8 are required to pass from the algebraic inductive (= direct) limit to its completion. ∎

THEOREM II.4.9. *Let* $\mathscr{D}$ *be the collection of all* C^**-algebras for which* $QT(A) = T(A)$ (i.e., *every* 2*-quasitrace is a trace). Then* $\mathscr{D}$ *contains all type I* C^**-algebras, and is closed under the formation of quotients, ideals, extensions, direct limits, and matrix rings.*

Proof. A quasitrace on a quotient clearly lifts, so $\mathscr{D}$ is closed under quotients, and is closed under extensions and formation of ideals by II.4.7 and [7; 2.10.4], and under direct limits by II.4.8.

To show that all type I C^*-algebras are in $\mathscr{D}$, we observe that commutative C^*-algebras are in $\mathscr{D}$, and that every type I can be constructed by extensions and limits of continuous trace C^*-algebras. Now II.4.10 below completes the proof. ∎

PROPOSITION II.4.10. *If* A *is a* C^**-algebra such that for all closed prime ideals* P, A/P *belongs to* $\mathscr{D}$, *then* A *belongs to* $\mathscr{D}$.

Proof. Let τ be an extreme point of $QT(A)$ (A may be assumed unital), and D its corresponding lower semicontinuous dimension function. Then D is an extreme point of the face LSCDF(A), hence is extreme in $DF(A)$. We shall show that $\ker D$ is prime.

Let $\bar{D}$ be the corresponding dimension function on $\bar{A} = A/\ker D$. Complete $\bar{A}$ at the rank metric to obtain the regular ring $\bar{R}$, and observe that $\bar{D}$ is obviously extremal in LSCDF($\bar{A}$). Now $\bar{D}$ extends to a rank function on $\bar{R}$. We now show that the image of $\bar{D}$ in $\mathbb{P}(\bar{R}) = \{\text{pseudo-rank functions on } \bar{R}\}$ is extremal.

If it were not, there would exist a regular ring pseudo-rank function E on $\bar{R}$ so that $E \leqslant k\bar{D}$. By I.3.8–proof, E restricts to a dimension function on $\bar{A}$, hence on A, G, such that G lies in the face generated by D in $DF(A)$. Hence $G = D$. As $\bar{A}$ is dense in $\bar{R}$, we would obtain that $E = \bar{D}$. Thus $\bar{D}$ is an extremal point of $\mathbb{P}(R)$. By [12; 19.14], $\bar{R}$ is simple.

Let I, J be ideals of $\bar{A}$ such that $IJ = 0$. Clearly, $\{r \in \bar{R} \mid Ir = 0 = rI\}$ is a two-sided ideal of $\bar{R}$, and is nonzero and proper if both I and J are. This would contradict the simplicity of $\bar{R}$; hence $\bar{A}$ must be prime.

By hypothesis, τ must be a trace. The natural map $T(A) \to QT(A)$ has image containing has all of the extreme points of $QT(A)$; by the Krein–Milman theorem and compactness of $T(A)$, the map must be onto. ∎

By applying the Cayley transform and its inverse, one can easily show that $T(A) = QT(A)$ for all C^*-algebras if and only if $T(B) = QT(B)$ for the single C^*-algebra, the full C^*-algebra of the free group on two generators, $B = C^*(\mathbb{F}_2)$.

COROLLARY II.4.11. *A unital stably finite C^*-algebra in $\mathscr{D}$ admits a trace; in particular, this applies to those unital stably finite C^*-algebras in the class closed under extensions, ideals, quotients, direct limits, and matrix rings, generated by type I C^*-algebras.*

Proof. By II.4.9, $T(A) = QT(A)$; by [14; 2.4], $QT(A) \neq \varnothing$. ∎

The second part of II.4.11 is of course well-known, and admits a simple direct proof.

III. K_0^* AND RELATED TOPICS

The group K_0^* defined in [4] in analogy with the usual construction of K_0 for C^*-algebras, has been used as a technical tool to prove existence of dimension functions. The construction of K_0^* for nonsimple C^*-algebras is discussed in [14]; the main feature that distinguishes this case from the simple situation, is that stable finiteness does not guarantee that the natural pre-ordering is a partial ordering. A main result of [4] is that there is a duality between $K_0^*(A)$ and $DF(A)$, in the sense that there is a natural bijection between the states (pre-order preserving homomorphisms into $\mathbb{R}$) of K_0^* and $DF(A)$.

Unfortunately (except in rather special cases, cf. I.2), $DF(A)$ is unmanageably large. The set $QT(A)$ is much more tractable, and because of its correspondence with LSCDF(A), there is hope that $QT(A)$ might serve as a predual for $K_0^*(A)$, in the sense that K_0^* may be realized as a set of affine functions on $QT(A)$.

This potential duality may be carried farther in some cases. When A is stably finite, $K_0(A)$ admits a natural partial ordering, and if $A \otimes \kappa$ has enough projections, one would expect that elements of $QT(A)$ would be determined on projections. Then $QT(A)$ could be viewed as a set of states on $K_0(A)$. Indeed, if A is an AF algebra, $QT(A) = T(A)$, and the duality is complete, as $T(A)$ can be identified with the state space of $K_0(A)$.

So if A has many projections, we can hope for relationships

$$K_0(A) \leftrightarrow QT(A) \leftrightarrow K_0^*(A) \leftrightarrow DF(A),$$

where the double-headed arrows represent dualities. We shall show these dualities hold for a class of C^*-algebras containing all simple AF algebras, and obtain en passant an explicit description of K_0^*. In the course of this, the isomorphism classes of the closed right ideals in such C^*-algebras are determined (K_0^* plays a similar role with respect to closed right ideals that K_0 plays with respect to projection-generated right ideals).

In III.1, we obtain a portion of the duality in a more general situation, and derive some consequent results about K_0 and the set of dimension functions; then in III.2, we establish the right ideal isomorphism class results, and in III.3, we describe the structure of K_0^*.

Throughout, A will denote a stably finite *unital* C^*-algebra, although this is done for convenience only—the corresponding results hold for non-unital C^*-algebras, with only the obvious modifications necessary.

III.1. *Representation of K_0^* by Affine Functions*

In this section, we shall assume that A is stably (HP), i.e., $M_n A$ has property (HP) (Definition I.1.9) for all n.

There is a natural order-preserving homomorphism from $K_0(A)$ into the pre-ordered $K_0^*(A)$. This is not generally one to one [4; p. 153, 154]; even when it is one to one, it is not generally an order-isomorphism onto its image.

Let S be the state space of $K_0(A)$. If p is a projection in $M_\infty A$, we obtain a continuous affine function $\hat{p}$ on S by evaluation. For a in $M_\infty A$, set $\hat{a} = \sup\{\hat{p} \mid p \in (a^* M_\infty Aa)^-\}$.

Observe that since there exists an integer n such that a belongs to $M_n A$, $\hat{a}$ is less than or equal the constant function n, and is thus bounded.

LEMMA III.1.1. *If $\{p_n\}$ is any increasing approximate identity for $(a^* M_\infty Aa)^-$ consisting of projections, then $\hat{a} = \sup \hat{p}_n$; thus $\hat{a}$ is affine and lower semicontinuous.*

Proof. If q is a projection in $(a^* M_\infty Aa)^-$, then $\{p_n q p_n\}$ converges to q; thus for sufficiently large n, q is equivalent to a subprojection of p_n, and thus $\hat{q} \leqslant \hat{p}_n$. ∎

It is also clear that if $a \perp b$, then $(a+b)\hat{} = \hat{a} + \hat{b}$ (observe that $((a+b)^* M_\infty A(a+b))^- \simeq (a^* M_\infty Aa)^- \oplus (b^* M_\infty Ab)^-$).

LEMMA III.1.2. *If $[a] \leqslant [b]$ in $K_0^*(A)$, then $\hat{a} \leqslant \hat{b}$.*

Proof. If $[a] \leqslant [b]$, there is a c in $M_\infty A$, with $a \perp c$, $b \perp c$, and $a + c \lesssim b + c$. So we may assume that $a \lesssim b$. If p is a projection in $(a^* M_\infty Aa)^-$, then $p \lessapprox a$ by [3; 1.9], so $p \lessapprox b$, and therefore $p \lesssim b$. If $\{q_n\}$is an approximate identity for $(b^* M_\infty Ab)^-$ consisting of projections, then $p \lessapprox bq_n \lessapprox q_n$ for n sufficiently large. By [3; 1.7], p is equivalent to a subprojection of q_n, so $\hat{p} \leqslant \hat{q}_n$. ∎

Therefore the function $\Lambda: a \to \hat{a}$ yields a pre-order preserving homomorphism (also denoted Λ) from $K_0^*(A)$ into the set of bounded affine functions on S with the ordinary ordering. Each point x in S defines a state on $K_0^*(A)$ by evaluation, and hence a dimension function on A by [4].

THEOREM III.1.3. *The mapping above is an affine bijection between S and* LSCDF(A); *the induced affine bijection between S and $QT(A)$ is a homeomorphism.*

Proof. Let x in S be fixed. If a is a positive element of $M_\infty A$, $\hat{a} = \sup\{\hat{p} \mid p \precsim a\}$. But if $p \precsim a$, then $p \precsim f_\varepsilon(a)$ for some $\varepsilon > 0$, so $\hat{p} \leqslant (f_\varepsilon(a))\hat{} \leqslant \hat{a}$, and in particular $\hat{x}(p) \leqslant \hat{x}(f_\varepsilon(a)) \leqslant \hat{x}(a)$. Therefore by Proposition I.1.5, the dimension function corresponding to x is lower semicontinuous dimension function on A is completely determined by its values on projections; thus the map is one to one. If D is any lower semicontinuous dimension function on A, then D induces a state on $K_0^*(A)$, and hence by composition a state on $K_0(A)$; and the map is thus onto.

Let $\{\tau_\alpha\}$ be a net in $QT(A)$ with $\{\tau_\alpha\} \to \tau_0$, and let x_α, x_0 be the corresponding points of S. Let p be a projection in $M_\infty A$. Since $\tau(p) = D_\tau(p)$ for every τ in $QT(A)$, we have $\tau_\alpha(p) = D_{\tau_\alpha}(p) = \hat{x}_\alpha(p) \to \tau_0(p) = \hat{x}_0(p)$ for every p, and hence $x_\alpha \to x_0$ in S. A compactness argument yields continuity of the inverse map. ∎

COROLLARY III.1.4. *Let A be a unital, stably finite, stably* (HP) *C^*-algebra. Then the state space of $K_0(A)$ is a simplex.*

This corollary slightly generalizes the corresponding result for AF algebras [9; 1.7].

COROLLARY III.1.5. *Let A be as in* III.1.4, *and D a dimension function on A. There is a unique lower semicontinuous dimension function D on A which agrees with D on projections. Hence there is a (generally discontinuous) affine retraction from $DF(A)$ onto* LSCDF(A).

III.2. *Description of Right Ideals*

We begin with a general result about module isomorphism of right ideals in C^*-algebras.

PROPOSITION III.2.1. *Let B be a C^*-algebra, containing elements a, b. Then the following are equivalent:*

(1) *There is a continuous module isomorphism of $(aB)^-$ onto $(bB)^-$.*

(2) *There is an isometric module isomorphism of $(aB)^-$ onto $(bB)^-$.*

(3) *There is a sequence (u_n) in $(bAa^*)^-$, with $\|u_n\| \leqslant 1$, such that for all x in $(aB)^-$, $\{u_n x\}$ converges, and $\psi(x) = \lim u_n x$ is an isometric module isomorphism of $(aB)^-$ onto $(bB)^-$.*

(4) *There is an isomorphism φ from $(aBa^*)^-$ onto $(bBb^*)^-$, and a sequence (u_n) in $(bBa^*)^-$ with $\|u_n\| \leqslant 1$, such that $(u_n x)$ converges for all x in $(aB)^-$, $(u_n^* y)$ converges for all y in $(bB)^-$, $(u_n x u_n^*)$ converges to $\varphi(x)$ for all x in $(aBa^*)^-$, and $(u_n^* y u_n)$ converges for all y in $(bBb)^-$ to $\varphi^{-1}(y)$.*

Proof. The implications $(3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$ are trivial.

$(1) \Rightarrow (4)$. Let $w\, (aB)^- \to (bB)^-$ be a continuous module isomorphism. Represent B on a Hilbert space, and let $w(a) = u\, |w(a)|$ be the polar decomposition. As $RP(u) = RP(a)$, u^*u is a unit for $(aBa^*)^-$. If x belongs to $(aB)^-$, then ux is an element of B by [2; 2.1]; so if x lies in $(aBa^*)^-$, then uxu^* belongs to B. Similarly (as w^{-1} is continuous), we obtain that u^*y belongs to B for y in $(bB)^-$, and that uxu^* lies in $(bBb^*)^-$ for all x in $(aBa^*)^-$.

Set $\varphi(x) = uxu^*$; then φ is an isomorphism of $(aBa^*)^-$ onto $(bBb^*)^-$. If $u_n = f_{1/n}(|b|)\, u f_{1/n}(|a|)$, then $u_n x \to ux$ for all x in $(aB)^-$, and $u_n x u_n^* \to \varphi(x)$ for all x in $(aBa^*)^-$.

$(4) \Rightarrow (3)$. Taking the same sequence (u_n), it is clear that $\psi(x) = \lim u_n x$ is a module homomorphism of $(aB)^-$ into $(bB)^-$ which is isometric as, for all x in $(aB)^-$,

$$\begin{aligned}\|\psi(x)\|^2 &= \lim \|u_n x\|^2 = \lim \|u_n x x^* u_n^*\| = \|\varphi(xx^*)\| \\ &= \|xx^*\| = \|x\|^2.\end{aligned}$$

Now ψ is onto, since if y belongs to $(bB)^-$, the sequence $(u_n^* y)$ converges to an element x of $(aB)^-$ with $\psi(x) = y$. ∎

One observation that is worth being made at this point is that a mapping sending a to b (a, b elements of a C^*-algebra A) extends (uniquely) to a continuous module homomorphism $(aA)^- \to (bA)^-$ if and only if there exists an integer K such that $b^*b \leqslant Ka^*a$.

In order to characterize completely the isomorphism clases of closed right ideals by means of the homomorphism Δ described in III.1, we are required to make additional assumptions on A (already assumed to be unital and stably (HP):

(1) The C^*-algebra A has the cancellation property for finitely generated projective modules, i.e., for V, W_1, W_2 such modules,

$$V \oplus W_1 \simeq V \oplus W_2 \qquad \text{implies} \quad W_1 \simeq W_2.$$

(2) The partially ordered abelian group $K_0(A)$ is unperforated, that is, if x is an element of $K_0(A)$ such that $nx \geqslant 0$ for some positive integer n, then $x \geqslant 0$.

Of course, AF algebras satisfy both properties, as well as all of the previous ones.

LEMMA III.2.2 [9; 1.4]. *Let (G, u) be an unperforated partially ordered abelian group with order unit u, and suppose x belongs to G. If $f(x) > 0$ for all states f of (G, u), then x belongs to G^+. Thus if B is a stably finite unital C^*-algebra having $K_0(B)$ unperforated, then for any x in $K_0(B)$ with $\hat{x} \gg 0$, $x \in K_0(B)^+$.*

PROPOSITION III.2.3. *Let B be a unital C^*-algebra with the cancellation property for finitely generated projective modules. Then B is stably finite. If p is a projection in $M_n B$, and q a projection in $M_\infty B$ with $[q] \leqslant [p]$ in $K_0(B)$, then q is equivalent to a subprojection of p; if additionally, q lies in $M_n B$, the equivalence may be unitarily implemented within $M_n B$.*

Proof. This is completely routine. ∎

In the presence of (HP), the cancellation property is equivalent to unitary 1-stable range (definition below).

PROPOSITION III.2.4. *Let B be a unital C^*-algebra with* (HP). *The following are equivalent*:

(1) *B has unitary 1-stable range, i.e., if $aB + bB = B$ (equivalently, $aa^* + bb^*$ is invertible), then there is a unitary u in B so that $a + bu$ is invertible.*

(2) *The invertible elements of B are dense in B.*

(3) *B has the cancellation property for finitely generated projection modules.*

Proof. That $(1) \Leftrightarrow (2)$ is proved in [21], and $(1) \Rightarrow (3)$ in [11; Corollary 1, p. 201].

$(3) \Rightarrow (2)$. Let b be an element of B, and choose $\varepsilon > 0$. Represent B on a Hilbert space, and let $b = u\,|b|$ be the polar decomposition. Let p be a projection in $(b^*Bb)^-$ such that $\|p\,|b| - |b|\| < \varepsilon$. Then $\||b|\,p - |b|\| < \varepsilon$, so $\|bp - b\| = \|u\,|b|\,p - u\,|b|\| < \varepsilon$. Then up belongs to B, so $q = upu^*$ also is in B. Further,

$$\|qb - b\| = \|upu^*u\,|b| - u\,|b|\| = \|up\,|b| - u\,|b|\| < \varepsilon,$$

so $\|qbp - b\| < 2\varepsilon$. Setting $c = qbp$, we see that c^*c is invertible in pBp, and cc^* is invertible in qBq. In particular, $p \sim q$, so by cancellation $1 - p \sim 1 - q$. Find v so that $v^*v = 1 - p$, and $vv^* = 1 - q$. Then $d = c + \varepsilon v$ is invertible, and $\|d - b\| < 3\varepsilon$. ∎

Remarks III.2.5. If B is stably finite and unital, it is not generally true that $K_0(B)$ is unperforated—for example, $B = C(\mathbb{RP}^2)$ [16; IV.6.47]. This example can also be used to obtain stably finite unital C^*-algebras C, D such that $C \not\simeq D$, but $M_2C \simeq M_2D$ (c.f. [20], [5; 1.11]).

The cancellation property does not hold for commutative C^*-algebras generally —non-equivalent vector bundles can be stably equivalent. It is not known whether such phenomena can occur for simple C^*-algebras, or in the presence of (HP). Although evidence suggests that K_0^* is better behaved than K_0, torsion or perforation may well occur here too.

In the (K_0) torsion-free situation, there is a partial result available on unperforation.

PROPOSITION III.2.6. *Let G be a partially ordered abelian group with the property that every quotient by an order-ideal is torsion-free. Let x be an element of G; if $nx \geqslant 0$ for some positive integer n, then $mx \geqslant 0$ for all sufficiently large m.*

Proof. Let n be a fixed integer for which $y = nx \geqslant 0$, and let H be the order-ideal (=hereditary subgroup) generated by y. Then x lies in H (as the quotient group G/H is torsion-free), and y is an order unit for H; thus there exists an integer N such that $Ny \geqslant kx$ for $k = 1, 2,..., n-1$. Thus $Nnx \geqslant kx$ or $(Nn-k)x \geqslant 0$ for $k = 1, 2,..., n-1$; if $m \geqslant Nn$, then mx is a sum of one of these and a mulitple of nx. ∎

We can now begin the classification of countably (hence singly) generated closed right ideals of A.

PROPOSITION III.2.7. *Let A be a unital C^*-algebra having* (HP) *and cancellation. For elements a, b in A, there is an isometric module homomorphism of $(aA)^-$ into $(bA)^-$ if and only if*

> *for some (or any) approximate identities (p_n), (q_n) for $(aAa^*)^-$, $(bAb^*)^-$, respectively, for every n there exists an m such that $p_n \lesssim q_m$.*

There is a continuous module isomorphism of $(aA)^-$ onto $(bA)^-$ if and only if

> *for each n, there is an m so that both $p_n \lesssim q_m$ and $q \lesssim p_m$.*

Proof. If there is an embedding of $(aA)^-$ into $(bA)^-$ (as A-modules), then the proof of III.2.1 yields an embedding φ of $(aAa^*)^-$ into $(bAb^*)^-$. If $v_n = \lim_k u_k p$, then $v_n \in A$, $v_n^* v_n = p_n$, and $v_n v_n^* = \varphi(p_n) \in (bAb^*)^-$. So if m is sufficiently large, $\varphi(p_n)$ is equivalent to a subprojection of q_m. If $(aA)^- \simeq (bA)^-$, the same argument applies to the inverse map. Conversely, suppose each p_n is subisomorphic to some q_m. By passing to a subsequence, we may assume $m=n$ for each n. Let w_1 be a partial isometry with $w_1^*w_1=p_1$, and $w_1w_1^*=r_1\leqslant q_1$. By cancellation, $p_2-p_1\lesssim q_2-r_1$; let w_2 be a partial isometry with $w_2^*w_2=p_2-p_1$ and $w_2w_2^*=r_2\leqslant q_2-r_1$. We may continue inductively, to obtain w_n so that $w_n^*w_n=p_n-p_{n-1}$ and $w_nw_n^*=r_n\leqslant q_n-r_1-\cdots-r_{n-1}$. The w_n (or rather left multiplication by the w_n) define isometric module isomorphisms from $(p_n-p_{n-1})A$ onto r_nA, and hence from $\bigcup p_nA=\oplus(p_n-p_{n-1})A$ to $\oplus r_nA\subseteq(bA)^-$. This extends isometrically to $(\bigcup p_nA)^-=(aA)^-$. If each q_n is subordinate (with respect to $\lesssim$) to a p_m, by relabelling we may assume $p_1\lesssim q_1\lesssim p_2\lesssim q_2\lesssim$ ..., and now the standard interweaving argument can be used to build an isomorphism (onto) from $(\bigcup p_nA)^-$ to $(\bigcup q_nA)^-$—namely, $w_1=v_1$ as above, w_2^* a partial isometry from q_2-r_1 to a subprojection s_2 of p_2-p_1, w_3 from $p_3-s_2-p_1$ to a subprojection of q_3-q_2, etc., and two inverse maps $(\bigcup p_nA)\rightleftarrows(\bigcup q_nA)$ are simultaneously built up. ∎

COROLLARY III.2.8. *Suppose (in addition to the hypotheses of* III.2.7) *that $K_0(A)$ is unperforated. If $(aAa^*)^-$ and $(bAb^*)^-$ have approximate identities (p_n) and (q_n) with $A(p_n-p_{n-1})\gg 0$ and $A(q_m-q_{m-1})\gg 0$ for all m, n, then $(aA)^-$ is isometrically isomorphic to a submodule of $(bA)^-$ if and only if*

$$A(a)\leqslant A(b);$$

and $(aA)^-\simeq(bA)^-$ if and only if

$$A(a)=A(b).$$

Proof. We have that $\hat{p}_n\ll\sup\hat{q}_m=\hat{b}$, so $\hat{p}_n\ll\hat{q}_m$ for sufficiently large m (by compactness of the state space). Thus $p_n\lesssim q_m$ by III.2.2 and III.2.3. ∎

This result has an application when A is simple. We first require a description of closed finitely generated right ideals.

LEMMA III.2.9. *Let B be a C^*-algebra, and a an element of B. The following are equivalent*:

(1) *The right ideal aB is closed*;

(2) *there exists a projection p in B such that $aB=pB$*;

(3) *the subalgebra aBa^* is closed*;

(4) *there exists a projection p in B so that $aBa^*=pBp$.*

Proof. The implications $(2)\Rightarrow(1)$, $(2)\Leftrightarrow(4)$, and $(4)\Rightarrow(3)$ are trivial.

$(1)\Rightarrow(2)$. If aB is closed, then $(aa^*)=ab$ for some b in B; thus $(aa^*)^{1/2}=abb^*a^*\leqslant\|b\|^2\,aa^*$. By functional calculus, for sufficiently small

$\varepsilon > 0$, $f_\varepsilon(aa^*)$ is a projection p which is a unit for aa^*. Now p belongs to aB, and $pa = a$, so a lies in pB. The proof of (3) $\Rightarrow$ (4) is similar. ∎

COROLLARY III.2.10. *Let B be a C^*-algebra, I a closed right ideal of B. Then I is finitely generated (as a right ideal, not necessarily as a closed right ideal) if and only if $I = pB$ for some projection p in B.*

Proof. Write $I = \sum_1^n s_i B$. In $M_n B$, consider the right ideal generated by

$$s = \begin{bmatrix} s_1 s_2 \cdots s_n \\ \text{O} \end{bmatrix}.$$

A simple computation shows that $sM_nB = e_{11}M_nI$, so sM_nB is closed, and thus by III.2.9, there is a projection p in sM_nB such that $pM_nB = sM_nB$. But p has nonzero elements only in the first row, and since $p = p^*$, its only nonzero entry must occur in the (1, 1) position, and that entry q must be a projection. It is clear that $qB = I$ follows from $sM_nB = pM_nB$. ∎

COROLLARY III.2.11. *Let A be a simple unital C^*-algebra with* (HP) *and cancellation, and so that $K_0(A)$ is unperforated. Let a, b be elements of A such that neither aA nor bA is closed. Then $(aA)^-$ is isometrically[2] module-isomorphic to a submodule of $(bA)^-$ if and only if $\hat{a} \leqslant \hat{b}$; and $(aA)^-$ is (isometrically) module-isomorphic to $(bA)^-$ if and only if $\hat{a} = \hat{b}$.*

Proof. By III.2.9, $(aAa^*)^-$ and $(bAb^*)^-$ possess strictly increasing approximate identities, whose successive differences must be order units in $K_0(A)$ (as A is simple). Now apply III.2.8. ∎

LEMMA III.2.12 (Folklore). *If B is a C^*-algebra, and p,q, are projections of B, then $pB \lesssim qB$ if and only if p and q are linked via a partial isometry.*

Via III.2.12, III.2.11 yields a complete classification of all isomorphism classes of countably generated closed right ideals (here "countably generated closed," means, the closure of a countably generated right ideal) in a simple C^*-algebra with all the hypotheses in III.2.11. Of course, if A is separable, every closed right ideal is countably generated, and in general, every countably generated closed right ideal is singly generated.

We have that, if aA is not closed,

$(aA)^-$ is isometrically module-embeddable in $(bA)^-$ if and only if $\hat{a} \leqslant \hat{b}$;

[2] "Isometrically" can be replaced by a continuous module isomorphism with closed image.

$(aA)^-$ is (isometrically) module-isomorphic to $(bA)^-$ if and only if $\hat{a}=\hat{b}$, and bA is not closed.

By going up to matrix rings, we can show by the same methods, that if aA is closed while bA is not, then $\hat{a}\leqslant\hat{b}$ implies

$$aA \perp (bA)^- \lesssim (bA)^- \perp (bA)^-,$$

where $\perp$ indicates orthogonal direct sum—this may be realized concretely as the row space $\{(x, y) \mid x \in aA, y \in (bA)^-\}$—and $\hat{a}=\hat{b}$ implies $aA \perp (bA)^- \simeq (bA)^- \perp (bA)^-$.

One can also ask, given two closed right ideals $(aA)^-$, $(bA)^-$, is there a surjective A-module homomorphism f: $(aA)^- \to (bA)^-$? It can be shown that since $(bA)^- = (\bigoplus p_i A)^-$ with $\{p_i\}$ an orthogonal set of projections, any such f will split continuously, i.e.,

$$\text{there exists } g\colon (ba)^- \to (aA)^- \text{ so that } fgf=f \text{ and } gfg=g.$$

The maps f and g can be straightened out to yield f', g' with g' an isometry onto its image. It follows that there exists a closed submodule $(cA)^-$ of $(aA)^-$ so that $(aA)^- = (cA)^- + g'((bA)^-)$ and $(cA)^- \cap g'((bA)^-) = \{0\}$. We thus deduce $\hat{a}=\hat{c}+\hat{b}$; hence the formal difference $\hat{a}-\hat{b}$ must be lower semicontinuous; if aA, bA are both not closed, this is sufficient as well.

For general unital C^*-algebras A, K_0^* is still intimately related to the classifications of closed right ideals. For example, it follows (from the definitions) that

(a) If there is a continuous A-module homomorphism $(aA)^- \to (bA)^-$ with dense image, then $D(a) \geqslant D(b)$ for all dimension functions D of A.

(b) If $(aA)^-$ is embedded continuously as an A-submodule of $(bA)^-$ then $D(a) \leqslant D(b)$.

(c) If $(aA)^- = (bA)^- + (cA)^-$ (in particular, the right side must be assumed closed) and $(bA)^- \cap (cA)^- = (0)$, then $D(a) = D(b) + D(c)$ for all lower semicontinuous dimension functions D.

Returning to our examples satisfying (HP), etc., let us calculate the isomorphism types of closed non-principal right ideals in a UHF algebra A. Then $K_0(A)$ is a rank one dense subgroup of $\mathbb{R}$, and since the isomorphism classes are in bijection with the pointwise suprema, we see that $(0,1]$ is a complete listing. Furthermore, if M is a maximal right ideal, then $\{\hat{p} \mid p=p^2=p^* \in M\}=1$, since by Powers' theorem $\mathrm{Aut}(A)$ acts transitively on the maximal right ideals. Hence all maximal right ideals are module isomorphic (this latter does not follow directly from the transitivity alone, since $*$-algebra automorphisms are generally not module homomorphisms).

III.3. *Characterization of K_0^**

We now apply the results of the previous sections to obtain the promised description of K_0^*. There is one additional assumption required on A; however, this is vacuous in many cases:

DEFINITION III.3.1. If B is a unital (stably finite) C^*-algebra with stable (HP), then B is *K_0^*-continuous* if for every $\varepsilon > 0$, there is a projection p in B so that $0 \ll \hat{p} \leqslant \varepsilon$.

The term K_0^*-continuous is an analogy with the description of finite W^*-algebras. It follows from I.1.14 that an infinite dimensional simple C^*-algebra (with (HP) etc.) is always K_0^*-continuous—more is true (III.3.4).

THEOREM III.3.2. *Let A be a unital stably finite K_0^*-continuous C^*-algebra with stable* (HP) *and cancellation, and having $K_0(A)$ unperforated. Then the function Λ from $K_0^*(A)$ into the set of bounded affine functions on $QT(A)$ is an order-isomorphism onto its image.*

Proof. It suffices to show that if $\hat{a} \leqslant \hat{b}$, then $[a] \leqslant [b]$ in $K_0^*(A)$—so we must find c in $M_\infty A$ orthogonal to both a and b, such that $a + c \lesssim b + c$. It is clear from Proposition III.2.1 that if $(dB)^-$ is isometrically isomorphic to a submodule of $(eB)^-$, then $d \lesssim e$; so it suffices to find c in $M_k A$ (for sufficiently large k) orthogonal to a and b, with $((a+c)M_k A)^-$ isometrically module-isomorphic to a submodule of $((b+c)M_k A)^-$. The hypotheses of cancellation and K_0^*-continuity imply that A has a sequence of orthogonal full projections, and thus contains a sequence of increasing projections r_n with $\Lambda(r_m - r_{m-1}) \gg 0$. Set $k = n+1$, and $c = e_{(n+1),(n+1)} \otimes (\sum_1^\infty 2^{-m} r_m)$. If (p'_m) and (q'_m) are approximate identities for $(aM_n Aa^*)^-$ and $(bM_n Ab^*)^-$, then $(p_m = p'_m + r_m)$ and $(q_m = q'_m + r_m)$ are approximate identities for $((a+c)M_k A(a+c)^*)^-$ and $((b+c)M_k A(b+c)^*)^-$ respectively; further, $\Lambda(p_m - p_{m-1}) \gg 0$ and $\Lambda(q_m - q_m - 1) \gg 0$, for all m. Since $\Lambda(a+c) \leqslant \Lambda(b+c)$, the result follows from III.2.8. ∎

It is highly likely that the above result holds without the hypothesis of K_0^*-continuity.

To complete the description of $K_0^*(A)$, the range of Λ must be calculated. The image of the positive cone (by III.3.2, $K_0^*(A)$ is now known to be a partially ordered, rather than just a pre-ordered group), $\Lambda(K_0^*(A)^+)$, consists of all countable increasing pointwise suprema of functions of the form $\hat{p}$, for p a projection in $M_\infty A$; so each such function is lower semicontinuous. The function Λ is also pointwise onto—if $\tau \in QT(A)$ and $\lambda > 0$, there is an a in $M_\infty A$ such that $\hat{a}(\tau) = \lambda$.

COROLLARY III.3.3. *Let A be a C^*-algebra satisfying the hypotheses of*

III.3.2, *and suppose that* $QT(A)$ *is metrizable and* $\{\hat{p} \mid p = p^2 = p^* \in M_\infty A\}$ *is dense in* $\mathrm{Aff}(QT(A))^+$. *Then* $K_0^*(A)$ *is order-isomorphic to the group of differences of bounded lower semicontinuous affine functions on* $QT(A)$, *equipped with the pointwise ordering.*

Proof. The image contains $\mathrm{Aff}(QT(A))$, and it contains countable increasing pointwise suprema of elements of $\mathrm{Aff}(QT(A))$ that are bounded above; since $K_0^*(A)$ is generated by its positive cone, the result follows. ∎

LEMMA III.3.4. *Let* A *be a unital stably finite* C^**-algebra having stable* (HP), *cancellation, and* $K_0(A)$ *unperforated. Suppose* A *admits no finite dimensional representations. Then*

$$\{\hat{p} \mid p = p^2 = p^* \in M_\infty A\} \text{ is dense in } \mathrm{Aff}(QT(A))^+.$$

(*In particular,* $K_0^*(A)$ *is continuous.*)

Proof. We first show that $K_0(A)$ is a dimension group. This is done by establishing Riesz decomposition along the lines of [25; II.10.3]; the proof can be adapted almost verbatim, because the cancellation and unperforated properties have been hypothesized. (If A is already AF, this is of course well-known.)

Next, any irreducible finite dimensional representation would induce a trace τ, which in turn yields the map $K_0(\tau)\colon K_0(A) \to \mathbb{R}$, a state of $K_0(A)$, with discrete range. Conversely, assume that there is a state $t\colon K_0(A) \to \mathbb{R}$ with discrete range. We may assume that t is pure, as in the proof of [24; Lemma 4.4]. This lifts back to a dimension function on A (III.1.5), and thus a quasitrace τ is obtained, such that $t = K_0(\tau)$. Since $K_0(\tau)(K_0(A))$ is discrete, $A/\mathrm{Ker}\,\tau$, must be finite dimensional (and simple).

Under our hypotheses, therefore, $K_0(A)$ admits no state with cyclic image. By [24; Corollary 4.9], the image of $K_0(A)^+$ is dense in $\mathrm{Aff}(S(K_0(A))^+$, where $S(K_0(A)$ is the state space of $K_0(A)$—this latter is naturally identifiable with $QT(A)$, and the result follows. ∎

COROLLARY III.3.5. *Let* A *be a unital* AF C^**-algebra with no finite dimensional representations. Then the conclusion of* III.3.3 *holds* (*and* $QT(A) = \mathrm{T}(A)$). *If* A *has exactly* n *pure traces, then* $K_0^*(A) \simeq \mathbb{R}^n$ *with the usual ordering.*

If A is not K_0^*-continuous, the image of A is harder to describe; the function A will not be pointwise onto. The interested reader should consult [24; 4.8] for a description of the norm closure of $K_0(A)$.

Acknowledgment

Both authors would like to acknowledge valuable conversations and correspondence with Joachim Cuntz on the contents of this paper.

References

1. S. Berberian, "Baer *-Rings," Springer-Verlag, Heidelberg/New York, 1972.
2. B. Blackadar and J. Cuntz, The structure of stable algebraically simple C^*-algebras, *Amer. J. Math.*, in press.
3. J. Cuntz, The structure of multiplication and addition in simple C^* algebras, *Math. Scand.* **40** (1977), 215–233.
4. J. Cuntz, Dimension functions on simple C^*-algebras, *Math. Ann.* **233** (1978), 145–153.
5. J. Cuntz, K-Theory for certain C^*-algebras, *Ann. of Math.* **113** (1981), 181–197.
6. J. Cuntz and G. Pedersen, Equivalence and traces on C^*-algebras, *J. Funct. Anal.* **33** (1979), 135–164.
7. J. Dixmier, Les C^*-algèbres et leurs représentations, "Gauthiers–Villars, Paris, 1969.
8. J. Dixmier and A. Douady, Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. France* **91** (1963), 227–284.
9. E. G. Effros, D. E. Handelman, and C.-L. Shen, Dimension groups and their affine representations, *Amer. J. Math.* **102** (1980), 385–407.
10. G. Elliott, On the classification of inductive limits of sequences of semisimple finite dimensional algebras, *J. Algebra* **38** (1976), 29–44.
11. L. Fuchs, On a substitution property for modules, *Monatsh. Math.* **75** (1971), 198–204.
12. K. R. Goodearl, "Von Neumann Regular Rings," Pitman, London, 1979.
13. K. Goodearl and D. Handelman, Rank functions and K_0 of regular rings, *J. Pure Appl. Algebra* **7** (1976), 195–216.
14. D. Handelman, Homomorphisms of C^* algebras to finite AW^* algebras, *Michigan Math. J.* **28** (1981), 229–240.
15. R. Kadison and G. Pedersen, Equivalence in operator algebras, *Math. Scand.* **27** (1979), 205–222.
16. M. Karoubi, "K-Theory," Springer-Verlag, Heidelberg/New York, 1978.
17. F. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936), 116–229.
18. G. Pedersen, "C^*-Algebras and their Automorphism Groups." Academic Press, New York, 1979.
19. G. Pedersen, The linear span of projections in simple C^*-algebras, *J. Operator Theory* **4** (1980), 289–296.
20. J. Plastiras, C^*-algebras isomorphic after tensoring. *Proc. Amer. Math. Soc.* **66** (1977), 276–278.
21. A. G. Robertson, Stable range in C^*-algebras, to appear.
22. E. Thoma, Über unitäre Darstellungen abzählbar, diskreter Gruppen, *Math. Ann.* **153** (1964), 111–138.
23. D. Handelman, Rings with involution as partially ordered abelian groups, *Rocky Mountain J.*, to appear.
24. K. R. Goodearl and D. E. Handelman, Metric completions of partially ordered abelian groups, *Indiana J. Math.* **29** (1980), 861–895.
25. K. R. Goodearl, D. E. Handelman, and J. W. Lawrence, Affine representations of Grothendieck groups and applications to Rickart C^*-algebras and $\aleph_0$-continuous regular rings, *Mem. Amer. Math. Soc.* **234** (1980).