Chapter 4

Bott periodicity

4.1 What we are going to prove

The aim of this section is to show the following formula:

$$\mathrm{KK}(A, S^2B) \cong \mathrm{KK}(A, B) \cong \mathrm{KK}(S^2A, B)$$

for all graded σ-unital C*-algebras A and B (with A separable). In fact, we are going to show that for all $n \in \mathbb{N}$ the graded C*-algebra $\mathbb{C}_n$ and the trivially graded C*-algebra $\mathcal{C}_0(\mathbb{R}^n)$ are KK-equivalent; this implies that $\mathbb{C}_2$ and S^2 are KK-equivalent and hence $\mathbb{C}_2 \hat{\otimes} A$ and S^2A are KK-equivalent (and likewise for B). Hence the formula follows from the corresponding formula for $\mathbb{C}_2$.

First note that it suffices to consider the case $n = 1$ because if x is a KK-equivalence from S to $\mathbb{C}_1$ then $x \hat{\otimes} x$ is a KK-equivalence from $S^2 \cong S \otimes S$ to $\mathbb{C}_2 \cong \mathbb{C}_1 \hat{\otimes} \mathbb{C}_1$, etc.

Note that it suffices to find an equivalence β between the algebras $\mathbb{C}$ and $\mathbb{C}_1 \hat{\otimes} \mathcal{C}_0(\mathbb{R}) \cong \mathcal{C}_0(\mathbb{R}, \mathbb{C}_1)$ because in this case

$$1_{\mathbb{C}_1} \otimes \beta \in \mathrm{KK}(\mathbb{C}_1 \hat{\otimes} \mathbb{C}, \mathbb{C}_1 \hat{\otimes} \mathbb{C}_1 \hat{\otimes} \mathcal{C}_0(\mathbb{R}))$$

is a KK-equivalence between $\mathbb{C}_1 \hat{\otimes} \mathbb{C} \cong \mathbb{C}_1$ and $\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1 \hat{\otimes} \mathcal{C}_0(\mathbb{R}) \cong \mathrm{M}_2(\mathbb{C}) \hat{\otimes} \mathcal{C}_0(\mathbb{R})$ where we take the standard even grading on $\mathrm{M}_2(\mathbb{C})$; the latter algebra is KK-equivalent to $\mathcal{C}_0(\mathbb{R})$ because $\mathrm{M}_2(\mathbb{C})$ is gradedly Morita equivalent to $\mathbb{C}$.

So we are looking for elements $\alpha \in \mathrm{KK}(\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1), \mathbb{C})$ and $\beta \in \mathrm{KK}(\mathbb{C}, \mathcal{C}_0(\mathbb{R}, \mathbb{C}_1))$ such that $\alpha \hat{\otimes}_{\mathbb{C}} \beta = 1_{\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1)}$ and $\beta \hat{\otimes}_{\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1)} \alpha = 1_{\mathbb{C}}$.

4.2 The elements α and β

Let us describe the element $\alpha \in \mathrm{KK}(\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1), \mathbb{C})$ first. Observe that $\mathbb{R}$ is homeomorphic to the open intervall $I = (-\pi, \pi)$, so we can replace $\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1)$ with $A := \mathcal{C}_0(I, \mathbb{C}_1)$. The element α is now given by the triple (H, ϕ, F) where H is the Hilbert space $\mathrm{L}^2(I) \oplus \mathrm{L}^2(I) \cong \mathrm{L}^2(I) \otimes \Lambda\, \mathbb{C}$ (if we equip $\Lambda\, \mathbb{C}$ with the canonical inner product making it a complex Hilbert space). The action

ϕ of $A = \mathcal{C}_0(I, \mathbb{C}_1)$ on $H = \mathrm{L}^2(I, \Lambda\,\mathbb{C})$ is given by a pointwise Clifford action: We just have to specify the action of $\mathbb{C}_1$ on $\Lambda\,\mathbb{C}$; the generator $1 = (1,0) \in \mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}_1$ acts as identity on $\mathbb{C} \oplus \mathbb{C} \cong \Lambda\,\mathbb{C}$ and the generator $(0,1) \in \mathbb{C}_1$ acts as the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

.

Now we have to specify the operator F. Let d be the operator on $\mathcal{C}_0^\infty(I, \Lambda\,\mathbb{C})$ which sends a function $t \mapsto (f(t), g(t))$ to $t \mapsto (0, \frac{d}{dt} f(t))$ (the de Rahm derivative). Let d^* be it's adjoint and $D := d + d^*$. We would like to define $F := D$, but D is an unbounded operator on $\mathrm{L}^2(I, \Lambda\,\mathbb{C})$, so we have to make it bounded.

The reason why we work on I and not on $\mathbb{R}$ is that we now can use Fourier series instead of Fourier transforms on $\mathbb{R}$. We can identify $\mathrm{L}^2(I)$ with $\mathrm{L}^2(S^1)$ and by Fourier analysis with $\ell^2(\mathbb{Z})$. Hence $\mathrm{L}^2(I, \Lambda\,\mathbb{C})$ can be identified with $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$. The operator D is now given by the matrix

$$\begin{pmatrix} 0 & in \\ -in & 0 \end{pmatrix}$$

where we write in for the operator which maps the basis vector e_n to ine_n.

We replace this operator by the matrix

$$F := \begin{pmatrix} 0 & i\,\mathrm{sign}(n) \\ -i\,\mathrm{sign}(n) & 0 \end{pmatrix}$$

where $\tilde{d} := i\,\mathrm{sign}(n)$ is the operator which maps e_n to $-ie_n$ if $n < 0$, 0 if $n = 0$ and ie_n if $n > 0$. Note that we have

$$1 - F^2 = 1 - \begin{pmatrix} i(-i)\,\mathrm{sign}(n)^2 & 0 \\ 0 & i(-i)\,\mathrm{sign}(n)^2 \end{pmatrix} = \begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix}$$

where p_0 is the orthogonal projection given by e_0. Hence $1 - F^2$ is compact.

We have to show that the graded commutator $[f, F]$ is compact where f denotes the multiplication operator given by a function f in $A = \mathcal{C}_0(I, \mathbb{C}_1)$. We can actually show this for all functions $f \in \mathcal{C}(S^1, \mathbb{C}_1)$ if we identify I with $S^1 \setminus \{-1\}$ in the obvious way.

First we consider the case that $f(t) = \sigma(t) = (0,1)$ for all $t \in S^1$. It is straighforward to see that $\sigma F = -F\sigma$, so the graded (!) commutator $[\sigma, F] = \sigma F + F\sigma$ vanishes. Because every odd element of $\mathcal{C}(S^1, \mathbb{C}_1)$ can be written as a product of an even element with σ it hence suffices to consider functions $(f_0, 0)$ of the form $f(t) = (f_0(t), 0)$. Because the map which sends f_0 to $[(f_0, 0), F]$ is continuous and linear it suffices to consider functions f_0 of the form $f_0(t) = e^{ikt}$ with $k \in \mathbb{Z}$.

Mulitplication by e^{ikt} on $\mathrm{L}^2(S^1)$ corresponds to the shift operator $s_k \colon e_n \mapsto e_{n+k}$ on $\ell^2(\mathbb{Z})$ after taking the Fourier transform. Hence the commutator $[s_k, F]$ is a finite rank operator and therefore compact.

We have shown that (H, ϕ, F) is in $\mathbb{E}(A, \mathbb{C})$ and therefore defines an element $\alpha \in \mathrm{KK}(A, \mathbb{C})$.

Now we come to the element $\beta \in \mathrm{KK}(\mathbb{C}, A)$. It is given by a triple $(A, 1, v\cdot) \in \mathbb{E}(\mathbb{C}, A)$:

We consider A as a Hilbert module over itself and let $\mathbb{C}$ act on it by scalar multiplication (so $1 \in \mathbb{C}$ acts as identity on A). The operator on A is given by (Clifford) multiplication by an odd element v of $\mathcal{C}_b(I, \mathbb{C}_1)$ (i.e., a bounded multiplier). To this end, let v be the function $t \mapsto (0, \sin(t/2))$; note that $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$, and actually, we could have chosen any continuous function on $[-\pi/2, \pi/2]$ with these properties instead of $\sin$. The $\sin$ function will soon turn out to be a good choice, however.

Pointwise Clifford multiplication by v defines an odd linear continuous operator on A. Note that $v^2(t) = (\sin^2(t/2), 0)$ so $(1 - v^2)(t) = (\cos^2(t/2), 0)$ which is an element of A. So multiplication by $1 - v^2$ is compact. The commutator $[z, v\cdot]$ vanishes for all $z \in \mathbb{C}$. Hence $(A, 1, v\cdot)$ is in $\mathbb{E}(\mathbb{C}, A)$ and defines an element $\beta \in \mathrm{KK}(\mathbb{C}, A)$.

4.3 The product $\beta\hat{\otimes}_{\mathcal{C}_0(\mathbb{R},\mathbb{C}_1)}\alpha = 1_{\mathbb{C}}$

We now use a Lemma from Blackadars book (Lemma 18.10.1) to calculate the Kasparov product of β and α. We use it in the following form (without readjusting the notation):

Lemma. *Let A, B, C be graded σ-compact C*-algebras, let A be separable, and let $x_1 := (E_1, \phi_1, T_1) \in \mathbb{E}(A, B)$ and let $x_2 := (E_2, \phi_2, T_2) \in \mathbb{E}(B, C)$ such that $T_1^* = T_1$ and $\|T_1\| \leq 1$. Let G be any T_2-connection of degree 1 on $E_{12} := E_1\hat{\otimes}_B E_2$. Define $\phi_{12}(a) := \phi_1(a)\hat{\otimes}1$ for all $a \in A$ and*

$$T_{12} := T_1\hat{\otimes}1 + [(1 - T_1^2)^{1/2}\hat{\otimes}1]G.$$

If $[T_{12}, \phi_{12}(a)] \in \mathrm{K}_C(E_{12})$ for all $a \in A$, then $(E_{12}, \phi_{12}, T_{12})$ is in $\mathbb{E}(A, C)$ and represents the Kasparov product of $[x_1]$ and $[x_2]$.

We use this lemma for $(\mathbb{C}, A, \mathbb{C})$ instead of (A, B, C) and $x_1 = (H, \phi, F)$ and $x_2 = (A, 1, v\cdot)$. First we determine E_{12}, i.e. $A\hat{\otimes}_A H$: Because A acts non-degenerately on H (i.e. $AH = H$), we can (and will) identify $A\hat{\otimes}_A H$ and H. If we regard an odd operator G on H as an operator also on $A\hat{\otimes}_A H$, then G is an F-connection if and only if $aG - (-1)^{\partial a}Fa$ and $aF - (-1)^{\partial a}Ga$ are compact for all $a \in A$. So it is easy to see that F is an F-connection in this sense because we have already checked that the graded commutator $[a, F]$ is always compact. So the lemma applies and we obtain that

$$\tilde{F} := (v\cdot)\hat{\otimes}1 + ((1 - v\cdot)^{1/2}\hat{\otimes}1)F$$

is an operator on $H = \mathrm{L}^2(I, \Lambda\,\mathbb{C})$ such that $(H, 1, \tilde{F}) \in \mathbb{E}(\mathbb{C}, \mathbb{C})$ is homotopic to a Kasparov product of $(A, 1, v\cdot)$ and (H, ϕ, F). The operator $(v\cdot)\hat{\otimes}1$ can be identified with the canonical action of the odd element $v = (t \mapsto \sin(t/2)\sigma) \in A = \mathcal{C}_0(I, \mathbb{C}_1)$ on $H = \mathrm{L}^2(I, \Lambda\,\mathbb{C})$ (where $\sigma = (0, 1) \in \mathbb{C}_1$)). And $((1 - v\cdot)^{1/2}\hat{\otimes}1)$ can be identified with the canonical action of the even element $t \mapsto \cos(t/2)1$ of A on H. So we have

$$\tilde{F} := \sin(t/2)\sigma + \cos(t/2)F.$$

We hence have to calculate the Fredholm index of the operator

$$T := \sin(t/2) + \cos(t/2)\tilde{d}$$

from $\mathrm{L}^2(I)$ to itself, where $\tilde{d}$ is the operator which, in the Fourier picture, sends e_n to $i\,\mathrm{sign}(n)e_n$. To make our calculations more pleasant we compute the (unchanged) index of the operator

$$S := 2ie^{it/2}T = (e^{it}-1) + i(e^{it}+1)\tilde{d}.$$

For all $n \in \mathbb{Z}$ we calculate (in the Fourier picture):

$$S(e_n) = \begin{cases} -2e_n, & n > 0, \\ e_1 - e_0, & n = 0, \\ 2e_{n+1}, & n < 0. \end{cases}$$

In other words, if $x \in \ell^2(\mathbb{Z})$ then

$$(Sx)_n = \begin{cases} -2x_n, & n > 1, \\ x_0 - 2x_1, & n = 1 \\ 2x_{-1} - e_0, & n = 0, \\ 2e_{n+1}, & n < 0. \end{cases}$$

From this it follows that the kernel of S is the span of $e_1 + e_{-1} + 2e_0$, and on the other hand, we have

$$e_n = \begin{cases} S(-1/2e_n), & n > 1, \\ S(e_1 + 1/2e_{-1}), & n = 1 \\ S(1/2e_{-1}), & n = 0, \\ S(1/2e_{n-1}), & n < 0, \end{cases}$$

so S is surjective (because we can define a split in an obvious way). So the index of S (and thus of T) is 1. So $\beta\hat{\otimes}_A\alpha = 1 \in \mathrm{KK}(\mathbb{C},\mathbb{C})$.

4.4 The product $\alpha\hat{\otimes}_{\mathbb{C}}\beta = 1_{\mathcal{C}_0(\mathbb{R},\mathbb{C}_1)}$

Instead of calculating the product on the level of cycles, we use the commutativity of the (general) product over $\mathbb{C}$ and a trick which is a variant of Atiyah rotation trick. In the calculations, we suppress tensor products by $\mathbb{C}$ (and hence also the canonical flip homomorphisms between tensor products by $\mathbb{C}$ from the left and from the right). Observe that

$$\alpha\hat{\otimes}_{\mathbb{C}}\beta = \beta\hat{\otimes}_{\mathbb{C}}\alpha = (\beta\hat{\otimes}1_A)\hat{\otimes}_{A\hat{\otimes}A}(1_A\hat{\otimes}\alpha) = (\beta\hat{\otimes}1_A)\hat{\otimes}_{A\hat{\otimes}A}([\Sigma_{A,A}]\hat{\otimes}_{A\hat{\otimes}A}(\alpha\hat{\otimes}1_A))$$

where $\Sigma_{A,A}$ is the automorphism of $A\hat{\otimes}A$ flipping the factors. If we can show that $\Sigma_{A,A}$ is homotopic to an isomorphism of the form $\mathrm{Id}_A\hat{\otimes}\psi$ where ψ is an automorphism of A, then we are done because then

$$\begin{aligned} (\beta\hat{\otimes}1_A)\hat{\otimes}_{A\hat{\otimes}A}[\Sigma_{A,A}]\hat{\otimes}_{A\hat{\otimes}A}(\alpha\hat{\otimes}1_A) &= (\beta\hat{\otimes}1_A)\hat{\otimes}_{A\hat{\otimes}A}(1_A\hat{\otimes}[\psi])\hat{\otimes}_{A\hat{\otimes}A}(\alpha\hat{\otimes}1_A) \\ &= (\beta\hat{\otimes}1_A)\hat{\otimes}_{A\hat{\otimes}A}(\alpha\hat{\otimes}1_A)\hat{\otimes}_{\mathbb{C}\hat{\otimes}A}(1_{\mathbb{C}}\hat{\otimes}[\psi]) \\ &= ((\beta\hat{\otimes}_A\alpha)\hat{\otimes}1_A)\hat{\otimes}_A[\psi] \\ &= (1_{\mathbb{C}}\hat{\otimes}1_A)\hat{\otimes}_A\psi = 1_A\hat{\otimes}_A[\psi] = [\psi]. \end{aligned}$$

This shows that $\alpha\hat{\otimes}_{\mathbb{C}}\beta$ is an automorphism of A while $\beta\hat{\otimes}_A\alpha = 1_{\mathbb{C}}$. So α is a right inverse of β and β also has a left inverse, so α is also a left inverse and $[\psi]$ is the identity in $\mathrm{KK}(A, A)$.

So what is left to show is that $\Sigma_{A,A}$ actually **is** homotopic to some $1_A\hat{\otimes}\psi$. To this end, we first identify $A\hat{\otimes}A$ with $\mathcal{C}_0(\mathbb{R}^2, \mathbb{C}_2)$ (if you like, you can think of this algebra as an algebra of sections in the complex Clifford bundle over $\mathbb{R}^2$).

Now observe that every linear isometry U of $\mathbb{R}^2$ induces a canonical graded $*$-automorphism $\tilde{U}$ of $\mathcal{C}_0(\mathbb{R}^2, \mathbb{C}_2)$: If f is in $\mathcal{C}_0(\mathbb{R}^2, \mathbb{C}_2)$, then $\tilde{U}(f) := \mathrm{Cliff}_{\mathbb{C}}(U) \circ f \circ U^{-1}$ where $\mathrm{Cliff}_{\mathbb{C}}(U)$ is the canonical unital automorphism of $\mathbb{C}_2$ induced by U given by the universal property of the Clifford algebra.

If U is the identity of $\mathbb{R}^2$, then $\mathrm{Cliff}_{\mathbb{C}}(U)$ is the identity on $\mathbb{C}_2$ and $\tilde{U}$ is of course the identity on $A \otimes A$.

On the other hand, if U is the map $(x, y) \mapsto (-y, x)$, then we obtain the following automorphism $\mathrm{Cliff}_{\mathbb{C}}(U)$: Let e_1, e_2 denote the standard basis vectors in $\mathbb{R}^2$ and let e denote the standard basis vector in $\mathbb{R}$. Let Φ denote the canonical unital isomorphism $\mathbb{C}_1\hat{\otimes}\mathbb{C}_1 \cong \mathbb{C}_2$, it sends $e\hat{\otimes}1$ to $e_1 \in \mathbb{C}_2$, $1\hat{\otimes}e$ to $e_2 \in \mathbb{C}_2$ and $e\hat{\otimes}e$ to $e_1e_2 \in \mathbb{C}_2$. Now $\mathrm{Cliff}_{\mathbb{C}}(U)$ sends e_1 to e_2, e_2 to $-e_1$ and hence e_1e_2 to $-e_2e_1 = e_1e_2$. So $\Phi^{-1} \circ \mathrm{Cliff}_{\mathbb{C}} \circ \Phi$ is the same as $\Sigma_{\mathbb{C}_1,\mathbb{C}_1} \circ (1 \otimes \mathrm{Cliff}_{\mathbb{C}}(-\mathrm{Id}_{\mathbb{R}}))$ (note that the graded flip $\Sigma_{\mathbb{C}_1,\mathbb{C}_1}$ sends $e\hat{\otimes}e$ to $-e\hat{\otimes}e$).

Similarly, you calculate that $\tilde{U}$ can be identified with $\Sigma_{A,A} \circ (\mathrm{Id}_A \hat{\otimes}\psi)$ where $\psi := -\tilde{\mathrm{Id}}_{\mathbb{R}}$ is the automorphism of $\mathcal{C}_0(\mathbb{R}, \mathbb{C}_1)$ induced by $-\mathrm{Id}_{\mathbb{R}}$ defined analogously to $\tilde{U}$.

Now observe that the automorphism $\tilde{V}$ of $A\hat{\otimes}A$ depends continuously on the isometry V of $\mathbb{R}^2$. Moreover, the above-mentioned isometry U is homotopic to the identity via a rotation. Hence $\tilde{U}$ is homotopic to the identity. It follows, that $\Sigma_{A,A} \circ (\mathrm{Id}_A \hat{\otimes}\psi)$ is homotopic to the identity, and after multiplying with $\Sigma_{A,A}$ from the left we see that $\mathrm{Id}_A \hat{\otimes}\psi$ is homotopic to $\Sigma_{A,A}$. Hence we are done.