# 1) Hilbert modules & adjoinable operators.

Let $\mathcal{B}$ be a $C^*$-algebra (all $C^*$-algebras are complex).

1.1) **Definition**: A (right) pre-Hilbert module $E$ over $\mathcal{B}$ is a complex vector space $E$ which is at the same time a right $\mathcal{B}$-module (+compatibility) and is equipped with a map

$$\langle \cdot , \cdot\cdot \rangle : E \times E \to \mathcal{B}$$

such that:

1) $\langle \cdot , \cdot\cdot \rangle$ is sesquilinear (linear in the second comp.)
2) $\forall b \in \mathcal{B}\ \forall e,f \in E$: $\langle e, fb\rangle = \langle e,f\rangle b$.
3) $\forall e,f \in E$: $\langle e,f\rangle^* = \langle f,e\rangle \in \mathcal{B}$.
4) $\forall e \in E$: $\left(\langle e,e\rangle = 0 \Leftrightarrow e = 0\right) \wedge \langle e,e\rangle \geq 0$.

Define $\|e\| := \sqrt{\|\langle e,e\rangle\|}$ for all $e \in E$.

[Exercise: $\|e\|$ defines a norm on $E$.]

If $E$ is complete, then we call $E$ a Hilbert $\mathcal{B}$-module.

$E$ is called full if $\mathrm{cl}\langle E,E\rangle = \mathcal{B}$.

1.2) Examples: a) If $\mathcal{B} = \mathbb{C}$, then a Hilbert $\mathcal{B}$-module is the same as a Hilbert space.

b) $\mathcal{B}$ itself is a Hilbert $\mathcal{B}$-module:

module action: $\forall e \in \mathcal{B}, b \in \mathcal{B}$: $e \cdot b := \underset{\text{in } \mathcal{B}}{e \cdot b}$

inner product: $\forall e \in \mathcal{B}, f \in \mathcal{B}$: $\langle e,f\rangle := e^* f$.

c) more generally: $I \subseteq \mathcal{B}$ closed right ideal
$\Rightarrow I_{\mathcal{B}}$ Hilbert $\mathcal{B}$-module

d) $(E_i)_{i \in I}$ family of pre-Hilbert $\mathcal{B}$-modules
$I$ ↑ index set

$\Rightarrow \bigoplus_{i \in I} E_i$ (= direct sum) is a pre-Hilbert $\mathcal{B}$-module.

$$\langle (e_i), (f_i) \rangle := \sum_{i \in I} \langle e_i, f_i \rangle_{E_i}$$

Because the completion of a pre-Hilbertmodule is a Hilbertmodule we can form the completion of $\bigoplus_{i \in I} E_i$, also called $\bigoplus_{i \in I} E_i$.

e) Special case of d): $I = \mathbb{N}$, $E_i = \mathcal{B}$. Define
$\mathbb{H}_{\mathcal{B}} := \bigoplus_{i \in \mathbb{N}} \mathcal{B}$, ~~pre~~-Hilbert $\mathcal{B}$-module.

[Exercise: Let $\ell^2(\mathbb{N}, \mathcal{B})$ be the space of sequences $(b_n)_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} \|b_n\|^2 < \infty$. Show:
$\ell^2(\mathbb{N}, \mathcal{B}) \subseteq \mathbb{H}_{\mathcal{B}}$. Find an example such that
$\mathbb{H}_{\mathcal{B}} \not\subseteq \ell^2(\mathbb{N}, \mathcal{B})$.]

1.3. Lemma: If $E$ is a pre-Hilbert $\mathcal{B}$-module, then:
$$\forall e, f \in E: \quad \|e\| \, \|f\| \geq \|\langle e, f \rangle\|.$$

Pf: If $f \neq 0$: Define $b := \frac{-\langle f, e \rangle}{\|f\|^2}$, use $\langle e + fb, e + fb \rangle \geq 0$. ∎

1.3,5 Lemma: $\overline{E\mathcal{B}} = E$ Pf: Ex.

1.4) Remark: Let $H$ be a Hilbert space and $T \in L(H)$.
Then $T^*$ is the unique operator in $L(H)$ such that
$$\langle x, Ty\rangle = \langle T^*x, y\rangle \quad \text{for all } x, y \in H.$$
Such a $T^*$ always exists and $T \mapsto T^*$ turns $L(H)$ into a $C^*$-algebra.

1.5) Def: Let $E_B$ and $F_B$ be Hilbert modules. Let $T$ be a map from $E$ to $F$. Then $T^*: F \to E$ is called an adjoint of $T$ if
$$\forall e \in E, f \in E: \quad \langle Te, f\rangle_F = \langle e, T^*f\rangle \in B.$$
If such a $T^*$ exists then $T$ is called adjoinable (or adjointable). The set of all such $T$ is denoted by $L(E,F)$.

1.6. Exercise: Find $E$, $F$ and $T$ a cont.&lin. $E \to F$ such that $T$ is not adjoinable.

1.7. Proposition: Let $E$, $F$ and $T$ be as above. Let $T$ be adjoinable.

1.) $T^*$ is unique, $T^*$ is also adjoinable and $(T^*)^* = T$.

2.) $T$ is linear, $B$-linear and continuous.

3.) $\|T\|^2 = \|T^*T\| = \|TT^*\| = \|T^*\|^2$.

Pf: [Exercise. For the continuity part you need some functional analysis.]

1.8. Proposition: If $E$ is a Hilbert $B$-module then $L(E) := L(E,E)$ is a $C^*$-algebra.

Pf: Just as for Hilbert spaces (use the Cauchy-Schwarz-ineq.).

[Remark: $L(E,F)$ is a Banach space.]

1.9 <u>Definition</u>: Let $E_B$ and $F_B$ be Hilbert modules. For all $e \in E$ and $f \in F$ define

$$\vartheta_{f,e} : E \to F, \quad e' \mapsto f \cdot \langle e, e' \rangle_E .$$

1.10 <u>Proposition</u>: In the above situation:

1) $\vartheta_{f,e} \in L(E,F)$ and $\vartheta_{e,f} = \vartheta_{f,e}^*$.

2) $\forall T \in L(F), S \in L(E)$:

$$T \circ \vartheta_{f,e} = \vartheta_{Tf,e} \quad \text{and} \quad \vartheta_{f,e} \circ S = \vartheta_{f,S^*e} .$$

<u>Pf</u>: ■

1.11 <u>Definition</u>: Define $K(E,F)$ to be the closed linear span of $\{\vartheta_{f,e} : e \in E, f \in F\} \subseteq L(E,F)$. Elements of $K(E,F)$ are called <u>compact operators</u>.

1.12 <u>Proposition</u>:

$$L(F) K(E,F) = K(E,F),$$
$$K(E,F) \cdot L(E) = K(E,F),$$
$$K(E,F)^* = K(F,E).$$

In particular, $K(E) := K(E,E)$ is a closed, $*$-closed ideal of $L(E)$.

1.13 <u>Lemma</u>: Let $E,F$ be Hilbert $\mathbb{B}$-modules. Then

$$K_{\mathbb{B}}(E,F) = \{ T \in L(E,F) \mid TT^* \in K(F) \}$$

<u>Proof</u>: "$\subseteq$" is obvious.

"$\supseteq$": Let $(u_\lambda)_{\lambda \in \Lambda}$ be a bounded approximate unit for $K(F)$. Then (using $u_\lambda = u_\lambda^*$):

$$\| u_\lambda T - T \|^2 = \| u_\lambda TT^* u_\lambda - u_\lambda TT^* - TT^* u_\lambda + TT^* \|,$$

so $\quad TT^* \in K(F) \Rightarrow u_\lambda T \to T$ in $L(E,F)$.

Because $u_\lambda T \in K(E,F)$ we have $T \in K(E,F)$.

1.14 <u>Examples</u>:

a) If $\mathbb{B} = \mathbb{C}$ and $H$ is a Hilbert $\mathbb{C}$-module then $K(H)$ is the usual algebra of compact operators.

b) If $\mathbb{B}$ is arbitrary and if you regard $\mathbb{B}$ is a Hilbert $\mathbb{B}$-module, then $\mathbb{B} \cong K(\mathbb{B})$.

<u>Pf</u>: Define $\Psi : \mathbb{B} \to L(\mathbb{B}),\ b \mapsto b\,\bullet$,

then $\Psi$ is a $*$-homomorphism,

$\Psi(b^* c) = \vartheta_{b,c}$ for all $b, c \in \mathbb{B}$, so

$\Psi(\mathbb{B} \cdot \mathbb{B}) \subseteq K(\mathbb{B})$, but $\mathbb{B} \cdot \mathbb{B} \doteq \mathbb{B}$.

c) If $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$, then

$K(E,F) = \bigoplus_{i=1,2} \bigoplus_{j=1,2} K(E_i, F_j)$ and every

$T \in K(E,F)$ can be expressed as a matrix

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

d) consequence: $K(\mathbb{B}^m, \mathbb{B}^n) \cong \mathrm{Mat}_{m \times n}(\mathbb{B})$.

1.15. Definition. If $\mathbb{B}$ is a $C^*$-algebra then we define $M(\mathbb{B}) := L(\mathbb{B})$.

$M(\mathbb{B})$ is called the multiplier algebra of $\mathbb{B}$.
Ex: $M(C_0(X)) = C_b(X)$.

1.16 Proposition If $E$ is a Hilbert $\mathbb{B}$-module, then

$$M(K(E)) \cong L(E).$$

Sketch of Proof [compare Coro. 1.1.16 in Jensen/Thomsen or Thm 15.2.12 in Wegge-Olsen]:
If $T \in L(E)$, then $S \mapsto T \circ S$ defines an element $T \cdot$ of $M(K(E))$. This defines a $*$-homomorphism $\Psi$.
$\Psi$ is injective: Let $T \in L(E)$ such that $\Psi(T) = 0$.
Let $e \in E$. Then $\langle \Psi(T)(\vartheta_{e,Te})(Te), \Psi(T)\vartheta_{e,Te}(Te) \rangle$
$= \langle (T \cdot \vartheta_{e,Te})(Te), (T\vartheta_{e,Te})(Te) \rangle = \langle Te, Te \rangle^3$, so
$Te = 0$, so $T = 0$.
If $m \in M(K(E))$ and $e \in E$, then define

$$T(e) := \lim_{\varepsilon \to 0} m\left(\vartheta_{e,e}\right)(e) \cdot \left(\langle e,e\rangle + \varepsilon\right)^{-1}.$$ 7)

Then this is a well-defined element of $L(E)$ and $\Psi(T) = m$, so $\Psi$ is surjective. $\boxtimes$

1.17 <u>Definition</u> Let $\varphi : \mathcal{B} \to \mathcal{B}'$ be a $*$-homomorphism (between $C^*$-algs)
Let $E_{\mathcal{B}}$ and $E'_{\mathcal{B}'}$ be Hilbert modules.
A <u>homomorphism</u> with coefficient map $\varphi$ from $E_{\mathcal{B}}$ to $E'_{\mathcal{B}'}$ is a map $\Phi : E \to E'$ such that

1.) $\Phi$ is $\mathbb{C}$-linear,

2.) $\forall e \in E \; \forall b \in \mathcal{B} : \; \Phi(eb) = \Phi(e)\varphi(b) \in E'$

3.) $\forall e, f \in E : \; \langle \Phi(e), \Phi(f) \rangle = \varphi(\langle e, f \rangle) \in \mathcal{B}'$.

We denote such a map also by $\Phi_{\varphi}$ to emphasize $\varphi$.

1.18 <u>Remark</u>: From 3.) it follows that $\|\Phi(e)\| \leq \|e\|$ f.a. $e \in E$.
and equality holds when $\varphi$ is inj.

1.19 <u>Remark</u>: There is an obvious <u>composition</u> of homomorphisms with coefficient maps:

$$\Phi_{\varphi} : E_{\mathcal{B}} \to E_{\mathcal{B}'}, \quad \Psi_{\chi} : E'_{\mathcal{B}'} \to E''_{\mathcal{B}''}$$

$$\Rightarrow \quad (\Psi \circ \Phi)_{\chi \circ \varphi} : E_{\mathcal{B}} \to E''_{\mathcal{B}''}.$$

Also : $(\mathrm{Id}_E)_{\mathrm{Id}_{\mathcal{B}}} : E_{\mathcal{B}} \to E_{\mathcal{B}}$

1.20 <u>Definition</u>: Two Hilbert $\mathcal{B}$-modules $E_{\mathcal{B}}$ and $E'_{\mathcal{B}}$ are called <u>isomorphic</u> if there is a homomorphism $\Phi_{\mathrm{id}_{\mathcal{B}}} : E_{\mathcal{B}} \to E'_{\mathcal{B}}$ which is bijective. Then $\Phi^{-1}_{\mathrm{id}_{\mathcal{B}}} : E'_{\mathcal{B}} \to E_{\mathcal{B}}$. Write $E \cong E'$.
[Note that in this case: $\Phi \in \mathcal{L}(E, E')$ and $\Phi^* = \Phi^{-1}$.]