c) Also the pull-back is a special case of the Kasparov product. Assume that we have shown that the product is well-defined on the level of homotopy classes:

Let $\varphi: A \to B$ a $*$-hom. From Exercise 3.19 one can assume wlog that $\phi_2: B \to L_C(E_2)$ is non-degenerate. Then $B \hat{\otimes}_B E_2 \cong E_2$ and we can regard T_2 as a T_2-connection. The action of A on E_2 under this identification is $\phi_2 \circ \varphi$. It is easy to see that we obtain an element in $0 \# T_2$ which is isomorphic to $\varphi^*(E_2)$.

d) In particular: „ $1_A \hat{\otimes}_A x = x = x \hat{\otimes}_B 1_B$ " for all $x \in KK(A,B)$.

3.23 Proof of Thm. 3.21 (see Section 3.2 of my 2004' notes)

Also: the product lifts to a biadditive, (associative) map on the level of KK $\to$ see section 3.3 of my 2004' notes

3.30 Lemma (Ex. 18.4.2 d) Blackadar) A, B, C as above,

$\mathcal{E}_1 = (E_1, \phi_1, T_1) \in \mathbb{E}(A,B)$ with $T_1^* = T_1$ and $\|T\| \le 1$ and $\mathcal{E}_2 = (E_2, \phi_2, T_2) \in \mathbb{E}(B,C)$

Let G be any T_2-connection of degree 1 on $E_{12} = E_1 \hat\otimes_B E_2$.

Define

$$T_{12} := T_1 \hat\otimes 1 + \overbrace{\left[(1-T_1^2)^{\frac{1}{2}} \hat\otimes 1\right]}^{=:L} G.$$

Then $\phi_{12}(a)(T_{12}^2 - 1)$ and $\phi_{12}(a)(T_{12} - T_{12}^*)$ are in $K_C(E_{12})$ and

$\phi_{12}(a)\,[T_{12}, T_1 \hat\otimes 1]\,\phi_{12}(a)^* \geq 0 \mod K_C(E_{12})$ for all $a \in A$.

Suppose $[T_{12}, \phi_{12}(a)] \in K(E_{12})$ f.a. $a \in A$, then $\mathcal{E}_{12} := (E_{12}, \phi_{12}, T_{12}) \in \mathbb{E}(A,C)$ and $\mathcal{E}_{12}$ is op. hom. to an element of $\mathcal{E}_1 \# \mathcal{E}_2$.

Pf.

$$\phi_{12}(a)(T_{12}^2 - 1) = \phi_{12}(a)\Big[T_1^2 \hat\otimes 1 + (T_1 \hat\otimes 1) L G + L G (T_1 \hat\otimes 1) + L G L G - 1\Big]$$

Now $\phi_{12}(a)(T_1 \hat\otimes 1) L G = \phi_{12}(a) L (T_1 \hat\otimes 1) G$

and $\phi_{12}(a) L \in K_B(E_1) \hat\otimes 1$, so $\phi_{12}(a) L (T_1 \hat\otimes 1) \in K_B(E_1) \hat\otimes 1$.

so $\left[\phi_{12}(a) L (T_1 \hat\otimes 1), G\right] \in K_C(E_{12})$

and hence

$$\phi_{12}(a) L (T_1 \hat\otimes 1) G \overset{\mathrm{mod}\,K}{\equiv} (-1)^{\partial a} G \phi_{12}(a) L (T_1 \hat\otimes 1)$$

$$\overset{\mathrm{mod}\,K}{\equiv} - \phi_{12}(a) L G (T_1 \hat\otimes 1)$$

Similarly $\phi_{12}(a) L G L G \equiv (-1)^{\partial a} G \phi_{12}(a) L^2 G \equiv (-1)^{\partial a + \partial a} \phi_{12}(a) L^2 G^2$

So $\phi_{12}(a)(T_{12}^2 - 1) \equiv \phi_{12}(a)\left((T_1^2 - 1) \hat\otimes 1 + ((1 - T_1^2) \hat\otimes 1) G^2\right)$

$$= \left[\phi_1(a)(T_1^2 - 1)\right] \hat\otimes 1 \cdot \underbrace{(1 - G^2)}_{0\text{-connection}} \in K_C(E_{12})$$

Similarly for $\phi_{12}(a)\left(T_{12} - T_{12}^*\right) \in K_C(E_{12})$
and $\phi_{12}(a)\left[T_{12}, T_1 \hat{\otimes} 1\right]\phi_{12}(a)^* \geq 0 \mod K_C(E_{12})$.

Now find M and N as in the existence proof of the product such that
$$\tilde{T}_{12} := M^{\frac{1}{2}}(T_1 \hat{\otimes} 1) + N^{\frac{1}{2}} G \quad \text{defines a}$$
Kasparov product $\tilde{\mathcal{E}}_{12} = (E_{12}, \phi_{12}, \tilde{T}_{12}) \in \mathbb{E}(A, C)$ of $\mathcal{E}_1$ and $\mathcal{E}_2$.

$\mathcal{E}_{12}$ is operator homotopic to $\tilde{\mathcal{E}}_{12}$ via:
$$T_t = \left[tM + (1-t)\right]^{\frac{1}{2}}(T_1 \hat{\otimes} 1) + \left[tN + (1-t)\left((1-T_1^2)^{\frac{1}{2}} \hat{\otimes} 1\right)\right]^{\frac{1}{2}} G.$$

The general form of the product:

Let A_1, A_2, B_1, B_2 and D be graded σ-unital C^*-algebras and

$x \in KK(A_1, B_1 \hat{\otimes} D)$, $y \in KK(D \hat{\otimes} A_2, B_2)$.

If A_1 and A_2 are separable, then we define

$$x \underset{D}{\otimes} y := \underbrace{(x \hat{\otimes} 1_{A_2})}_{\in KK(A_1 \hat{\otimes} A_2,\, B_1 \hat{\otimes} D \hat{\otimes} A_2)} \underset{B_1 \hat{\otimes} D \hat{\otimes} A_2}{\hat{\otimes}} \underbrace{(1_{B_1} \hat{\otimes} y)}_{\in KK(B_1 \hat{\otimes} D \hat{\otimes} A_2,\, B_1 \hat{\otimes} B_2)} \in KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

Check Blackadar, 18.9.1 for functorial properties of this general product.

If $\mathbb{C} = D$, then we obtain a product

$$\underset{\mathbb{C}}{\otimes} : KK(A_1, B_1) \otimes KK(A_2, B_2) \longrightarrow KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

It is commutative in the following sense. Let

$$\Sigma_{A_1, A_2} : A_1 \hat{\otimes} A_2 \to A_2 \hat{\otimes} A_1, \quad a_1 \hat{\otimes} a_2 \mapsto (-1)^{\partial a_1 \partial a_2} a_2 \hat{\otimes} a_1$$

and define Σ_{B_1, B_2} analogously. Then

$$x \underset{\mathbb{C}}{\otimes} y = \Sigma_{B_1, B_2}^{-1} \circ y \underset{\mathbb{C}}{\otimes} x \circ \Sigma_{A_1, A_2}$$