

13-07-09 (Münster) : KK^{ban} and the Baum-Connes Conjecture

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* BC with coefficient (?):

$$\begin{array}{ccc} K_*^{top}(\underline{\Sigma}G, \mathcal{B}) & \longrightarrow & K_*(\mathcal{B} \rtimes_r G) \\ & \searrow \cong ? & \uparrow \cong ? \\ & & K_*(A(G, \mathcal{B})) \end{array}$$

* \mathcal{E} : class of groups, contains:

- G locally compact acting geometrically (continuously, properly, isometrically) on:
 - * complete, simply connected Riemann mfd's with s.c. $\leq b$
 - * affine buildings
 - * uniformly locally finite, weakly geodesic, weakly bolic metric spaces.
- G discrete, acting geometrically on weakly

G a-T-menable

* for \mathcal{C} : γ -element, BC + coefficients is injective
Lafforgue : \mathcal{B}^{oo} with coeff. is injective.

$\mathcal{C}' = \left\{ \begin{array}{l} G \text{ loc. compact } \downarrow \text{ geometrically on} \\ + \text{ compl. simply connected Riem. mfd's, s.c. } \leq 0 \\ \text{s.c. is bdd below + derivative of the curvature} \\ \text{tensor is bounded} \\ + \text{ uniformly locally finite, weakly geodesics,} \\ \text{weakly holic metric spaces + holicity} \\ \text{condition } \int \end{array} \right.$

* $\mathcal{C}' \subset \mathcal{C}$: For \mathcal{C}' : Boost is surjective.

Definition: An alg. norm $\|\cdot\|_A$ on $C_c(G)$ is called "unconditional"

if : $\forall f_1, f_2 \in C_c(G) : (\forall g \in G, |f_1(g)| \leq |f_2(g)|) \Rightarrow (\|f_1\|_A \leq \|f_2\|_A)$.

The completion of $C_c(G)$ with respect to $\|\cdot\|_A$ is denoted by $A(G)$.

Now, on $C_c(G, B)$ with B a Banach algebra, define

$$\|f\|_{A(G, B)} := \|g \mapsto \|f(g)\|_B\|_A$$

Question: What is $KK^{\text{ban}}(A, B)$ for Banach algebras A, B ?

1. What are B -Hilbert modules?

Definition 1.1: A right Banach B -module E is a right B -module which is, at the same time a Banach space such that $\forall e \in E$ and $\forall b \in B$ one has

$$\|eb\| \leq \|e\| \|b\|.$$

Definition 1.2: A right Banach B -pair is a pair $E = (E^{\leftarrow}, E^{\rightarrow})$ together with a map $\langle \cdot, \cdot \rangle : E^{\leftarrow} \times E^{\rightarrow} \rightarrow B$ such that

(i) E^{\leftarrow} is a left Banach B -module;

(ii) E^{\rightarrow} is a right " " " "

(iii) $\langle \cdot, \cdot \rangle$ is \mathbb{C} -bilinear and satisfies:

$$(a) \langle b e^{\leftarrow}, e^{\rightarrow} \rangle = b \langle e^{\leftarrow}, e^{\rightarrow} \rangle,$$

$$(b) \langle e^{\leftarrow}, e^{\rightarrow} b \rangle = \langle e^{\leftarrow}, e^{\rightarrow} \rangle b$$

$$(c) \|\langle e^{\leftarrow}, e^{\rightarrow} \rangle\|_B \leq \|e^{\leftarrow}\|_{E^{\leftarrow}} \|e^{\rightarrow}\|_{E^{\rightarrow}},$$

for all $b \in B$, $e^{\leftarrow} \in E^{\leftarrow}$ and $e^{\rightarrow} \in E^{\rightarrow}$.

If B is a C^* -algebra and \mathcal{H} is a Hilbert B -module then $(\mathcal{H}^*, \mathcal{H})$ is a Banach B -pair (under $b e := e b^*$).

2. What are adjointable operators?

Definition 2.1: Let E_B and F_B be Banach B -pairs. An operator from E to F is a pair $T = (T^{\leftarrow}, T^{\rightarrow})$ such that

$$(1) T^{\leftarrow} \in \mathcal{L}_B(F^{\leftarrow}, E^{\leftarrow}) ; \text{ i.e. } E^{\leftarrow} \xleftarrow{T^{\leftarrow}} F^{\leftarrow}$$

$$(2) T^{\rightarrow} \in \mathcal{L}_B(E^{\rightarrow}, F^{\rightarrow}) ; \text{ i.e. } E^{\rightarrow} \xrightarrow{T^{\rightarrow}} F^{\rightarrow}$$

$$(3) \forall e^{\rightarrow} \in E^{\rightarrow}, f^{\leftarrow} \in F^{\leftarrow} : \langle T^{\leftarrow} f^{\leftarrow}, e^{\rightarrow} \rangle_B = \langle f^{\leftarrow}, T^{\rightarrow} e^{\rightarrow} \rangle_B$$

The space of all operators from E to F is called $\mathcal{L}_B(E, F)$ (Banach space with norm $\|T\| = \sup\{\|T^{\leftarrow}\|, \|T^{\rightarrow}\|\}$).

3. What are compact operators?

Definition 3.1: Let E and F be B -pairs. Let $e^{\leftarrow} \in E^{\leftarrow}$, $f^{\rightarrow} \in E^{\rightarrow}$. Define:

$$|f^{\rightarrow}\rangle \langle e^{\leftarrow}|^{\rightarrow} : E^{\rightarrow} \rightarrow F^{\rightarrow}, e^{\rightarrow} \mapsto f^{\rightarrow} \langle e^{\leftarrow}, e^{\rightarrow} \rangle.$$

$$|f^{\rightarrow}\rangle \langle e^{\leftarrow}|^{\leftarrow} : F^{\leftarrow} \rightarrow E^{\leftarrow}, f^{\leftarrow} \mapsto \langle f^{\leftarrow}, f^{\rightarrow} \rangle e^{\leftarrow}.$$


This defines an element $|f\rangle\langle c| \in \mathcal{L}_B(E, F)$.

Call an element in the closed linear span $\mathcal{K}_B(E, F)$ of these operators "compact".

Examples 3.2. 1) Let X be a Banach space. Then (X', X) is a Banach \mathbb{C} -pair, and $\mathcal{K}_{\mathbb{C}}(X', X)$ consists of approximate operators on X .

$$2) \mathcal{K}_B(B) = \text{cl} \left\{ \underbrace{|b\rangle\langle c|}, b \in B \right\} = \text{cl} \{ b \cdot | b \in B \} \subseteq \mathcal{L}_B(B)$$

↳ multiplication with b from the other side

[ $b \mapsto b \cdot$, $B \rightarrow \mathcal{K}_B(B)$ is neither surjective, nor injective in general.

4. What is $\mathbb{E}^{\text{ban}}(A, B)$?

It is the set of triples (E, ϕ, T) , where E is a graded B -pair, $\phi: A \rightarrow \mathcal{L}_B(E)$ is a Banach algebra homomorphism (i.e.: $\langle \epsilon a, e \rangle = \langle \epsilon, a \bar{e} \rangle$) and T is an odd element in $\mathcal{L}_B(E)$ such that:

$$a) \forall a \in A: [\phi(a), T] \in \mathcal{K}_B(E),$$

$$b) \forall a \in A: \phi(a) (T^2 - 1) \in \mathcal{K}_B(E).$$

5. What is $KK^{\text{ban}}(A, B)$?

Define $KK^{\text{ban}}(A, B) := \mathbb{E}^{\text{ban}}(A, B) / \mathbb{E}^{\text{ban}}(A, B[0, 1])$.

One shows that $KK^{\text{ban}}(A, B)$ is an abelian group which acts on K -theory: $KK^{\text{ban}}(A, B) \rightarrow \text{Hom}(K_0(A), K_0(B))$.

■ We have to point out that there is no Kasparov product known for KK^{ban} .

However, there is an isomorphism

$$KK^{ban}(\mathbb{C}, B) \cong K_0(B).$$