# THE GREEN-JULG THEOREM

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Abstract. We give a short $KK$-theoretic proof of the Green-Julg Theorem, i.e., we show that for any compact group $G$ and any $G$-$C^*$-algebra $B$ the group $KK^G_*(\mathbb{C}, B)$ is canonically isomorphic to $K_*(B \rtimes G)$.

Let $G$ be a locally compact group and let $A$ and $B$ be two $G$-$C^*$-algebras. Then the equivariant $KK$-groups $KK^G(A, B) =: KK^G_0(A, B)$ are defined as the set of all homotopy classes of triples $(\mathcal{E}, \Phi, T)$, where

(1) $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a $\mathbb{Z}_2$-graded Hilbert $B$-module endowed with a grading preserving action $\gamma : G \to \mathrm{Aut}(\mathcal{E})$;
(2) $\Phi = \begin{pmatrix} \Phi_0 & 0 \\ 0 & \Phi_1 \end{pmatrix}$ is a $G$-equivariant $*$-homomorphism;
(3) $T = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ is an operator in $\mathcal{L}(\mathcal{E})$ such that

$$[\Phi(a), T], (T^* - T)\Phi(a), (T^2 - 1)\Phi(a), (\mathrm{Ad}\,\gamma_s(T) - T)\Phi(a) \in \mathcal{K}(\mathcal{E})$$

for all $a \in A$ and $s \in G$.

If $G$ is compact, then it was shown by Kasparov that we may assume without loss of generality that the operator $T$ is $G$-equivariant (by replacing $T$ by $T^G = \int_G \mathrm{Ad}\,\gamma_s(T)\, ds$ if necessary) and that $\Phi$ is nondegenerate. In particular, if $A = \mathbb{C}$, then $KK^G(\mathbb{C}, B)$ can be described as the set of homotopy classes of pairs $(\mathcal{E}, T)$ such that $T$ is $G$-invariant and

$$T^* - T,\ T^2 - 1 \in \mathcal{K}(\mathcal{E}).$$

Similarly, we can describe $K_0(B \rtimes G) = KK(\mathbb{C}, B \rtimes G)$. We start with the following easy lemma:

**Lemma 0.1.** *Let $G$ be a compact group and let $B$ be a $G$-$C^*$-algebra. Suppose that $\mathcal{E}_B$ is a $G$-equivariant Hilbert $B$-module. Then $\mathcal{E}_B$ becomes a pre-Hilbert $B \rtimes G$-module if we define the right action of $B \rtimes G$ on $\mathcal{E}_B$ and the $B \rtimes G$-valued inner products by the formulas*

$$e \cdot f := \int_G \gamma_s\big(e \cdot f(s^{-1})\big)\, ds \quad \text{and} \quad \langle e_1, e_2\rangle_{B\rtimes G}(s) := \langle e_1, \gamma_s(e_2)\rangle_B$$

*for $e, e_1, e_2 \in \mathcal{E}$ and $f \in C(G, B) \subseteq B \rtimes G$. Denote by $\mathcal{E}_{B\rtimes G}$ its completion. Moreover, if $(\mathcal{E}_B, T)$ represents an element of $KK^G(\mathbb{C}, B)$ with $T$ being $G$-invariant, then $T$ extends to an operator on $\mathcal{E}_{B\rtimes G}$ such that $(\mathcal{E}_{B\rtimes G}, T)$ represents an element of $KK(\mathbb{C}, B \rtimes G)$.*[1]

*Proof.* First note that the above defined right action of $C(G, B)$ on $\mathcal{E}_B$ extends to an action of $B \rtimes G$. For this we observe that the pair $(\Psi, \gamma)$, with $\Psi : B \to \mathcal{L}_{\mathcal{K}(\mathcal{E})}(\mathcal{E}_B)$ given by the formula $\Psi(b)(e) = e \cdot b^*$, is a covariant homomorphism of $(B, G, \beta)$ on the left Hilbert

[1] We are grateful to Walther Paravicini for pointing out a mistake in a previous version of this lemma!

$\mathcal{K}(\mathcal{E}_B)$-module $\mathcal{E}_B$. Then $e \cdot f = \big(\Psi \times \gamma(f)\big)(e)$ for $f \in C(G,B)$, and the right hand side clearly extends to all of $B \rtimes G$.

It is easily seen that $\langle \cdot, \cdot \rangle_{B \rtimes G}$ is a well defined $B \rtimes G$-valued inner product which is compatible with the right action of $B \rtimes G$ on $\mathcal{E}_B$. So we can define $\mathcal{E}_{B \rtimes G}$ as the completion of $\mathcal{E}_B$ with respect to this inner product.

If $T \in \mathcal{L}_B(\mathcal{E}_B)$, then $T$ determines an operator $T^G \in \mathcal{L}_{B \rtimes G}(\mathcal{E}_{B \rtimes G})$ by the formula

$$T^G(e) = \int_G \gamma_t\big(T(\gamma_{t^{-1}}(e))\big)\, dt, \quad \text{for } e \in \mathcal{E}_B \subseteq \mathcal{E}_{B \rtimes G},$$

and one checks that $T \to T^G$ is a $*$-homomorphism from $\mathcal{L}_B(\mathcal{E}_B)$ to $\mathcal{L}_{B \rtimes G}(\mathcal{E}_{B \rtimes G})$. In particular, it follows that every $G$-invariant operator on $\mathcal{E}_B$ extends to an operator on $\mathcal{E}_{B \rtimes G}$. If $e_1, e_2 \in \mathcal{E}_B \subseteq \mathcal{E}_{B \rtimes G}$, then a short computation shows that the corresponding finite rank operator $\Theta_{e_1,e_2} \in \mathcal{K}(\mathcal{E}_{B \rtimes G})$ is given by the formula

$$\Theta_{e_1,e_2} = \tilde{\Theta}^G_{e_1,e_2}$$

if $\tilde{\Theta}_{e_1,e_2} \in \mathcal{K}(\mathcal{E}_B)$ denotes the corresponding finite rank operator on $\mathcal{E}_B$. This easily implies that the remaining part of the lemma. □

**Theorem 0.2** (Green-Julg Theorem)**.** *Let $G$ be a compact group and let $B$ be a $G$-$C^*$-algebra. Then the map*

$$\mu : KK^G(\mathbb{C}, B) \to KK(\mathbb{C}, B \rtimes G); \mu\big([(\mathcal{E}_B, T)]\big) = [(\mathcal{E}_{B \rtimes G}, T)]$$

*is an isomorphism.*

*Proof.* Note first that we can apply the same formula to a homotopy, so the map is well defined. We now define a map $\nu : KK(\mathbb{C}, B \rtimes G) \to KK^G(\mathbb{C}, B)$ and show that it is inverse to $\mu$.

For this let $L^2(G,B)$ denote the Hilbert $B$-module defined as the completion of $C(G,B)$ with respect to the $B$-valued inner product

$$\langle f, g \rangle_B = \int_G \beta_s\big(f(s^{-1})^* g(s^{-1})\big)\, ds$$

and the right action of $B$ on $L^2(G,B)$ given by $(f \cdot b)(t) = f(t)\beta_t(b)$ for $f \in C(G,B), b \in B$. There is a well defined left action of $B \rtimes G$ on $L^2(G,B)$ given by convolution when restricted to $C(G,B) \subseteq B \rtimes G$ (and $C(G,B) \subseteq L^2(G,B)$). We even have $B \rtimes G \subseteq \mathcal{K}\big(L^2(G,B)\big)$. To see this we simply note that $\mathcal{K}\big(L^2(G,B)\big) = C(G,B) \rtimes G$ by Green's imprimitivity theorem (where $G$ acts on $C(G)$ by left translation), and $B \rtimes G$ can be viewed as a subalgebra of $C(G,B) \rtimes G$ in a canonical way.

Let $\sigma : G \to \operatorname{Aut}\big(L^2(G,B)\big)$ be defined by

$$\sigma_s(f)(t) = f(ts); \quad f \in C(G,B).$$

Then $\sigma$ is compatible with the action $\beta$ of $G$ on $B$. Moreover, a short computation shows that the homomorphism of $B \rtimes G$ into $\mathcal{L}\big(L^2(G,B)\big)$ given by convolution is equivariant with respect to the trivial $G$-action on $B \rtimes G$ and the action $\operatorname{Ad}\sigma$ on $L^2(G,B)$. Assume now that $(\tilde{\mathcal{E}}, T)$ represents an element of $KK(\mathbb{C}, B \rtimes G)$. Then $\tilde{\mathcal{E}} \otimes_{B \rtimes G} L^2(G,B)$ equipped with the action $\operatorname{id} \otimes \sigma$ is a $G$-equivariant Hilbert $B$-module and $(\tilde{\mathcal{E}} \otimes L^2(G,B), T \otimes 1)$

represents an element of $KK^G(\mathbb{C}, B)$ (here we use the fact that $B \rtimes G \subseteq \mathcal{K}\big(L^2(G,B)\big)$). Thus we define

$$\nu : KK(\mathbb{C}, B \rtimes G) \to KK^G(\mathbb{C}, B);\;\; \nu\big([(\tilde{\mathcal{E}}, T)]\big) = [(\tilde{\mathcal{E}} \otimes_{B\rtimes G} L^2(G,B), T \otimes 1)].$$

Again, applying the same formula to homotopies implies that $\nu$ is well defined.

To see that $\nu$ is an inverse to $\mu$ one checks:

(a) Let $\mathcal{E}_B$ be a Hilbert $B$-module and let $\mathcal{E}_{B\rtimes G}$ be the corresponding Hilbert $B \rtimes G$-module as described in Lemma 0.1. Then

$$\mathcal{E}_B \odot C(G,B) \to \mathcal{E}_B; e \otimes f \mapsto e \cdot f = \int_G \gamma_s\big(e \cdot f(s^{-1})\big)\, ds$$

extends to a $G$-equivariant isometric isomorphism between $\mathcal{E}_{B\rtimes G} \otimes_{B\rtimes G} L^2(G,B)$ and $\mathcal{E}_B$.

(b) Let $\tilde{\mathcal{E}}$ be a Hilbert $B \rtimes G$-module. Then

$$\tilde{\mathcal{E}} \odot C(G,B) \to \tilde{\mathcal{E}}; e \otimes f \mapsto e \cdot f$$

determines an isometric isomorphism

$$\big(\tilde{\mathcal{E}} \otimes_{B\rtimes G} L^2(G,B)\big)_{B\rtimes G} \cong \tilde{\mathcal{E}}$$

as Hilbert $B \rtimes G$-modules.

Both results follow from some straightforward computations. Note that for the proof of (b) one should use the fact that for all $x \in B \rtimes G$ and $f, g \in C(G,B)$ the element of $B \rtimes G$ given by the continuous function $s \mapsto \langle x^* \cdot f, \sigma_s(g)\rangle_B$ coincides with $f^* * x * g$, where we view $f, g$ as elements of $B \rtimes G$. This follows by direct computations for $x \in C(G,B)$, and since both expressions are continuous in $x$, it follows for all $x \in B \rtimes G$. Finally, it is trivial to see that the operators match up in both directions. □

**Remark 0.3.** We should remark, that the above Theorem 0.2 is a special case of a more general result for crossed products by proper actions due to Kasparov and Skandalis (see [2, Theorem 5.4]). There also exist important generalizations to proper groupoids by Tu [3, Proposition 6.25] and Paravicini [4], where the latter provides a version within Lafforgue's Banach KK-theory. For the original proof of the Green-Julg theorem for compact groups we refer to [1].

## References

[1] P. Julg. *K-thorie équivariante et produits croisés.* C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 13, 629–632.

[2] G. Kasparov and G. Skandalis. *Groups acting properly on "bolic" spaces and the Novikov conjecture.* Annals of Math. **158** (2003), 165–206.

[3] J.-L. Tu. *La conjecture de Novikov pour les fouilletages hyperboliques.* $K$-Theory **16** (1999), no. 2, 129–184.

[4] Walther Paavicini. Ph.D. Dissertation. Münster, January 2007.

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