Category of Compact Quantum Semigroups

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1.Compact quantum semigroup definition

$(\mathcal{A},\Delta)$ is called *a compact quantum semigroup*, if $\mathcal{A}$ is a unital C^*-algebra and $\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is a unital *-homomorphism, satisfying *coassociativity* property:

$$(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta.$$

In this case Δ is called *a comultiplication.*
When $\mathcal{A}$ is a commutative C^*-algebra, by Gelfand-Naimark theorem it is isomorphic to $C(P)$ for a compact space P. Then the comultiplication Δ gives a semigroup structure on P:

$$f(x\cdot y)=\Delta(f)(x,y).$$

When $\mathcal{A}$ is non-commutative, it does not correspond to any compact semigroup. But algebra $\mathcal{A}$ is understood as *an algebra of functions on a compact **quantum** semigroup.*

2. A representation of C(G)

Let G be a compact Abelian group, Γ – a discrete abelian group, isomorphic to a group of characters of G.
Consider Hilbert space $L^2=L^2(G,d\mu)$, where μ is a shift-invariant normed measure. Denote by $\{e_a\}_{a\in\Gamma}$ an orthonormal basis of L^2, corresponding to the characters $\chi^a, a\in\Gamma$ of the group G.
For $f\in C(G)$ define an operator $L_f\colon L^2\to L^2$ by

$$L_f g = f\cdot g.$$

We obtain representation $f\to L_f$ of the algebra $C(G)$ in $B(L^2)$, with the image being a commutative C^*-algebra.

3. A deformation of the functions algebra.

Take a subsemigroup $S\subset\Gamma$ with zero, which generates Γ.
Let $H_S\subset L^2$ be a Hilbert space with basis $\{e_a\}_{a\in S}$, and $P_S\colon L^2\to H_S$ – a projection. Define for $f\in C(G)$:

$$T_f = P_S L_f P_S. \tag{1}$$

A reduced semigroup C^a*st*-algebra $\mathcal{C}^*_{red}(S)$ is a C^*-algebra generated by all T_f for $f\in C(G)$.

4. Example. Deformation parameter

Suppose Γ is a group of integers $\mathbb{Z}$. As an example of such semigroup S we could choose a semigroup of non-negative integers $\mathbb{Z}_+\subset\Gamma$. The Pontryagin dual of $\mathbb{Z}$, its group of characters, would be a unit circle $G=S^1$. For $S=\mathbb{Z}_+$, $C^a st_{red}(S)=\mathcal{T}$ is the Toeplitz algebra.
Note, that for the same group Γ we could choose other semigroups, e.g. $S=\{0,2,3,4,\dots\}$ or $S=\{0,3,6,7,8,\dots\}$. Indeed, all such semigroups generate $\mathbb{Z}$, since they contain elements with difference between them equal to unit. But the corresponding C^*-algebras are not canonically isomorphic.
This shows that the result of deformation depends on the choice of S, not only on G. That is why we call S a deformation parameter.

5. Inverse semigroup

Isometric operators $T_a=P_S L_{\chi^a} P_S$ for $a\in S$ and T_a^* are generators in $\mathcal{C}^*_{red}(S)$.
A finite product $T_a, T_b^*, a,b\in S$ is called *a monomial.* Linear combinations of monomials are dense in $\mathcal{C}^*_{red}(S)$. Monomials form an inverse semigroup denoted by $Mon(S)$.

6. A compact quantum semigroup

Define comultiplication $\Delta\colon Mon(S)\to Mon(S)\otimes Mon(S)$ by $\Delta(V)=V\otimes V$.

Theorem 1.

The map Δ extends to a unital *-homomorphism $\Delta\colon \mathcal{C}^*_{red}(S)\to\mathcal{C}^*_{red}(S)\otimes\mathcal{C}^*_{red}(S)$. The pair $(\mathcal{C}^*_{red}(S),\Delta)$ is a compact quantum semigroup.

$(\mathcal{C}^*_{red}(S),\Delta)$ is a compact quantum **group** if and only if $S=\Gamma$. In this case $(\mathcal{C}^*_{red}(\Gamma),\Delta)\cong G$.

$$S\to QS=(\mathcal{C}^*_{red}(S),\Delta),$$
$$\Gamma\to G.$$

7. Quantum semigroup morphisms

We say that π^* is *a **morphism*** taking a compact quantum semigroup $(\mathcal{A}_2,\Delta_2)$ to $(\mathcal{A}_1,\Delta_1)$, if there exists a *-homomorphism $\pi\colon\mathcal{A}_1\to\mathcal{A}_2$, verifying the equation:

$$(\pi\otimes\pi)\Delta_1=\Delta_2\pi$$

π^* is called a dual morphism to π.
If π is a surjection, then $(\mathcal{A}_2,\Delta_2)$ is called *a compact quantum subsemigroup* in $(\mathcal{A}_1,\Delta_1)$, and π^* is an embedding.

8. Quantum subgroup

Consider compact abelian group G and a subsemigroup S in the dual group. In the classical compact quantum group $(C(G),\Delta_G)$ the comultiplication is the following for all $f\in C(G),\ x,y\in G$.

$$\Delta_G(f)(x,y)=f(x\cdot y),$$

Theorem 2.

Compact group $G=(C(G),\Delta_G)$ is a compact quantum subgroup in $QS=(\mathcal{C}^*_{red}(S),\Delta)$. There exists a non-ergodic action of G on QS, given by a C^*-dynamical system $(G,\alpha,\mathcal{C}^*_{red}(S))$. And a quantum projective spave $P=QS/G$ is a classical compact semigroup.

9. Inverse semigroup duals

Let $\mathcal{S}_{inv}$ be a category of inverse semigroups from the class defined in Section 5, morphisms are inverse semigroup morphisms. Denote by $\mathcal{QS}_{red}$ a category of compact quantum semigroups $(C^*_{red}(S),\Delta)$ for all $S\in\mathcal{S}_{inv}$.

Theorem 3.

Category $\mathcal{QS}_{red}$ is dual to $\mathcal{S}_{inv}$.

This duality extends the Pontryagin duality theorem. When $S=\Gamma$ the dual quantum semigroup is a compact group of characters G.

10. The dual algebra

Denote $\mathfrak{A}=C^*_{red}(S)$, and $\mathfrak{A}^*$ — a dual space of $\mathfrak{A}$. The comultiplication Δ generates the Banach unital algebra structure on $\mathfrak{A}^*$, with multiplication given by

$$(\phi*\psi)(A)=(\phi\otimes\psi)\Delta(A)$$

$\phi,\psi\in\mathfrak{A}^*$, $A\in\mathfrak{A}$.
Let $\widetilde{C(S)}$ be a closed linear span of T_a and T_a^* for $a\in S$, and $\widetilde{C(S)}^\perp$ – a space of functionals with zero value on $\widetilde{C(S)}$.

Theorem 4.

There exists a short exact split sequence

$$0\to\widetilde{C(S)}^\perp\to\mathfrak{A}^*\to M(G)\to 0,$$

where $M(G)$ is an algebra of regular Borel measures on group G with convolution multiplication.

11. Haar state

A Haar state in $\mathfrak{A}^*$ is a state $h\in\mathfrak{A}^*$, such that for any functional $\phi\in\mathfrak{A}^*$ we have

$$h*\phi=\phi*h=\lambda_\phi h,\ \lambda_\phi\in\mathbb{C}.$$

The Haar state is unique if it exists.

Theorem 5.

There exists a Haar state h in $\mathfrak{A}^*$.