Partial Crossed Product Description of the Cuntz-Li Algebras

Giuliano Boava

Groups, Dynamical Systems and C^*-Algebras

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Cuntz-Li Algebras: Definition

- R integral domain with finite quotients, i.e., $R/(m)$ is finite, for all $m \neq 0$ in R, which is not a field.

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- R integral domain with finite quotients, i.e., $R/(m)$ is finite, for all $m \neq 0$ in R, which is not a field.

Definition (Cuntz-Li, 2010)

The **Cuntz-Li algebra of** R, denoted by $\mathfrak{A}[R]$, is the universal C^*-algebra generated by isometries $\{s_m \mid m \in R^\times\}$ and unitaries $\{u^n \mid n \in R\}$ subject to the relations

(CL1) $s_m s_{m'} = s_{mm'}$;

(CL2) $u^n u^{n'} = u^{n+n'}$;

(CL3) $s_m u^n = u^{mn} s_m$;

(CL4) $\displaystyle\sum_{l+(m)\in R/(m)} u^l s_m s_m^* u^{-l} = 1.$

Cuntz-Li Algebras: Properties

- There is a natural projection $p_{m,m'} : R/(m') \longrightarrow R/(m)$ whenever $m \leq m'$.

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Theorem (Cuntz-Li, 2010)

$\overline{\mathrm{span}}\{u^n s_m s_m^* u^{-n} \mid m \in R^\times,\ n \in R\}$ is a commutative C^*-algebra and its spectrum is homeomorphic to $\hat{R}$.

Cuntz-Li Algebras: Properties

Theorem (Cuntz-Li, 2010)

$\mathfrak{A}[R]$ *is simple and purely infinite.*

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$\mathfrak{A}[R]$ *is simple and purely infinite.*

Theorem (Cuntz-Li, 2010)

$\mathfrak{A}[R]$ *is a crossed product by a semigroup.*

Contents

Partial Action

Definition

A **partial action** α of a (discrete) group G on a C^*-algebra $\mathcal{A}$ is a collection $(\mathcal{D}_t)_{t\in G}$ of ideals of $\mathcal{A}$ and $*$-isomorphisms $\alpha_t : \mathcal{D}_{t^{-1}} \longrightarrow \mathcal{D}_t$ such that

(PA1) $\mathcal{D}_e = \mathcal{A}$;

(PA2) $\alpha_t^{-1}(\mathcal{D}_t \cap \mathcal{D}_{s^{-1}}) \subseteq \mathcal{D}_{(st)^{-1}}$;

(PA3) $\alpha_s \circ \alpha_t(x) = \alpha_{st}(x), \ \ \forall\, x \in \alpha_t^{-1}(\mathcal{D}_t \cap \mathcal{D}_{s^{-1}})$.

Partial Crossed Product

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- $\mathcal{L}$ is a $*$-algebra with the operations $(a_s\delta_s)(a_t\delta_t) = \alpha_s(\alpha_{s^{-1}}(a_s)a_t)\delta_{st}$ and $(a_t\delta_t)^* = \alpha_{t^{-1}}(a_t^*)\delta_{t^{-1}}$.

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Definition

The **full partial crossed product** and the **reduced partial crossed product** of $\mathcal{A}$ by G through α, denoted by $\mathcal{A}\rtimes_\alpha G$ and $\mathcal{A}\rtimes_{\alpha,\mathrm{r}} G$, are the completion of $\mathcal{L}$ under certain C^*-norms.

Partial Representation

Definition

A **partial representation** π of a (discrete) group G into a unital C^*-algebra $\mathcal{B}$ is a map $\pi : G \longrightarrow \mathcal{B}$ such that, for all $s, t \in G$,

(PR1) $\pi(e) = 1$;

(PR2) $\pi(t^{-1}) = \pi(t)^*$;

(PR3) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$.

Universal Property of $\mathcal{A} \rtimes_\alpha G$

Definition

Let $\pi : G \longrightarrow \mathcal{B}$ be a partial representation of G into a unital C^*-algebra $\mathcal{B}$ and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism. We say that the pair (φ, π) is α-**covariant** if:

(COV1) $\varphi(\alpha_t(x)) = \pi(t)\varphi(x)\pi(t^{-1})$, for all $t \in G$ e $x \in \mathcal{D}_{t^{-1}}$;

(COV2) $\varphi(x)\pi(t)\pi(t^{-1}) = \pi(t)\pi(t^{-1})\varphi(x)$, for all $x \in \mathcal{A}$ e $t \in G$.

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(COV2) $\varphi(x)\pi(t)\pi(t^{-1}) = \pi(t)\pi(t^{-1})\varphi(x)$, for all $x \in \mathcal{A}$ e $t \in G$.

Proposition

If (φ, π) is α-covariant pair, then there exists a unique $*$-homomorphism $\varphi \times \pi : \mathcal{A}\rtimes_\alpha G \longrightarrow \mathcal{B}$ such that

$$(\varphi \times \pi)(a_t\delta_t) = \varphi(a_t)\pi(t), \quad \forall\, t \in G,\ \forall\, a_t \in \mathcal{D}_t.$$

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Partial Group Algebra

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Definition (Exel-Laca-Quigg, 2002)

The **partial group algebra of** G, denoted by $C_{\mathrm{p}}^*(G)$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ subject to the relations

$$\mathcal{R}_{\mathrm{p}} = \{[e] = 1\} \cup \{[t^{-1}] = [t]^*\}_{t \in G} \cup \{[s][t][t^{-1}] = [st][t^{-1}]\}_{s,t \in G}.$$

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Partial Group Algebra with Relations

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Definition (Exel-Laca-Quigg, 2002)

The **partial group algebra of G with relations** $\mathcal{R}$, denoted by $C_{\mathrm{p}}^*(G, \mathcal{R})$, is defined to be the universal C^*-algebra generated by the set $\mathcal{G}$ with the relations $\mathcal{R}_{\mathrm{p}} \cup \mathcal{R}$.

Theorems

Theorem (Exel-Laca-Quigg, 2002)

$C_{\mathrm{p}}^*(G) \cong C(X) \rtimes_\alpha G$, *where* $X = \{\xi \subseteq G \,|\, e \in \xi\}$.

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Theorem (Exel-Laca-Quigg, 2002)

$C^*_{\mathrm{p}}(G) \cong C(X) \rtimes_\alpha G$, where $X = \{\xi \subseteq G \,|\, e \in \xi\}$.

Theorem (Exel-Laca-Quigg, 2002)

$C^*_{\mathrm{p}}(G, \mathcal{R}) \cong C(\Omega) \rtimes_\alpha G$, where
$\Omega = \{\xi \in X \,|\, f(t^{-1}\xi) = 0, \ \forall f \in \mathcal{R}, \ \forall t \in \xi\}$.

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- Set of relations $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where $\mathcal{R}_1 = \{\varepsilon_{(n,1)} = 1 \mid n \in R\}$, $\mathcal{R}_2 = \left\{\varepsilon_{(0,\frac{1}{m})} = 1 \,\middle|\, m \in R^{\times}\right\}$ and $\mathcal{R}_3 = \left\{ \sum_{l+(m) \in R/(m)} \varepsilon_{(l,m)} = 1 \,\middle|\, m \in R^{\times} \right\}$.

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- R integral domain with finite quotients which is not a field.
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- Set of relations $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where
  $\mathcal{R}_1 = \left\{\varepsilon_{(n,1)} = 1 \mid n \in R\right\}$, $\mathcal{R}_2 = \left\{\varepsilon_{\left(0,\frac{1}{m}\right)} = 1 \,\middle|\, m \in R^{\times}\right\}$
  and $\mathcal{R}_3 = \left\{ \sum_{l+(m)\in R/(m)} \varepsilon_{(l,m)} = 1 \,\middle|\, m \in R^{\times} \right\}$.
- Partial group algebra $C^*_{\mathrm{p}}(K \rtimes K^{\times}, \mathcal{R})$.

Partial Group Algebra Description

Proposition (B.-Exel, 2013)

There exists a $*$*-isomorphism*

$$\begin{array}{rcl} \mathfrak{A}[R] & \longrightarrow & C^*_{\mathrm{p}}(K \rtimes K^\times, \mathcal{R}) \\ u^n & \longmapsto & [n, 1] \\ s_m & \longmapsto & [0, m] \\ s^*_{m'} u^n s_m & \longleftarrow & \left[\frac{n}{m'}, \frac{m}{m'}\right]. \end{array}$$

Sketch of the Proof

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(s^*_{p'} u^q s_p)(s^*_{m'} u^n s_m)(s^*_m u^{-n} s_{m'}) \longleftarrow [s][t][t]^*.
\end{aligned}
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Partial Crossed Product Description

Corollary

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- Extend the partial order from $R^\times$ to $K^\times$. For $w, w' \in K^\times$, $w \leq w'$ if there exists $r \in R$ such that $w' = wr$.
- Consider the fractional ideals $(w) = wR$, $w \in K^\times$.

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- Consider the fractional ideals $(w) = wR$, $w \in K^{\times}$.
- There is a natural projection $p_{w,w'} : (R + (w'))/(w') \longrightarrow (R + (w))/(w)$ whenever $w \leq w'$.

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- $\hat{R}_K = \varprojlim\{(R + (w))/(w),\ p_{w,w'}\}$.
- Clearly, $\hat{R}_K \cong \hat{R}$.

Partial Crossed Product Description

Proposition

Ω is homeomorphic to $\hat{R}_K$ and, hence, to $\hat{R}$.

Partial Crossed Product Description

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Corollary

There exists a $*$-isomorphism

$$\begin{array}{rcl} \mathfrak{A}[R] & \longrightarrow & C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times \\ u^n & \longmapsto & 1\delta_{(n,1)} \\ s_m & \longmapsto & 1_{(0,m)}\delta_{(0,m)}, \end{array}$$

where $1_{(u,w)}$ is the characteristic function of $\{(u_{w'} + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) = u + (w)\}$.

Partial Crossed Product Description

Proposition

The partial action θ on $\hat{R}_K$ is topologically free and minimal.

Partial Crossed Product Description

Proposition

The partial action θ on $\hat{R}_K$ is topologically free and minimal.

Corollary

$\mathfrak{A}[R]$ *is simple.*

Contents

Bost-Connes Algebra

Definition (Bost-Connes, 1995)

The **Bost-Connes algebra**, denoted by $C_{\mathbb{Q}}$, is the universal C^*-algebra generated by isometries $\{\mu_m \mid m \in \mathbb{N}^*\}$ and unitaries $\{e_\gamma \mid \gamma \in \mathbb{Q}/\mathbb{Z}\}$ subject to the relations

(BC1) $\mu_m \mu_{m'} = \mu_{mm'}$;

(BC2) $\mu_m \mu^*_{m'} = \mu^*_{m'} \mu_m$, if $(m, m') = 1$;

(BC3) $e_\gamma e_{\gamma'} = e_{\gamma+\gamma'}$;

(BC4) $e_\gamma \mu_m = \mu_m e_{m\gamma}$;

(BC5) $\mu_m e_\gamma \mu^*_m = \frac{1}{m} \sum e_\delta$, where the sum is taken over all $\delta \in \mathbb{Q}/\mathbb{Z}$ such that $m\delta = \gamma$.

Partial Crossed Product Description

- Taking $R = \mathbb{Z}$, we have $\mathfrak{A}[\mathbb{Z}] \cong C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes_{\alpha} \mathbb{Q} \rtimes \mathbb{Q}^*$.

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- There is a natural embedding $\mathbb{Q}^*_+ \hookrightarrow \mathbb{Q} \rtimes \mathbb{Q}^*$ given by $q \mapsto (0, q)$.
- Restricting α to $\mathbb{Q}^*_+$, we obtain the partial crossed product $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes \mathbb{Q}^*_+$.

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- There is a natural embedding $\mathbb{Q}_+^* \hookrightarrow \mathbb{Q} \rtimes \mathbb{Q}^*$ given by $q \mapsto (0, q)$.
- Restricting α to $\mathbb{Q}_+^*$, we obtain the partial crossed product $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes \mathbb{Q}_+^*$.

Theorem

The Bost-Connes algebra $C_{\mathbb{Q}}$ is $*$-isomorphic to $C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes \mathbb{Q}_+^*$.

Partial Crossed Product Description

One side of the isomorphism is given by

$$
\begin{aligned}
C_{\mathbb{Q}} &\longrightarrow C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes \mathbb{Q}_+^* \\
\mu_m &\longmapsto 1_{(0,m)}\delta_m \\
e(n/m) &\longmapsto \sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(-\frac{ln}{m}\cdot 2\pi i\right) 1_{(l,m)}\delta_1 .
\end{aligned}
$$

Partial Crossed Product Description

The other side is given by

$$
\begin{array}{rcl}
C(\hat{\mathbb{Z}}_{\mathbb{Q}}) \rtimes \mathbb{Q}_+^* & \longrightarrow & C_{\mathbb{Q}} \\
\delta_{m/m'} & \longmapsto & \mu_{m'}^* \mu_m \\
1_{(n/m',m/m')} & \longmapsto & \displaystyle\frac{1}{m} \sum_{l+(m)\in\mathbb{Z}/(m)} \exp\left(\frac{nl}{m} \cdot 2\pi i\right) e\left(\frac{lm'}{m}\right).
\end{array}
$$

THE END!