Hypergroupoids and their C*-algebras

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Introduction

Rohit Holkar visited Orleans last March and brought to my attention an observation and a question.

Observation: Let X be a proper G-space, where G is a groupoid. If X is not free, then the quotient $(X * X)/G$ is no longer a groupoid.

Question: What kind of object is it?

The answer is that it is a hypergroupoid and that there is a nice theory of locally compact hypergroupoids with Haar system which extends the case of groupoids.

Groupoid equivalence

Recall the usual setting of groupoid equivalence:

Definition

Two topological groupoids G, H are said to be equivalent if there exists a topological space X endowed with a left principal action of G and a right principal action of H such that the moment maps give identification maps

$$r : X/H \to G^{(0)} \quad \text{and} \quad s : G\backslash X \to H^{(0)}.$$

Then

$$G \simeq (X * X)/H, \qquad H \simeq G\backslash(X * X)$$

Thus, if X is a left principal G-space, $G\backslash(X * X)$ is a groupoid equivalent to G and this is the most general situation of groupoid equivalence.

Proper G-space

Suppose now that X is only a proper G-space. What kind of object is

$$H = G \backslash (X * X) \qquad ?$$

Let us be more precise. We assume that

- G carries a Haar system λ
- X carries a G-invariant r-system α; we say that (X, α) is a measured proper G-space.

Then the usual formulas of [MRW 87] or [R 87] define

- a convolution product on $C_c(H)$,
- a structure of $(C_c(G), C_c(H))$-bimodule on $C_c(X)$ with compatible left and right inner products.

Convolution formulas

Given measured proper G-spaces $(X,\alpha),(Y,\beta),(Z,\gamma)$ and $f \in C_c((X*Y)/G), g \in C_c((Y*Z)/G)$, we define

$$(\alpha,f,\beta)(\beta,g,\gamma) = (\alpha, f *_\beta g, \gamma)$$

where the convolution product is given by:

$$f *_\beta g[x,z] = \int f[x,y]g[y,z]d\beta^{rx(x)}(y)$$

One defines also

$$(\alpha,f,\beta)^* = (\beta,f^*,\alpha)$$

where the involution is given by $f^*[y,x] = \overline{f[x,y]}$.

The hypergroupoid C*-algebras

The full and the reduced norms on $C_c(G,\lambda)$ induce a norm on $C_c(H)$ and on $C_c(X)$ via the disintegration theorem. We thus obtain:

- the C*-algebras $C^*(H) = (\alpha, \alpha)$ and $C_r^*(H) = (\alpha, \alpha)_r$
- the C*-bimodules $C^*(X) = (\lambda, \alpha)_r$ and $C_r^*(X) = (\lambda, \alpha)_r$.

Definition

We say that $H = G\backslash(X * X)$ is a spatial hypergroupoid and that $C^*(H)$ and $C_r^*(H)$ are their full and reduced C*-algebras.

An elementary example

One of the simplest example one can think arises from the flip on the real line:

- $G = \mathbb{Z}_2$;
- $X = \mathbb{R}$ where $\mathbb{Z}_2$ acts by the flip $x \mapsto -x$ and α is the Lebesgue measure.

Identifying $H^{(0)} = X/G$ and $\mathbb{R}_+$ via $[x] = x^2$ and $H = X \times X/G$ and $\mathbb{C}$ via $[x, y] = (x + iy)^2$, we obtain on $C_c(\mathbb{C})$:

$$(f * g)(x^2 - z^2 + 2ixz) = \int f(x^2 - y^2 + 2ixy)g(y^2 - z^2 + 2iyz)dy$$

$$f^*(w) = \overline{f(-\overline{w})}$$

The source and range maps foliate $\mathbb{C}$ into families of parabolas with one degenerate parabola which is a half-line.

More examples

- Z is a free G-space $\Leftrightarrow$ H is a groupoid;
- Z is a transitive G-space $\Leftrightarrow$ H is a hypergroup.

1. A pair (G, K), where G is a locally compact group and K is a compact subgroup. The homogeneous space $X = G/K$ is a proper G-space equipped with an invariant measure α. Then, $(X \times X)/G$ is the double coset hypergroup $K \backslash G/K$. The full and the regular representations of G yield respectively the full and the reduced C*-algebras of this hypergroup.

Hecke C*-algebras

2. A Hecke pair (Γ, Λ) consists of
   - a countable discrete group Γ;
   - an almost-normal subgroup Λ.

This means that the left action of Λ on Γ/Λ has finite orbits. The Schlichting completion produces a totally disconnected locally compact group $G = \overline{\Gamma}$ and a compact subgroup $K = \overline{\Lambda}$ such that $C_c(\Lambda\backslash\Gamma/\Lambda) = C_c(K\backslash G/K)$ as $*$-algebras. This fits into the above general framework and gives some insight about natural C*-completions.

Groupoid quotients

3. A pair (G, K), where G is a locally compact groupoid with Haar system and K is a proper subgroupoid. Assume that the map $r : G/K \to G^{(0)}$ has a G-invariant system of measures α. Then $(X = G/K, \alpha)$ is a measured proper G-space. Thus we can construct the hypergroupoid $(X * X)/G = K \backslash G/K$ and its full and its reduced C*-algebras.

M. Laca, N. Larsen and S. Neshveyev [Non-Commutative Geometry, 2007] consider the case of a semi-direct groupoid $G = \Gamma \ltimes Y$ where a group Γ acts on a space Y and $H = \Lambda \ltimes Y$ where Λ is a subgroup of Γ acting properly on Y.

Locally compact hypergroups

Our spatial hypergroupoids $X * X/G$ should fit into a general theory of locally compact hypergroupoids with Haar systems.

The theory of abstract locally compact hypergroups is now well-established (Dunkl, Jewett, Spector are the names usually associated with it). This theory emphasizes the measure algebra $M(H)$ of a locally compact hypergroup and not the convolution function algebra $C_c(H)$. The earliest reference I have found to the hypergroup C*-algebra $C^*(H)$ is in K. Tzanev's thesis (2000).

An earlier appearance of the hypergroup C*-algebra $C^*(H)$ was given to me by S. Echterhoff during the workshop : P. Hermann, Induced representations of hypergroups, Math. Z. 211, 687-699 (1992).

It should be said here that the existence of a Haar measure on an arbitrary locally compact hypergroup has not been established yet. At this stage, it is better to assume its existence.

Axioms

The idea is very simple: we take the usual definition of a locally compact groupoid H but where the product of two composable elements x, y is no longer a third element but a probability measure $x * y$ with compact support. Thus we have the hypergroupoid H, its unit space $H^{(0)}$ identified to a subset of H, the range and source maps $r, s : H \to H^{(0)}$, the inverse map $x \mapsto x^{-1}$. We denote by $H^{(2)}$ the set of composable pairs . We assume that H is locally compact Hausdorff and make the usual continuity assumptions.

The product

We denote by $P(H)$ the space of probability measures on H. The product is a map $m : H^{(2)} \to P(H)$ such that

1. the support of $m(x, y)$ is a compact subset of $H^{r(x)}_{s(y)}$;
2. for all f bounded and Borel on $H^{(2)}$, the map $(x, y) \mapsto f(x * y) := \int f \, dm(x, y)$ is Borel;
3. for all $(x, y, z) \in H^{(3)}$, we have $\int m(x, .)dm(y, z) = \int m(., z)dm(x, y)$;
4. for all $x \in H$, $m(r(x), x) = m(x, s(x)) = \delta_x$;
5. for all $(x, y) \in H^{(2)}$, $m(x, y)^{-1} = m(y^{-1}, x^{-1})$.

Continuity assumptions

We need some assumptions expressing the continuity of the product. Examples of spatial hypergroupoids show that the natural condition:

$\forall f \in C_c(H), (x, y) \in H^{(2)} \mapsto f(x * y)$ is continuous

is too strong.

We define the left translation operator $L(x)f(y) := f(x^{-1} * y)$.

We require

- $L(x)$ maps $C_c(H^{s(x)})$ into $C_c(H^{r(x)})$;
- for all $f \in C_c(H)$ and $\epsilon > 0$, there exists a neighborhood U of $H^{(0)}$ in H such that $|f(x) - f(y^{-1})| \leq \epsilon$ as soon as the support of $m(x, y)$ meets U.

Haar systems

Definition

A Haar system on a locally compact hypergroupoid H is a system of Radon measures $\lambda = (\lambda^u)$ for the range map such that

1. for all $f \in C_c(H)$, $u \in H^{(0)} \mapsto \int f \, d\lambda^u$ is continuous;
2. for all $f \in C_c(H)$ and all $x \in H$, $\int f(x * y) d\lambda^{s(x)}(y) = \int f(y) d\lambda^{r(x)}(y)$.

Locally compact groups, commutative locally compact hypergroups, compact hypergroups are known to have a Haar measure, which is unique. Etale locally compact hypergroupoids also have a Haar system.

Haar systems for spatial hypergroupoids

Theorem

Let G be a locally compact groupoid and (X, α) a measured proper G-space. Then $H = (X * X)/G$ is a locally compact hypergroupoid with Haar system.

The disintegration of $\alpha^{r(x)}$ along the map $\varphi^x : X^{r(x)} \to H^{[x]}$ sending y to $[x, y]$ provides both the probability measures $m[x, y, z]$ defining the product and the measure $\lambda^{[x]} = \varphi^x_* \alpha^{r(x)}$. Explicitly,

$$f[x, y, z] = \int f[\zeta x, z] d\beta^y(\zeta)$$

where β^y is the normalized Haar measure of the isotropy group $G(y)$.

The $*$-algebra $C_c(H)$

The convolution product and the involution are defined by the usual formulas: for $f, g \in C_c(H)$, we set

$$(f * g)(x) = \int f(x * y) g(y^{-1}) d\lambda^{s(x)}(y)$$

$$f^*(x) = \overline{f(x^{-1})}$$

We require

- $\forall f, g \in C_c(H)$, $f * g \in C_c(H)$.

Proposition

Endowed with these operations and the inductive limit topology, $C_c(H)$ is a topological $*$-algebra.

The I-norm

As usual, the I-norm of $f \in C_c(H)$ is

$$\|f\|_I = \max\left(\sup_{u \in H^{(0)}} \int |f| d\lambda^u, \sup_{u \in H^{(0)}} \int |f^*| d\lambda^u \right)$$

It satisfies

$$\|f * g\|_I \leq \|f\|_I \|g\|_I \qquad \|f^*\| = \|f\|$$

The regular representations

We fix $u \in H^{(0)}$. Let $f \in C_c(H)$. Given $\xi \in C_c(H_u)$, we define $L_u(f)\xi \in C_c(H_u)$ by

$$L_u(f)\xi(x) = \int f(x * y)\xi(y^{-1})d\lambda^u(y)$$

We endow $C_c(H_u)$ with the scalar product $(\xi|\eta)_u = \int \overline{\xi(x)}\eta(x)d\lambda_u(x)$. Its completion is the Hilbert space $L^2(H_u, \lambda_u)$.

Proposition

For all $u \in H^{(0)}$, L_u is a $*$-representation of the $*$-algebra $C_c(H)$ on the Hilbert space $L^2(H_u, \lambda_u)$. Moreover $\|L_u(f)\| \le \|f\|_I$ for all $f \in C_c(H)$.

The reduced C*-algebra

We define the reduced norm as

$$\|f\|_r = \sup\{\|L_u(f)\| : u \in H^{(0)}\}$$

Definition

The reduced C*-algebra $C_r^*(H)$ is the completion of $C_c(H)$ for the reduced norm.

The full C*-algebra

We define the full norm

$$\|f\| = \sup\{\|L(f)\| : L \text{ non-degenerate and I-bounded } *\text{-representation}\}$$

Definition

The full C*-algebra $C^*(H)$ is the completion of $C_c(H)$ for the full norm.

Representations of a hypergroupoid

Definition

Let H be a Borel hypergroupoid. A Borel H-Hilbert bundle is a Borel Hilbert bundle $p : \mathcal{H} \to H^{(0)}$ and a Borel map

$$H * \mathcal{H} \to \mathcal{H} : \quad (x, \xi) \mapsto L(x)\xi$$

such that

1. $p(L(x)\xi) = r(x)$;
2. $L(x) : \mathcal{H}_{s(x)} \to \mathcal{H}_{r(x)}$ is a linear contraction;
3. if $u \in H^{(0)}$, $L(u) : \mathcal{H}_u \to \mathcal{H}_u$ is the identity;
4. $L(x * y) = L(x)L(y)$;
5. $L(x^{-1}) = L(x)^*$.

Quasi-invariant measures

Same definition as for Borel groupoids with Haar system:

Definition

A quasi-invariant measure of a Borel hypergroupoid with Haar system (H, λ) is a measure μ on $H^{(0)}$ such that $\mu \circ \lambda$ and $(\mu \circ \lambda)^{-1}$ are equivalent.

Its module is the Radon-Nikodym derivative $D = d(\mu \circ \lambda)^{-1}/d(\mu \circ \lambda)$. It satisfies

$$D(x * y) = D(x)D(y) \quad a.e.$$

The integral of a representation

Given a Borel H-Hilbert bundle $\mathcal{H}$ and a quasi-invariant measure μ for a locally compact hypergroupoid with Haar system (H, λ), we define for $f \in C_c(H)$

$$L(f) : (L^2(H^{(0)}, \mu, \mathcal{H})) \to (L^2(H^{(0)}, \mu, \mathcal{H}))$$

such that for all $\xi, \eta \in L^2(H^{(0)}, \mu, \mathcal{H})$:

$$\langle \xi, L(f)\eta \rangle = \int f(x)(\xi \circ r(x), L(x)\eta \circ s(x))D^{-1/2}(x)d(\mu \circ \lambda)(x)$$

Proposition

The above formula defines a non-degenerate and I-bounded $*$-representation of $C_c(H)$ in the Hilbert space $L^2(H^{(0)}, \mu, \mathcal{H})$.

Disintegration theorem

The same proof as in the case of a locally compact groupoid yields the following disintegration theorem:

Theorem

Let (H, λ) be a second countable locally compact hypergroupoid with Haar system. Then every non-degenerate and I-bounded $*$-representation of $C_c(H)$ in a Hilbert space is equivalent to the integral of a representation of H with respect to some quasi-invariant measure.

References

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