A homology theory for Smale spaces

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Hyperbolicity

An invertible linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ is *hyperbolic* if $\mathbb{R}^d = E^s \oplus E^u$, T-invariant, $C > 0, 0 < \lambda < 1$,

$$\|T^n v\| \leq C\lambda^n \|v\|, \quad n \geq 1 \quad v \in E^s,$$
$$\|T^{-n} v\| \leq C\lambda^n \|v\|, \quad n \geq 1 \quad v \in E^u,$$

Same definition replacing $\mathbb{R}^d$ by a vector bundle (over compact space).

M compact manifold, $\varphi : M \to M$ diffeomorphism is *Anosov* if $D\varphi : TM \to TM$ is hyperbolic.

Smale: M, φ *Axiom A*: replace TM above by $TM|_{NW(\varphi)} = E^s \oplus E^u$, where $NW(\varphi)$ is the set of non-wandering points. But $NW(\varphi)$ is usually a fractal, not a submanifold.

Smale spaces (D. Ruelle)

(X, d) compact metric space,

$\varphi : X \to X$ homeomorphism $0 < \lambda < 1$,

For x in X and $\epsilon > 0$ and small, there is a local stable set $X^s(x, \epsilon)$ and a local unstable set $X^u(x, \epsilon)$:

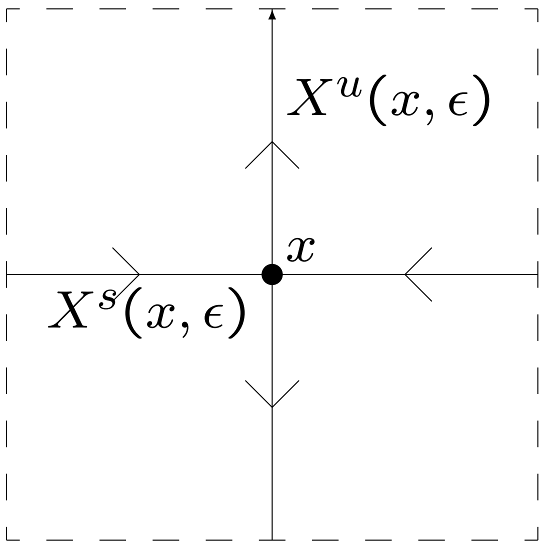
1. $X^s(x, \epsilon) \times X^u(x, \epsilon)$ is homeomorphic to a neighbourhood of x,

2. φ-invariance,

3.

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

That is, we have a local picture:



Global stable and unstable sets:

$$
\begin{aligned}
X^s(x) &= \{y \mid \lim_{n\to+\infty} d(\varphi^n(x), \varphi^n(y)) = 0\} \\
X^u(x) &= \{y \mid \lim_{n\to+\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}
\end{aligned}
$$

These are equivalence relations.

$X^s(x,\epsilon) \subset X^s(x)$, $X^u(x,\epsilon) \subset X^u(x)$.

Example 1

The linear map $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is hyperbolic. Let $\gamma > 1$ be the Golden mean,

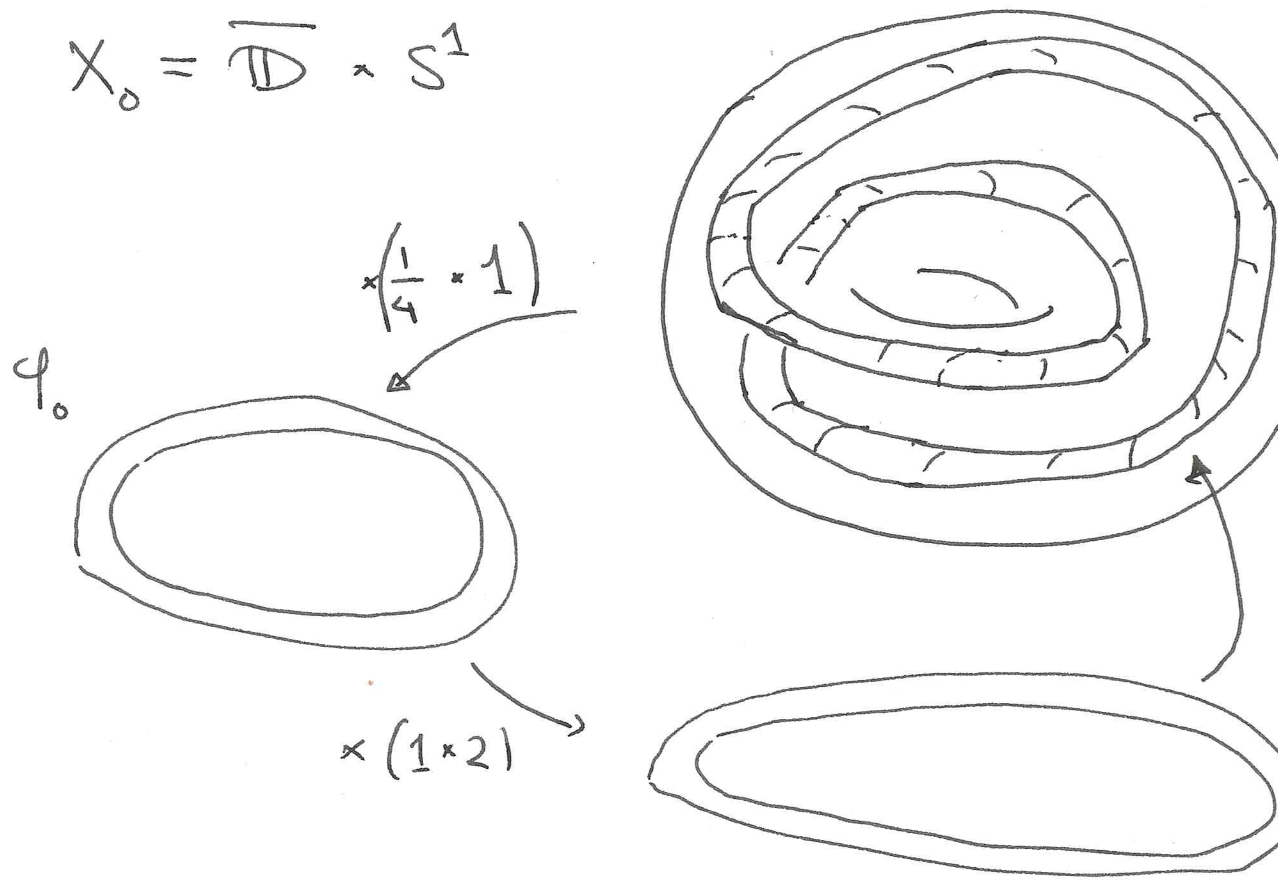
$$
\begin{aligned}
(\gamma, 1)A &= \gamma(\gamma, 1) \\
(-1, \gamma)A &= -\gamma^{-1}(-1, \gamma)
\end{aligned}
$$

As $det(A) = -1$, it induces a homeomorphism of $\mathbb{R}^2/\mathbb{Z}^2$ which is Anosov.

X^s and X^u are Kronecker foliations with lines of slope $-\gamma^{-1}$ and γ.

Example 2: 2^∞-solenoid

$$X_0 = \overline{\mathbb{D}} \times S^1$$



$$X = \bigcap_{n \geq 0} \varphi_0^n(X_0), \quad \varphi = \varphi_0 | X$$

$$X^s((x,y),\varepsilon) \cong \overline{\mathbb{D}} \times \{y\} \cap X$$ Cantor

$$X^u((x,y),\varepsilon) \cong \{x\} \times (y-\varepsilon, y+\varepsilon)$$

Example 3: Shifts of finite type (SFTs)

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then we have the shift space and shift map:

$$\begin{aligned} \Sigma_G &= \{(e^k)_{k=-\infty}^{\infty} \mid e^k \in G^1, \\ &\quad i(e^{k+1}) = t(e^k), \text{ for all } n\} \\ \sigma(e)^k &= e^{k+1}, \text{ "left shift"} \end{aligned}$$

The local product structure is given by

$$\Sigma^s(e,1) = \{(\ldots, *, *, *, e^0, e^1, e^2, \ldots)\}$$
$$\Sigma^u(e,1) = \{(\ldots, e^{-2}, e^{-1}, e^0, *, *, *, \ldots)\}$$

Smales spaces have a large supply of periodic points and it is interesting to count them.

Adjacency matrix of G: $G^0 = \{1, 2, \ldots, N\}$, A_G is $N \times N$ with

$$(A_G)_{i,j} = \#\text{edges from } i \text{ to } j$$

Theorem 1. *Let A_G be the adjancency matrix of the graph G. For any $p \geq 1$, we have*

$$\#\{e \in \Sigma_G \mid \sigma^p(e) = e\} = Tr(A_G^p).$$

This is reminiscent of the Lefschetz fixed-point formula for smooth maps of compact manifolds.

Question 2. *Is the right hand side actually the result of σ acting on some homology theory of (Σ_G, σ)?*

Positive answers by Bowen-Franks and Krieger.

Krieger's invariants for SFT's

W. Krieger defined invariants, which we denote by $D^s(\Sigma_G, \sigma), D^u(\Sigma_G, \sigma)$, for shifts of finite type by considering stable and unstable equivalence as groupoids and taking its groupoid C^*-algebra:

$$K_0(C^*(X^s)), K_0(C^*(X^s))$$

In this case, these are both AF-algebras and

$$D^s(\Sigma_G, \sigma) = \lim \mathbb{Z}^N \xrightarrow{A_G} \mathbb{Z}^N \xrightarrow{A_G} \cdots$$

(For the unstable, replace A_G with A_G^T.) Each comes with a canonical automorphism.

Returning to Smale spaces . . .

Bowen's Theorem

Theorem 3 (Bowen)**.** *For a non-wandering Smale space, (X, φ), there exists a SFT (Σ, σ) and*

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

with $\pi \circ \sigma = \varphi \circ \pi$, continuous, surjective and finite-to-one.

First, this means that SFT's have a special place among Smale spaces. Secondly, one can try to understand (X, φ) by investigating (Σ, σ). For example, they will have the same entropy. Of course, (Σ, σ) is not unique.

A. Manning used Bowen's Theorem to provide a formula counting the number of periodic points for (X, φ).

For $N \geq 0$, define

$$\Sigma_N(\pi) = \{(e_0, e_1, \ldots, e_N) \mid \pi(e_n) = \pi(e_0), 0 \leq n \leq N\}.$$

For all $N \geq 0$, $(\Sigma_N(\pi), \sigma)$ is also a shift of finite type. Observe that S_{N+1} acts on $\Sigma_N(\pi)$.

Theorem 4 (Manning)**.** *For a non-wandering Smale space (X, φ), (Σ, σ) as above and $p \geq 1$, we have*

$$\#\{x \in X \mid \varphi^p(x) = x\} = \sum_N (-1)^N Tr(\sigma_*^p : D^s(\Sigma_N(\pi))^{alt} \to D^s(\Sigma_N(\pi))^{alt}).$$

Question 5 (Bowen)**.** *Is there a homology theory for Smale spaces $H_*(X, \varphi)$ which provides a Lefschetz formula, counting the periodic points?*

In fact, the groups $D^s(\Sigma_N(\pi))^{alt}$ appear to be giving a chain complex.

Idea: for $0 \leq n \leq N$, let $\delta_n : \Sigma_N(\pi) \to \Sigma_{N-1}(\pi)$ be the map which deletes entry n.

Let $(\delta_n)_* : D^s(\Sigma_N(\pi))^{alt} \to D^s(\Sigma_{N-1}(\pi))^{alt}$ be the induced map and $\partial = \sum_{n=0}^{N}(-1)^n(\delta_n)_*$ to make a chain complex.

This is wrong: a map

$$\rho : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

between shifts of finite type does *not* always induce a group homomorphism between Krieger's invariants.

While it is true that ρ will map $R^s(\Sigma)$ to $R^s(\Sigma')$ the functorial properties of the construction of groupoid C^*-algebras is subtle.

Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces. For every y in Y, we have $\pi(Y^s(y)) \subseteq X^s(\pi(y))$.

Definition 6. *π is s-bijective if $\pi : Y^s(y) \to X^s(\pi(y))$ is bijective, for all y.*

Theorem 7. *If π is s-bijective then $\pi : Y^s(y, \epsilon) \to X^s(\pi(y), \epsilon')$ is a local homeomorphism.*

Theorem 8. *Let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map between SFT's.*

If π is s-bijective, then there is a map

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma).$$

If π is u-bijective, then there is a map

$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma).$$

Bowen's $\pi : (\Sigma, \sigma) \to (X, \varphi)$ is not s-bijective or u-bijective if X is a torus, for example.

A better Bowen's Theorem

Let (X, φ) be a Smale space. We look for a Smale space (Y, ψ) and a factor map

$$\pi_s : (Y, \psi) \to (X, \varphi)$$

satisfying:

1. π_s is s-bijective,

2. $dim(Y^u(y, \epsilon)) = 0$.

That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

This is a "one-coordinate" version of Bowen's Theorem.

Similarly, we look for a Smale space (Z, ζ) and a factor map $\pi_u : (Z, \zeta) \to (X, \varphi)$ satisfying $dim(Z^s(z, \epsilon)) = 0$, and π_u is u-bijective.

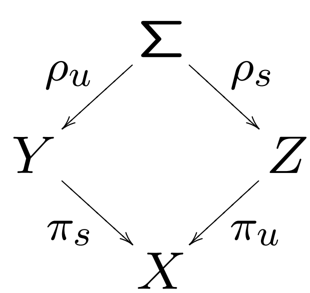
We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ a *s/u-bijective pair* for (X, φ).

Theorem 9. *If (X, φ) is a non-wandering Smale space, then there exists an s/u-bijective pair.*

Consider the fibred product:

$$\Sigma = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$$

with



$\rho_s(y, z) = z$ is s-bijective, $\rho_u(y, z) = y$ is u-bijective. Hence, Σ is a SFT.

For $L, M \geq 0$, we define

$$\begin{aligned} \Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ &\quad y_l \in Y, z_m \in Z, \\ &\quad \pi_s(y_l) = \pi_u(z_m)\}. \end{aligned}$$

Each of these is a SFT.

Moreover, the maps

$$\begin{aligned} \delta_{l,} : \ \Sigma_{L,M} &\to \ \Sigma_{L-1,M}, \\ \delta_{,m} : \ \Sigma_{L,M} &\to \ \Sigma_{L,M-1} \end{aligned}$$

which delete y_l and z_m are s-bijective and u-bijective, respectively.

This is the key point! We have avoided the issue which caused our earlier attempt to get a chain complex to fail.

We get a double complex:

$$
\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2})^{alt} & \longleftarrow & D^s(\Sigma_{1,2})^{alt} & \longleftarrow & D^s(\Sigma_{2,2})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1})^{alt} & \longleftarrow & D^s(\Sigma_{1,1})^{alt} & \longleftarrow & D^s(\Sigma_{2,1})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,0})^{alt} & \longleftarrow & D^s(\Sigma_{1,0})^{alt} & \longleftarrow & D^s(\Sigma_{2,0})^{alt} & \longleftarrow &
\end{array}
$$

$$
\begin{aligned}
\partial^s_N : \quad & \oplus_{L-M=N} D^s(\Sigma_{L,M})^{alt} \\
\to \quad & \oplus_{L-M=N-1} D^s(\Sigma_{L,M})^{alt}
\end{aligned}
$$

$$\partial^s_N = \sum_{l=0}^{L} (-1)^l \delta^s_{l,} + \sum_{m=0}^{M+1} (-1)^{m+M} \delta^{s*}_{,m}$$

$$H^s_N(\pi) = \ker(\partial^s_N)/Im(\partial^s_{N+1}).$$

Recall: beginning with (X, φ), we select an s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta\pi_u)$ construct the double complex and compute $H^s_N(\pi)$.

Theorem 10. *The groups $H^s_N(\pi)$ do not depend on the choice of s/u-bijective pair π.*

From now on, we write $H^s_N(X, \varphi)$.

Theorem 11. *The functor $H^s_*(X, \varphi)$ is covariant for s-bijective factor maps, contravariant for u-bijective factor maps.*

Theorem 12. *The groups $H^s_N(X, \varphi)$ are all finite rank and non-zero for only finitely many $N \in \mathbb{Z}$.*

We can regard $\varphi : (X, \varphi) \to (X, \varphi)$, which is both s and u-bijective and so induces an automorphism of the invariants.

Theorem 13. *(Lefschetz Formula) Let (X, φ) be any non-wandering Smale space and let $p \geq 1$.*

$$
\begin{aligned}
\sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^p : & \quad H^s_N(X, \varphi) \otimes \mathbb{Q} \\
\to & \quad H^s_N(X, \varphi) \otimes \mathbb{Q}] \\
= & \quad \#\{x \in X \mid \varphi^p(x) = x\}
\end{aligned}
$$

Example 1: Shifts of finite type

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is an s/u-bijective pair.

The double complex D_a^s is:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow &
\end{array}$$

and $H_0^s(\Sigma, \sigma) = D^s(\Sigma)$ and $H_N^s(\Sigma, \sigma) = 0, N \neq 0$.

Example 2: $\dim(\mathbf{X^s(x,\epsilon)}) = \mathbf{0}$.

(As an example, the solenoid we saw in example 2.)

We may find a SFT and s-bijective map

$$\pi_s : (\Sigma, \sigma) \to (X, \varphi).$$

The $Y = \Sigma, Z = X$ is an s/u-bijective pair and the double complex D^s is:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_0)^{alt} & \leftarrow & D^s(\Sigma_1)^{alt} & \leftarrow & D^s(\Sigma_2)^{alt} & \leftarrow &
\end{array}$$

Example 2': $(\mathbf{X}, \varphi) = \mathbf{2}^{\infty}$**-solenoid** (Bazett-P.)

An s/u-bijective pair is $Y = \{0, 1\}^{\mathbb{Z}}$, the full 2-shift, $Z = X$ and the double complex D^s is

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}[1/2] & \leftarrow & \mathbb{Z} & \leftarrow & 0 & \leftarrow &
\end{array}$$

and we get

$$H_0^s(X, \varphi) \cong \mathbb{Z}[1/2], H_1^s(X, \varphi) \cong \mathbb{Z},$$

$H_N^s(\Sigma_G, \sigma) = 0, N \neq 0, 1.$

Generalized 1-solenoids (Williams, Yi, Thomsen): Amini, P, Saeidi Gholikandi.

Example 3: Our Anosov example (Bazett-P.):

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

The double complex D^s looks like:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^2 & \leftarrow & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^3 & \leftarrow & \mathbb{Z}^2 & \leftarrow & 0 & \leftarrow &
\end{array}$$

and

N	$H^s_N(X,\varphi)$	φ^s
-1	$\mathbb{Z}$	1
0	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
1	$\mathbb{Z}$	$-1.$