Westfälische Wilhelms-Universität Münster

Groups, Geometry & Actions

# C*-Algebras Associated to Irreversible Algebraic Dynamics

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## Setting

**Irreversible algebraic dynamics** $(G,\mathcal{P},\theta)$**:**
Let $G$ be a discrete, abelian group and $\mathcal{P}$ a semigroup with unit $e$ isomorphic to $\mathbb{N}^k$ or $\bigoplus_{\mathbb{N}} \mathbb{N}$. Suppose further:

- $\mathcal{P} \overset{\theta}{\curvearrowright} G$ is an action by group monomorphisms with **finite cokernel**,
- $p \wedge q = e$ in $\mathcal{P} \iff \theta_p(G) + \theta_q(G) = G$ (**independence**),
- $\bigcap_{p\in\mathcal{P}} \theta_p(G) = \{0\}$ (**exactness**).

**C*-algebraic construction:**
Let $\mathcal{O}\,[G,\mathcal{P},\theta]$ be the universal C*-algebra generated by

- a unitary representation $(u_g)_{g\in G}$ of $G$ and
- an isometric representation $(s_p)_{p\in\mathcal{P}}$ of the semigroup $\mathcal{P}$

subject to the relations :

| | | |
|---|---|---|
| (CNP 1) | $s_p^* s_q = s_q s_p^*$ | for all rel. prime $p,q\in\mathcal{P}$. |
| (CNP 2) | $s_p u_g = u_{\theta_p(g)} s_p$ | for all $p\in\mathcal{P},\ g\in G$. |
| (CNP 3) | $\sum_{[g]\in G/\theta_p(G)} e_{p,g} = 1$ | for all $p\in\mathcal{P}$, |

where $e_{p,g} = u_g s_p s_p^* u_g^*$.

## Motivation

The case of a single endomorphism has been studied by Joachim Cuntz and Anatoly Vershik in [2], where they exhibit several interesting features of the algebra. Additionally, they considered an analogous construction for a polymorphism given by two independent endomorphisms. This revealed signs that multiply generated systems may yield intriguing algebras that go beyond the scope of singly generated systems.

In [3], Jeong Hee Hong, Nadia S. Larsen and Wojciech Szymański introduced a notion of finite type product systems of Hilbert bimodules in order to study the KMS structure on the corresponding Nica-Toeplitz algebras. It is already implicit in [3] that irreversible algebraic dynamics give rise to optimal examples for these product systems of finite type.

## Examples

- $G=\mathbb{Z}$, $\mathcal{P}\subset\mathbb{Z}^\times$ acting by multiplication is exact. $p,q$ are relatively prime in $\mathbb{Z}$ if and only if $p\mathbb{Z}+q\mathbb{Z}=\mathbb{Z}$.
- For $G=\mathbb{Z}^d$, $\theta_p\in M_d(\mathbb{Z})$ such that $|\det(\theta_p)|>1$ unless $p=e$. Relatively prime determinants yield a sufficient criterion for independence. Exactness is related to the eigenvalues of the matrices.
- $G=\mathcal{R}$ : the ring of integers of an algebraic number field, $\mathcal{P}\subset\mathcal{R}^\times$ satisfying $\mathcal{P}\cap\mathcal{R}^*=\{1\}$ and $\theta$ given by multiplication. Exactness is automatic and independence relates to prime ideal factorization.
- $G=\mathbb{Z}^2$ with $\mathcal{P}\cong\mathbb{N}^2$ acting via $\begin{pmatrix}1&-5\\1&1\end{pmatrix}$ and $\begin{pmatrix}1&5\\-1&1\end{pmatrix}$ also satisfies the hypotheses, although both generators have the same determinant. Note that the system is isomorphic to the one given by multiplication with $1+\sqrt{-5}$ and $1-\sqrt{-5}$ on $G=\mathbb{Z}[\sqrt{-5}]$, which is the ring of integers in $\mathbb{Q}(\sqrt{-5})$.

## Results

The **core** $\mathcal{F}$, given by as the fixed point algebra of the natural gauge action of $\widehat{\mathcal{P}^{-1}\mathcal{P}}$ and its **diagonal** $\mathcal{D}$, generated by $e_{p,g}$, play a dominant role for the structure of $\mathcal{O}[G,\mathcal{P},\theta]$:

$$\begin{array}{ccc} \mathcal{O}[G,\mathcal{P},\theta] & \cong & C(G_\theta)\rtimes(G\rtimes_\theta\mathcal{P}) \\ \cup & & \cup \\ \mathcal{F} & \cong & C(G_\theta)\rtimes G \\ \cup & & \cup \\ \mathcal{D} & \cong & C(G_\theta) \end{array}$$

where $G_\theta := \varprojlim_{\mathcal{P}} G/\theta_p(G)$.

**Facts:**

- $\mathcal{O}\,[G,\mathcal{P},\theta]$ is the Cuntz-Nica-Pimsner algebra of the product system of finite type associated to $(G,\mathcal{P},\theta)$.
- $\mathcal{F}$ is a generalized Bunce-Deddens algebra in the sense of [5] and hence has many regularity properties. According to [1], it is classified by its Elliott invariant.
- $\mathcal{O}\,[G,\mathcal{P},\theta]$ is a unital UCT Kirchberg algebra.
- The canonical representation of $\mathcal{O}\,[G,\mathcal{P},\theta]$ on $\ell^2(G)$ is faithful.
- $C^*(G)$ is a masa in $\mathcal{O}\,[G,\mathcal{P},\theta]$.
- $\mathcal{O}\,[G,\mathcal{P},\theta]$ is simple if and only if $\theta$ is exact.

## Current projects

Modifying (CNP 3) to get the natural Toeplitz extension $\mathcal{NT}[G,\mathcal{P},\theta]$, we obtain

$$\mathcal{NT}[G,\mathcal{P},\theta] \cong \mathcal{D}_{\mathcal{NT}}\rtimes(G\rtimes_\theta\mathcal{P}) \cong C^*(G\rtimes_\theta\mathcal{P}),$$

where the C*-algebra of the semigroup $G\rtimes_\theta\mathcal{P}$ is to be understood in the sense of Xin Li, see [4]. The structure of these and closely related algebras is being analysed in a joint project with Nathan Brownlowe and Nadia S. Larsen.

A modified construction that allows for endomorphisms with infinite cokernel is being developed. However, there is evidence that $C^*(G\rtimes_\theta\mathcal{P})$ may already be simple in this case, see [6].

Using $k$-graphs and a certain discretization procedure, a possibly different algebra is obtained. Since it is likely to be a unital UCT Kirchberg algebra, this difference can be measured in terms of K-theory.

## References


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