On panel-regular lattices in 2-dimensional buildings

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Overview

Definition

Let G be a topological group. A *uniform lattice* Γ is a discrete and cocompact subgroup of G.

Which groups do we consider?

G is a reductive group over a local field (e.g. $\mathrm{PSl}_n(\mathbb{Q}_p)$).

Goal

Construct uniform lattices in G *in a geometric way*.

What is a geometric construction?

Fact

- *G acts cocompactly on affine building X.*
- *X locally finite CAT(0) simplicial complex.*
- $\mathrm{Aut}(X)/G$ *"small"*

Proposition

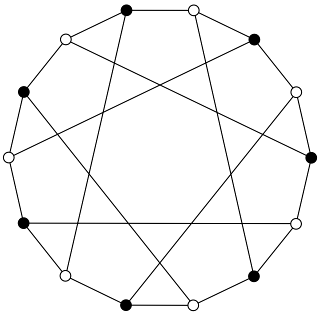
$\Gamma \leq G$ *or* $\Gamma \leq \mathrm{Aut}(X)$ *is a uniform lattice iff*

- *Γ acts cocompactly on X and*
- *every vertex stabilizer is finite.*

Idea

Building $X \to$ uniform lattice $\Gamma \leq \mathrm{Aut}(X) \to$ Show $\Gamma \leq G$.

Finite projective planes



A *finite projective plane* of order q is a (bipartite) graph

- of diameter 3
- and girth 6,

in which every vertex is adjacent to exactly $q+1$ other vertices.

$\bullet$ points
$\circ$ lines

The definition of an $\widetilde{A}_2$-building

Definition

A locally finite *building of type* $\widetilde{A}_2$:

- 2-dimensional CAT(0) simplicial complex
- vertex-3-colored
- vertex links: finite projective planes

2-simplices chambers

1-simplices panels

Examples

classical buildings associated to $\mathrm{PSl}_3(\mathbb{Q}_p)$ and $\mathrm{PSl}_3(\mathbb{F}_p((t)))$.

exotic buildings many more examples!

Known constructions

Theorem (Kantor-Liebler-Tits)

There are only 13 classical affine buildings which admit chamber-transitive lattices.

Example

Köhler-Meixner-Wester: in $\mathrm{Sl}_3(\mathbb{F}_2((t)))$

Construction (Cartwright-Mantero-Steger-Zappa)

Construction of vertex-transitive lattices in buildings of type $\tilde{A}_2$. Some of these lattices are contained in $\mathrm{Sl}_3(\mathbb{F}_q((t)))$.

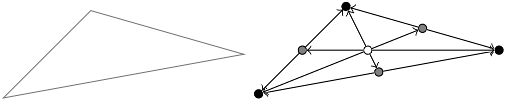
Panel-regular lattices

Facts

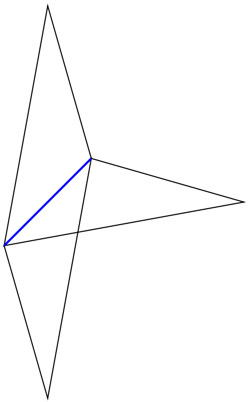
- Γ acts regularly on one type of panel $\Rightarrow$ Γ acts regularly on all types of panels.
- The vertex stabilizers Γ_v act point- and line-regularly on the link $\mathrm{lk}(v)$ (*Singer group*)

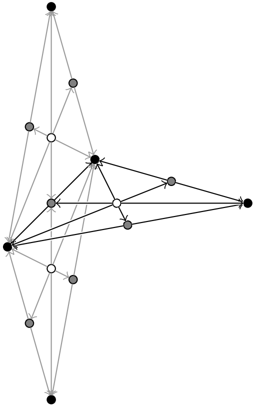
Situation

Γ acts panel-regularly on a locally finite $\tilde{A}_2$ building X. Consider the associated scwol $\mathcal{X}$



Fundamental domain





Classification result

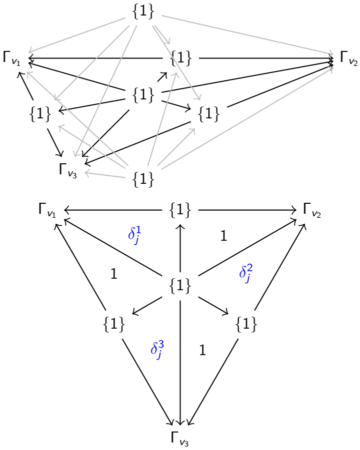
Theorem

Γ admits the quotient complex of groups $\Gamma \backslash\backslash \mathcal{X}$ with

- *Γ_{v_i} Singer groups*
- *twist elements δ^i_j, such that there are points p_i and lines l_i satisfying*

  $$p_i \sim \delta^i_j(l_i) \qquad \forall 0 \leq j \leq q.$$

  $D_i = \{\delta^i_j\}$ is a difference set.



Construction

Construction

The reverse construction with

- Γ_1, Γ_2, Γ_3 Singer groups of a fixed projective plane of order q
- D_1, D_2, D_3 difference sets

is also possible using the following theorems.

Theorem (Bridson-Haefliger)

A complex of groups in which every local development is CAT(0) is developable.

Theorem (Charney-Lytchak)

A CAT(0) metric space locally isometric to an $\tilde{A}_2$-building is globally isometric to an $\tilde{A}_2$-building.

Cyclic Singer groups

Theorem (Singer)

Every classical projective plane admits a cyclic Singer group.

Result

Write $\Gamma_i = \langle x_i \rangle$ and $D_i = \{x_i^{d_j^i}\}$.
We obtain the following presentation:

$$\Gamma = \langle x_1, x_2, x_3 : x_1^{q^2+q+1} = x_2^{q^2+q+1} = x_3^{q^2+q+1} = 1, \\ x_1^{d_j^1} x_2^{d_j^2} x_3^{d_j^3} = x_1^{d_k^1} x_2^{d_k^2} x_3^{d_k^3} \qquad \forall j,k \rangle$$

Remark

We also get a very explicit description of the associated building.

Determining the building

Problem

Which buildings do these lattices correspond to?

Partial solution (Cartwright-Mantero-Steger-Zappa)

The spheres of radius 2 in the buildings associated to $\mathrm{Sl}_3(\mathbb{Q}_p)$ and $\mathrm{Sl}_3(\mathbb{F}_p((t)))$ are different!

Theorem

Spheres of radius 2 in the constructed buildings are as in the building associated to $\mathrm{Sl}_3(\mathbb{F}_p((t)))$.

The relation to known lattices

Proposition

For $q = 2$ and a simple, symmetric choice of the difference sets, our lattice embeds into the chamber-transitive lattice in $\mathrm{Sl}_3(\mathbb{F}_2((t)))$ *constructed by Köhler, Meixner and Wester.*

Remark

Computer calculations so far seem to suggest that for $q = 3$, no lattice of the above construction embeds into $\mathrm{PSl}_3(\mathbb{F}_3((t)))$.

Summary of results

Theorem 1

Every panel-regular lattice on a building of type $\tilde{A}_2$ admits a presentation in terms of Singer groups and difference sets.

Theorem 2

The groups

$$\Gamma = \langle x_1, x_2, x_3 : x_1^{q^2+q+1} = x_2^{q^2+q+1} = x_3^{q^2+q+1} = 1, \\ x_1^{d_j^1} x_2^{d_j^2} x_3^{d_j^3} = x_1^{d_k^1} x_2^{d_k^2} x_3^{d_k^3} \qquad \forall j, k \rangle$$

are cocompact lattices in buildings of type $\tilde{A}_2$, not associated to $\mathrm{PSl}_3(k)$ for a field k of characteristic zero.

Important

This method can be used for other types of 2-dimensional buildings!

Extension to other types of two-dimensional affine buildings

Type $\tilde{C}_2$: vertex links are

- complete bipartite graphs
- generalized quadrangles

Singer quadrangles: see Shult, K. Thas and de Winter

Result

Can construct panel-regular lattices in buildings of type $\tilde{C}_2$, most of which are necessarily exotic.

Note

Not much is known about Singer *hexagons*. If there are any Singer hexagons, we could perhaps construct lattices in buildings of type $\tilde{G}_2$!

Thank you

Thank you for your attention!