Will Johnson's work: Shelah's conjecture for dp finite fields

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Outlook

The aim of this reading group is to study the proof of the following

Theorem (Johnson, 2020)

Every infinite dp finite field is algebraically closed, real closed, or admits a nontrivial henselian valuation.

Main sources (all available on ArXiv):

[1] Will Johnson, Dp finite fields I: infinitesimals and positive characteristic, 2019.
[2] Will Johnson, Dp finite fields II: the canonical topology and its relation to henselianity, 2019.
[3] Will Johnson, Dp finite fields III: inflators and directories, 2019.
[4] Will Johnson, Dp finite fields IV: the rank 2 picture, 2020.
[5] Will Johnson, Dp finite fields V: topological fields of finite weight, 2020.
[6] Will Johnson, Dp finite fields VI: the dp finite Shelah conjecture, 2020.

Stable fields

Let T be an $\mathcal{L}$-theory. We call T stable in case no $\mathcal{L}$-formula has the order property (OP) modulo T, i.e. there are no $M \models T$, a $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and sequences $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$ in M, such that we have

$$\varphi(a_i, b_j) \iff i < j.$$

A field K is called stable if its theory $\mathrm{Th}(K)$ in the language of rings $\mathcal{L}_{\mathrm{ring}} = \{+, \cdot, 0, 1\}$ is stable.

Conjecture (197x)

Every infinite stable field is separably closed.

Conversely: Every separably closed field is stable (Ershov/Wood).
A special case of stability is ω-stability.

Theorem (Macintyre, 1971)

Every infinite ω-stable field is algebraically closed.

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What's known about stable fields?

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Every infinite stable field is separably closed.

For now, we consider all fields in $\mathcal{L}_{\mathrm{ring}}$. The following special cases of the conjecture are known:

1971 Infinite ω-stable fields are algebraically closed (Macintyre).

1980 Infinite superstable fields are algebraically closed (Cherlin-Shelah).

2011 Infinite stable fields of weight 1 are separably closed (Krupinsky-Pillay).

2017 Infinite stable fields of finite dp rank are algebraically closed (Halevi-Palacin).

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NIP

Definition

Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory, $\varphi(\bar{x}, \bar{y})$ an $\mathcal{L}$-formula. We define

- VC-dim$(\varphi(\bar{x}, \bar{y})) \geq n$ if there are sequences $(a_i)_{i\in\{1,\ldots,n\}}$ und $(b_J)_{J\subseteq\{1,\ldots,n\}}$ (in a model of T) such that we have

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- VC-dim$(\varphi(\bar{x}, \bar{y})) = \sup\{n \in \mathbb{N} : \text{VC-dim}(\varphi(\bar{x}, \bar{y})) \geq n\} \in \mathbb{N} \cup \{\infty\}$

In case VC-dim$(\varphi(\bar{x}, \bar{y})) = \infty$, we say that $\varphi(\bar{x}, \bar{y})$ has the independece property (IP).

We say that T has NIP if no formula has IP.

Example:

- In $\mathrm{Th}(\mathbb{Q}, <)$ we have VC-dim$(x < y) = 1$.
- In $\mathrm{Th}(\mathbb{Z}; +, \cdot, 0, 1)$ we have VC-dim$(x|y) = \infty$.

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VC-Dimension of a relation

0	1	0	1	1	0	1	1	1	1	0	1	...
1	0	0	1	1	0	1	0	1	1	1	0	...
1	1	0	1	0	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	0	1	1	1	0	...
1	0	1	1	1	0	0	1	1	0	1	0	...
0	1	1	1	0	0	1	0	1	0	1	0	...
0	1	1	1	1	0	1	0	0	0	0	0	...
1	1	0	1	1	1	1	0	1	1	0	0	...
1	1	1	1	1	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	1	1	0	0	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Definition

A structure has NIP, if every definable relation $\varphi(\bar{x}, \bar{y})$ has finite VC-dimension.

VC-Dimension of a relation

0	1	0	1	1	0	1	1	1	1	0	1	...
1	0	0	1	1	0	1	0	1	1	1	0	...
1	1	0	1	0	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	0	1	1	1	0	...
1	0	1	1	1	0	0	1	1	0	1	0	...
0	1	1	1	0	0	1	0	1	0	1	0	...
0	1	1	1	1	0	1	0	0	0	0	0	...
1	1	0	1	1	1	1	0	1	1	0	0	...
1	1	1	1	1	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	1	1	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

VC-Dimension ≥ 2

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0	1	0	1	1	0	1	1	1	1	0	1	...
1	0	0	1	1	0	1	0	1	1	1	0	...
1	1	0	1	0	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	0	1	1	1	0	...
1	0	1	1	1	0	0	1	1	0	1	0	...
0	1	1	1	0	0	1	0	1	0	1	0	...
0	1	1	1	1	0	1	0	0	0	0	0	...
1	1	0	1	1	1	1	0	1	1	0	0	...
1	1	1	1	1	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	1	1	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

VC-Dimension ≥ 2

VC-Dimension ≥ 3

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0	1	0	1	1	0	1	1	1	1	0	1	...
1	0	0	1	1	0	1	0	1	1	1	0	...
1	1	0	1	0	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	0	1	1	1	0	...
1	0	1	1	1	0	0	1	1	0	1	0	...
0	1	1	1	0	0	1	0	1	0	1	0	...
0	1	1	1	1	0	1	0	0	0	0	0	...
1	1	0	1	1	1	1	0	1	1	0	0	...
1	1	1	1	1	0	1	1	1	1	1	1	...
0	1	0	1	1	0	1	1	1	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

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$<$	0	1	2	3	4	5	6	7	8	9	10	...
0	0	1	1	1	1	1	1	1	1	1	1	...
1	0	0	1	1	1	1	1	1	1	1	1	...
2	0	0	0	1	1	1	1	1	1	1	1	...
3	0	0	0	0	1	1	1	1	1	1	1	...
4	0	0	0	0	0	1	1	1	1	1	1	...
5	0	0	0	0	0	0	1	1	1	1	1	...
6	0	0	0	0	0	0	0	1	1	1	1	...
7	0	0	0	0	0	0	0	0	1	1	1	...
8	0	0	0	0	0	0	0	0	0	1	1	...
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A structure has NIP, if every definable relation $\varphi(\bar{x}, \bar{y})$ has finite VC-dimension.

The Shelah Conjecture

A theory has NIP (or: is dependent) if every formula has finite VC-dimension.

Definition

We call a field K NIP if its $\mathcal{L}_{\text{ring}}$-theory has NIP.

Examples

- Any separably closed field is NIP.
- Any real closed field is NIP.
- Any p-adically closed field (or a finite extension therof) is NIP. The field $\mathbb{C}((t))$ has NIP.
- The field $\mathbb{Q}$ has IP.

Shelah Conjecture for NIP fields

Every infinite NIP field is separably closed, real closed, or admits a nontrivial henselian valuation.

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History of the Shelah Conjecture

Shelah (Sh:783 and Sh:863) suggests strong dependence as a solution to the equation

$$\frac{x}{\textit{dependent}} = \frac{\textit{superstable}}{\textit{stable}}$$

He poses the following test question:

Conjecture (Shelah)

Infinite strongly dependent fields are algebraically closed, real closed or 'like the p-adic numbers, or a finite extension of such'.

Moreover, he asks for a classification of NIP fields.

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Infinite strongly dependent fields are algebraically closed, real closed or elementarily equivalent to a field admitting a valuation v, with strongly dependent value group and strongly dependent residue field, that eliminates field quantifiers in some Denef-Pas language[a], or a finite extension of such.

[a] this is believed to imply henselianity

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Comparing the Conjectures

Let T be any theory. If $\varphi(\bar{x}, \bar{y})$ has IP, then it also has the order property:

$$\text{stable} \Longrightarrow \text{NIP}$$

Proposition (J.-Koenigsmann)

The first statement implies the second:

- Every infinite NIP field is separably closed, real closed, or admits a nontrivial henselian valuation.
- Every infinite NIP field is separably closed, real closed, or admits a nontrivial definable valuation.

Here, a valuation v on a field K is called definable if $\mathcal{O}_v$ is an $\mathcal{L}_{\text{ring}}$-definable subset of K. Any field admitting a nontrivial definable valuation is unstable. Thus, we have:

$$\text{Shelah Conjecture for NIP fields} \Longrightarrow \text{Stable fields conjecture}$$

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Consequences of the Shelah Conjecture

Shelah Conjecture for NIP fields

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implies any of the following statements:

- Stable fields conjecture.
- Every infinite NIP field is separably closed, real closed, or admits a nontrivial definable valuation.
- Every infinite NIP field is separably closed, real closed, or admits a nontrivial definable henselian valuation.[1]
- (Henselianity conjecture) Any NIP valued field (K, v) is henselian.[1]

Note: A valued field (K, v) is called NIP if its theory in the language $\mathcal{L}_{\text{val}} = \mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$ is NIP.

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Dp-minimality

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- Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ has an ICT-pattern of depth 2 if there are $\mathcal{L}$-formulae $\varphi(x, \bar{y})$ and $\psi(x, \bar{y})$ and sequences $(\bar{a}_i)_{i\in\omega}$ and $(\bar{b}_j)_{j\in\omega}$ in $\mathcal{M}$ such that for all $i, j \in \omega$ the type

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is consistent.
- A theory is dp-minimal if no $\mathcal{M} \models T$ has an ICT-pattern of depth 2.

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- Any strongly minimal / o-minimal / weakly o-minimal / p-minimal / C-minimal theory is dp-minimal.
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Not dp-minimal: A chessboard

Example: Consider the language $\mathcal{L} = \{R, I\}$ with two unary function symbols and the $\mathcal{L}$-structure $\mathcal{C}$ with universe $\mathbb{C}$ and interpretations $R^{\mathcal{C}}(z) = \mathrm{Re}(z)$ and $I^{\mathcal{C}}(z) = \mathrm{Im}(z)$. Then the formulae $R(x) = y$ and $I(x) = y$ give an ICT pattern of depth 2 via the sequences $a_i = b_i = i$ for all $i \in \omega$:
For any $i, j \in \omega$, the type

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is obviously consistent.

Similar: Let Γ be an ordered abelian group with $\Gamma / p\Gamma$ infinite and choose any $\gamma \in \Gamma \setminus \{0\}$. Then, the formulae $\exists z : x = y + p \cdot z$ and $y - \gamma < x < y + \gamma$ give an ICT pattern of depth 2.

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Dp minimal fields

Theorem (Johnson)

A field K is dp-minimal if and only if K is perfect and there exists a valuation v on K such that:

1. v is henselian.
2. v is defectless (i.e., any finite valued field extension (L, v) of (K, v) is defectless, i.e., satisfies $[L : K] = [vL : vK][Lv : Kv]$).
3. The residue field Kv is either an algebraically closed field of positive characteristic or elementarily equivalent to a local field of characteristic 0.
4. The value group vK is almost divisible, i.e., $[vK : n(vK)] < \infty$ for all n.
5. If $\mathrm{char}(Kv) = p \neq 0$ then $[-v(p), v(p)] \subseteq p(vK)$.

In particular, the Shelah Conjecture holds for dp-minimal fields.

Structure of the proof, part 1

'$\Longleftarrow$': The fact that any field occuring in the classification is dp-minimal is (comparatively) easy: It uses the fact that all occuring residue fields are dp-minimal, all value groups are dp-minimal, and variants of

Theorem (Chernikov-Simon)

Let (K, v) be a henselian valued field of equicharacteristic 0. Then

$$\underbrace{(K,v) \text{ is dp-minimal}}_{\text{in } \mathcal{L}_{\mathrm{val}}} \Longleftrightarrow \underbrace{\Gamma_v \text{ is dp-minimal}}_{\text{in } \mathcal{L}_{\mathrm{oag}}} \text{ and } \underbrace{Kv \text{ is dp-minimal}}_{\text{in } \mathcal{L}_{\mathrm{ring}}}$$

Structure of the proof, part 2

'$\Longrightarrow$': The difficult part is to conjure up a valuation out of thin air. This is done in two steps:

Theorem (J.-Simon-Walsberg, Johnson)

Any dp-minimal valued field is henselian, i.e., the henselianity conjecture holds for dp-minimal fields.

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Any infinite dp-minimal field is algebraically closed, real closed or admits a nontrivial definable V-topology.

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Consequences of the characterization of dp-minimal fields

There are two striking consequences of Johnson's characterization:

1. Any dp-minimal field K has a bounded absolute Galois group (i.e., for each $n \in \mathbb{N}$, the field K has only finitely many extensions of degree n).
2. Any finite extension of a dp-minimal field is dp-minimal.[2]

Open Questions:

1. Is there a direct proof that dp-minimality implies boundedness? Does the same hold for inp-minimal fields?
2. Is there a direct proof that dp-minimality goes up to finite extensions (in $\mathcal{L}_{\text{ring}}$)? Does the same hold for inp-minimal fields?

[2] It follows immediately from the definition of dp rank that any finite extension of a dp-minimal field has finite dp-rank.

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Excursion: the canonical henselian valuation

Let K be any field. If K is not separably closed, one can order the non-trivial henselian valuations on it. Divide the class of henselian valuations on K into two subclasses, namely

$$H_1(K) = \{v \text{ henselian on } K \mid Kv \neq Kv^{sep}\}$$

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The Canonical Henselian Valuation

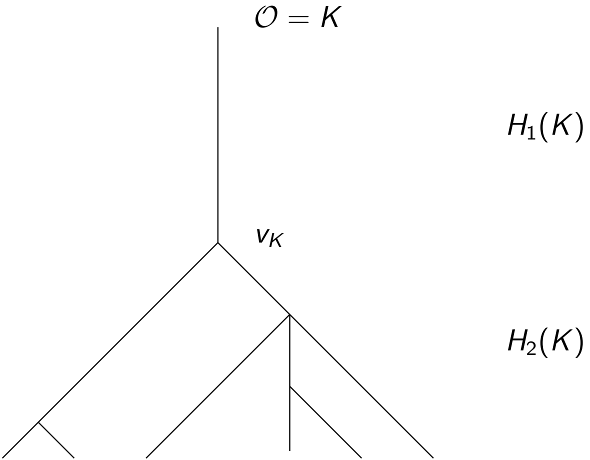

Abbildung: The canonical henselian valuation

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Then any valuation $v_2 \in H_2(K)$ is finer than any $v_1 \in H_1(K)$, i.e. $\mathcal{O}_{v_2} \subset \mathcal{O}_{v_1}$, and any two valuations in $H_1(K)$ are comparable.
Furthermore, if H_2 is non-empty, then there exists a unique coarsest v_K in H_2; otherwise there exists a unique finest $v_K \in H_1$. In any case, v_K is called the canonical henselian valuation.
If K admits a nontrivial henselian valuation, then v_K is nontrivial.

Rephrasing Johnson for dp-minimal fields

Let K be an infinite dp-minimal field and let v_K be the (possibly trivial) canonical henselian valuation on K. Then one of the following holds:

1. Kv_K is real closed or algebraically closed of characteristic 0 and $K \equiv \mathbb{R}((\Gamma))$ or $K \equiv \mathbb{C}((\Gamma))$ (as fields) where $\Gamma \equiv v_K K$ (as ordered abelian groups) and Γ is dp-minimal.
2. $\mathrm{char} K = p > 0$, Kv_K is algebraically closed, (K, v_K) is tame Kaplansky and $K \equiv \overline{\mathbb{F}}_p((\Gamma))$ (as fields) where $\Gamma \equiv v_K K$ (again, as ordered abelian groups) and Γ is dp-minimal.
3. $\mathrm{char}(K, Kv_K)$ is of mixed characteristic $(0, p)$ and Kv_K is finite. Then $K \equiv Q((\Gamma))$ (as fields) where Q is a finite extension of $\mathbb{Q}_p$ and Γ is dp-minimal.
4. $\mathrm{char}(K, Kv_K)$ is of mixed characteristic $(0, p)$ and Kv_K is infinite. In that case $Kv_K \models ACF_p$ and $K \equiv L((\Gamma))$ (as fields) where L is a field admitting a rank 1 valuation v turning it into a mixed characteristic tame Kaplansky field, with residue field as in (2) above and Γ is dp-minimal.

Moreover, any of the fields described in clauses $(1) - (4)$ is dp-minimal.

ICT patterns

Let T be any theory. We work in a monster model $\mathbb{C} \models T$.

Definition

Let $\Sigma(\bar{x})$ be a partial type and κ a cardinal (possibly finite). An ICT pattern of depth κ consists of a sequence of formulae $\varphi_\alpha(\bar{x}, \bar{y})$ and an array of parameters $(b_{\alpha,i})$ for $\alpha < \kappa$ and $i \in \omega$ such that for any $f : \kappa \to \omega$, the following type is consistent:

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Note: If $\varphi(\bar{x}, \bar{y})$ has IP, then by compactness we can construct ICT patterns of arbitrary depth κ for the constant sequence $(\varphi(\bar{x}, \bar{y}))_{\alpha<\kappa}$.

Fact (Shelah)

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in case there is such a supremum, and $\mathrm{dp}(\Sigma(\bar{x})) = \infty$ otherwise.

Fact/Definition

- T is NIP if and only if $\mathrm{dp}(x = x) < \infty$.
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An application: perfectness

Proposition (Johnson)

Any dp finite field is perfect.

Uses the following ingredients of dp rank on definable sets X and Y:

- $\mathrm{dp}(X) > 0$ if and only if X is infinite.
- If X and Y are in definable bijection, then $\mathrm{dp}(X) = \mathrm{dp}(Y)$.
- If $X \subset Y$, then $\mathrm{dp}(X) \leq \mathrm{dp}(Y)$.
- $\mathrm{dp}(X \times Y) = \mathrm{dp}(X) + \mathrm{dp}(Y)$.

Proof of the proposition: Since K^p is in definable bijection with K, we have $\mathrm{dp}(K^p) = \mathrm{dp}(K)$. K imperfect $\Rightarrow$ K is a definable K^p-vector space of dimension > 1. Thus, $K^p \times K^p$ injects definably into K. This shows

$$\mathrm{dp}(K) \geq 2 \cdot \mathrm{dp}(K^p) = 2 \cdot \mathrm{dp}(K).$$

Thus, $\mathrm{dp}(K) = 0$ and K is finite. Finite fields are perfect.

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- $\mathrm{dp}(X \times Y) = \mathrm{dp}(X) + \mathrm{dp}(Y)$.

Proof of the proposition: Since K^p is in definable bijection with K, we have $\mathrm{dp}(K^p) = \mathrm{dp}(K)$. K imperfect $\Rightarrow$ K is a definable K^p-vector space of dimension > 1. Thus, $K^p \times K^p$ injects definably into K. This shows

$$\mathrm{dp}(K) \geq 2 \cdot \mathrm{dp}(K^p) = 2 \cdot \mathrm{dp}(K).$$

Thus, $\mathrm{dp}(K) = 0$ and K is finite. Finite fields are perfect.

An application: perfectness

Proposition (Johnson)

Any dp finite field is perfect.

Uses the following ingredients of dp rank on definable sets X and Y:

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In fact, Shelah shows in Sh:863 that any strongly dependent field is perfect.

Shelah conjecture for dp finite fields

The main aim of this reading seminar is to understand the proof of

Theorem (Johnson, Dp finite fields VI: the dp finite Shelah conjecture)

Let K be an infinite dp finite field. Then K is either algebraically closed, real closed or admits a nontrivial henselian valuation.

Again, the proof follows the same two step pattern as in the dp-minimal case:

Theorem (Johnson)

1. Dp finite henselianity conjecture: any dp finite valued field is henselian.
2. Dp finite existence conjecture: Any infinite dp finite field is either algebraically closed, real closed or admits a nontrivial definable valuation.

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Consequences of the dp finite classification

Halevi-Palacin: An ordered abelian group Γ has finite dp rank in $\mathcal{L}_{\mathrm{oag}}$ if and only Γ/pΓ is finite for all but finitely many primes.

Corollary (Halevi-Hasson-J.)

A field K is dp finite if and only if K is perfect and there exists a valuation v on K such that:

1. v is henselian.
2. v is defectless (i.e., any finite valued field extension (L, v) of (K, v) is defectless, i.e., satisfies $[L : K] = [vL : vK][Lv : Kv]$).
3. The residue field Kv is either an algebraically closed field of positive characteristic or elementarily equivalent to a local field of characteristic 0.
4. The value group vK satisfies $[vK : p(vK)] < \infty$ for all but finitely many primes p.
5. If $\mathrm{char}(Kv) = p \neq 0$ then $[-v(p), v(p)] \subseteq p(vK)$.

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Dp rank versus strong dependence

Corollary

A dp finite field is dp-minimal if and only if it has bounded absolute Galois group.

Question: Is there a direct proof of this?

Theorem (Halevi-Palacin)

An ordered abelian group Γ is strongly dependent (in $\mathcal{L}_{\mathrm{oag}}$) if and only $\Gamma/p\Gamma$ is finite for all but finitely many primes.

Thus, one gets

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Assume that Shelah's conjecture holds for strongly dependent fields. Then any strongly dependent field has finite dp rank.

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Henselianity conjecture

Theorem (Johnson)

Let (K, v) be an NIP field of positive characteristic. Then v is henselian, i.e., the henselianity conjecture holds for NIP fields of positive characteristic.

Proof ingredients:

- Infinite NIP fields of characteristic $p > 0$ admit no Artin-Schreier extensions (Kaplan-Scanlon-Wagner)
- (K, v) valued field and L/K a finite normal extension. Then every extension of v to L is definable (identifying L with K^d for $d = [L : K]$)
- Let G be an NIP group. Then G^{00} exists, i.e., the intersection of all type-definable subgroups of G of bounded index is itself a subgroup of bounded index. (Shelah)

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