# THE MODEL THEORY OF COHEN RINGS

SYLVY ANSCOMBE AND FRANZISKA JAHNKE

Abstract. The aim of this article is to give an self-contained account of the algebra and model theory of Cohen rings, a natural generalization of Witt rings. Witt rings are only valuation rings in case the residue field is perfect, and Cohen rings arise as the Witt ring analogon over imperfect residue fields. Just as one studies truncated Witt rings to understand Witt rings, we study Cohen rings of positive characteristic as well as of characteristic zero. Our main results are a relative completeness and a relative model completeness result for Cohen rings, which imply the corresponding Ax–Kochen/Ershov type results for unramified henselian valued fields also in case the residue field is imperfect. The key to these results is a proof of relative quantifier elimination down to the residue field in an appropriate language which holds in any unramified henselian valued field.

## 1. Introduction

The aim of this paper is to give an introduction to the model theory of complete Noetherian local rings $A$ which have maximal ideal $pA$. From an algebraic point-of-view, the theory of such rings is classical. Under the additional hypothesis of regularity, they are valuation rings, and their study goes back to work of Krull ([Kru37]) and many others. Structure theorems were obtained by Hasse and Schmidt ([HS34]), although there were deficiencies in the case that $A/pA$ is not perfect. Further structural results were obtained by Witt ([Wit37]) and Teichmüller ([Tei36b]). In particular Teichmüller gave a brief but precise account of the structure of such rings, even in the case that $A/pA$ is imperfect. This was followed by Mac Lane ([Mac39c]), who improved upon Teichmüller's theory and proved relative structure theorems. Mac Lane built his work upon his study of Teichmüller's notion of $p$-independence in [Tei36a]. For further historical information, especially on this early period, the reader is encouraged to consult Peter Roquette's article [Roq03] on the history of valuation theory.

Turning away from the hypothesis of regularity, Cohen ([Coh46]) gave an account of the structure of such rings. In fact his context was even more general: he did not assume Noetherianity.

Despite all of this work, more modern treatments (e.g. Serre, [Ser79]) of this subject are often restricted to the case that $A/pA$ is perfect. Consequently, the literature on the model theory of complete Noetherian local rings is sparse. For example, [vdD14] also assumes that $A/pA$ is perfect.

We became interested in the model theory of complete Noetherian local rings when we tried to construct examples of NIP henselian valued fields with imperfect residue field. After getting acquainted with the algebra of these rings as scattered in the literature detailled above, we realized that with a bit of tweaking, the proof ideas of these (classical) results can be used gain an understanding of the model theory of such rings. First, this requires a careful recapitulation of the known algebraic (or structural) theory of such rings, bringing older results together in one framework. This overview is given in the first part of the article. In this first part, many of the proof ideas are inspired by the work of others (and we point to the original sources), but we take care to prove everything which cannot be cited directly from elsewhere.

The underlying definition of a Cohen ring is the following:

**Definition 1.1** (cf Definitions 2.3 and 2.7)**.** A **Cohen ring** is a complete Noetherian local ring $A$ with maximal ideal $pA$, where $p$ is the residue characteristic of $A$.

April 17, 2019.

A Cohen ring may either have characteristic 0 (in which case we call it strict) or $p^n$, where $p$ is the characteristic of the residue field $A/pA$. In section 1, we recall that for a given field $k$, Cohen rings of every possible characteristic exist. In the second section, we discuss and develop the machinery of multiplicative representatives and $\lambda$-representatives, namely good sections from the perfect core of the residue field, and respectively the residue field, into the Cohen ring $A$. For a precise definition see 3.1 and 3.4. We also comment on to what extent these sections are unique, see Theorems 3.3 and 3.7. In the fourth section, we prove that Teichmüller's embedding technique works in this context: we embed a Cohen ring with residue field $k$, with a choice of representatives, into the corresponding Cohen ring over the perfect hull of $k$ (see Theorem 4.1). Building on this and using ideas from Cohen, we show that any two Cohen rings of the same characteristic and over the same residue field, both equipped with representatives, are isomorphic. In fact there is a unique isomorphism which respects the choices of representatives and is the identity on the residue field (Cohen Structure Theorem, 6.3). In section 7, we develop an embedding lemma and relative structure theorems, which describe how two Cohen rings are related over a common substructure (see Corollary 7.2). These results are applied in our later work on quantifier elimination. In the final section of the first part of the paper, we compare Cohen rings to Witt rings.

In the second part we begin a model-theoretic study, including describing the complete theories of Cohen ring of a fixed characteristic, over a given residue field. Moreover, we prove quantifier elimination in appropriate languages. After introducing the language of the residue field, we show quantifier elimination for Cohen rings of finite characteristic (see Theorem 11.3). This is the key step to proving relative quantifier elimination in the strict case as, by a result of Bélair, we always have quantifier elimination of the quantifiers over the base field in an appropriate $\omega$-sorted language, where the sorts correspond exactly to the finite characteristic residue rings. From this quantifier elimination, we then deduce relative completeness. In particular, this result gives the following Ax–Kochen/Ershov principle:

**Theorem 1.2** (Cf Corollary 13.4)**.** *Let $(K_1, v_1)$ and $(K_2, v_2)$ be unramified henselian valued fields of mixed characteristic $(0, p)$. The following are equivalent.*

- **(i)** *$(K_1, v_1)$ and $(K_2, v_2)$ are $\mathcal{L}_{\mathrm{val}}$-elementarily equivalent,*
- **(ii)** *$\Gamma_{v_1} \equiv \Gamma_{v_2}$ and $k_1 \equiv k_2$.*

Note that this was already claimed by Bélair in [Bél99, Corollaire 5.2(1)]. However, since his proof crucially relies on Witt rings, it only works for perfect residue fields.

We also get the following relative model-completeness result (see section 12.2 for the definition of the language $\mathcal{L}_{\mathrm{ac},S}$):

**Theorem 1.3** (Cf Corollary 13.6). *Let $(K_1, v_1) \subseteq (K_2, v_2)$ be extension of unramified henselian valued fields of mixed characteristic $(0, p)$, viewed as an extension of $\mathcal{L}_{\mathrm{ac},S}$-structures. The following are equivalent.*

- **(i)** *$(K_1, v_1) \preceq (K_2, v_2)$ as $\mathcal{L}_{\mathrm{ac},S}$-structures,*
- **(ii)** *$\Gamma_{v_1} \preceq \Gamma_{v_2}$ and $k_1 \preceq k_2$.*

Finally, we conclude that in any unramified henselian valued field, the residue field is stably embedded, see Theorem 13.7.

# Part 1. The structure of Cohen rings

## 2. Pre-Cohen rings and Cohen rings

Throughout this paper, $A, B, C$ will denote rings, which will always have a multiplicative identity 1 and be commutative; and $k, l$ will be fields of characteristic $p$, which is a fixed prime number.

A ring $A$ is **local** if it has a unique maximal ideal, which we will usually denote by $\mathfrak{m}$. A local ring is equipped with the **local topology**[1], which is the ring topology defined by declaring the descending sequence of ideals $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \ldots$ to be a base of neighbourhoods of 0. The **residue field** of a local ring $A$, which we usually denote by $k$, is the quotient ring $A/\mathfrak{m}$, and the natural quotient map

$$\mathrm{res} : A \longrightarrow k$$

is called the **residue map**. The **residue characteristic** of $A$ is by definition the characteristic of $k$.

For the sake of clarity, since maps between residue fields of local rings are of central importance in this paper, it will be suggestive to work with pairs[2] $(A, k)$ consisting of a local ring $A$, together with its residue field $k$. Of course, such a pair is already determined by the local ring $A$, and this notation fails to explicitly mention the maximal ideal or the residue map. Without risk of confusion, we will also refer to such pairs as local rings.

**Lemma 2.1** (Krull, [Kru38, Theorem 2])**.** *Let $A$ be a Noetherian local ring. Then $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n = \{0\}$. In other words, $A$ is Hausdorff with respect to the local topology.*

*Remark* 2.2 (Other terminology). Before we give our main definitions, namely Definitions 2.3 and 2.7, we note that many closely related ideas have been named in the literature, both in original papers and textbooks. Mac Lane, in [Mac39c], works with '$p$-adic fields' and '$\mathfrak{p}$-adic fields'; whereas Cohen, in [Coh46], prefers to work with 'local rings' (which, for Cohen, are necessarily Noetherian), 'generalized local rings', and '$v$-rings'. Serre, in [Ser79, Chapter II, §5], defines a '$p$-ring' to be a ring $A$ which is Hausdorff and complete in the topology defined by a decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$ of ideals, such that $\mathfrak{a}_m\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$, and for which $A/\mathfrak{a}_1$ is a perfect ring of characteristic $p$. More recently, van den Dries, in [vdD14, p. 132], defines a 'local $p$-ring' to be a complete local ring $A$ with maximal ideal $pA$ and perfect residue field $A/pA$.

To minimise the risk of confusion with existing terminology, we will not work with $v$-rings, $p$-adic fields, $\mathfrak{p}$-adic fields, $p$-rings, or local $p$-rings. Instead, since Warner's point of view, in [War93, Chapter IX], is closer to our own, it is his definition of 'Cohen ring' that we adopt. We hope the reader will forgive us for this, but we feel that none of the other notions (several of which are arguably more standard in the literature) exactly captures the right context for this paper.

**Definition 2.3.** A **pre-Cohen ring** is a local ring $(A, k)$ such that $A$ is Noetherian and the maximal ideal $\mathfrak{m}$ is $pA$.

In particular, pre-Cohen rings are of residue characteristic $p$. Turning to the question of the characteristic of $A$ itself, we note that a pre-Cohen ring need not even be an integral domain. However, a pre-Cohen ring is either of characteristic 0 or of characteristic $p^n$, for some $n \in \mathbb{N}_{>0}$.

**Lemma 2.4.** *For a pre-Cohen ring $(A, k)$, the following are equivalent:*

- **(i)** *$A$ is of characteristic zero,*
- **(ii)** *$A$ is an integral domain,*
- **(iii)** *$A$ is a valuation ring.*

*In this case, the corresponding valuation $v_A$ on the quotient field of $A$ is of mixed characteristic $(0, p)$, has value group isomorphic to $\mathbb{Z}$, with $v_A(p)$ minimum positive, and has residue field $k$.*

*Proof.* This is a special case of [War93, 21.4 Theorem]. □

**Definition 2.5.** If any (equivalently, all) of the conditions of Lemma 2.4 are satisfied, then we say that $(A, k)$ is **strict**.

[1] The local topology is also known as the $\mathfrak{m}$-*adic topology*.

[2] In Part 2, when we adopt a more model-theoretic viewpoint, such pairs will naturally appear as two-sorted structures.

The word 'strict' is borrowed from Serre, [Ser79, II,§5].

*Remark* 2.6. In [Coh46], Cohen writes in terms of regular Noetherian local rings. A local ring is **regular** if its Krull dimension is equal to the number of generators of its unique maximal ideal. In the case of a pre-Cohen ring $(A, k)$, the maximal ideal is by definition generated by one element, namely $p$. Therefore, $(A, k)$ is regular if and only if its Krull dimension is 1, which in turn holds if and only if $(A, k)$ is strict.

A **morphism** of pre-Cohen rings, which we write as $\phi : (A_1, k_1) \longrightarrow (A_2, k_2)$, is a pair $\phi = (\phi_A, \phi_k)$ of ring homomorphisms $\phi_A : A_1 \longrightarrow A_2$ and $\phi_k : k_1 \longrightarrow k_2$, such that

**(i)** $\mathfrak{m}_1 = \phi_A^{-1}(\mathfrak{m}_2)$, i.e. $\phi_A$ is a morphism of local rings, and

**(ii)** $\phi_k \circ \mathrm{res} = \mathrm{res} \circ \phi_A$.

This is nothing more than a way of speaking about morphisms of local rings as pairs of maps, to match the pairs $(A, k)$. Every morphism $\phi_A$ of local rings induces a ring homomorphism $\phi_k : k_1 \longrightarrow k_2$ such that $(\phi_A, \phi_k)$ is a morphism of pre-Cohen rings. From now on, by 'morphism' we mean a morphism of pre-Cohen rings. We will often (but not always) be concerned with morphisms $\phi = (\phi_A, \phi_k)$ such that $k_2/\phi_k(k_1)$ is separable.

By an **embedding**, we mean a morphism $\phi = (\phi_A, \phi_k)$ such that $\phi_A$ is injective. In the obvious way, we write $(A_1, k_1) \subseteq (A_2, k_2)$ if $A_1$ is a subring of $A_2$, $k_1$ is a subfield of $k_2$, and the inclusion maps form an embedding $(A_1, k_1) \longrightarrow (A_2, k_2)$.

**Definition 2.7** (cf [War93, 21.3 Definition])**.** A pre-Cohen ring $(A, k)$ is a **Cohen** ring if it is also complete, i.e. complete with respect to the local topology.

**Example 2.8.** $(\mathbb{Z}_p, \mathbb{F}_p)$ is a strict Cohen ring.

**Lemma 2.9.** *Every pre-Cohen ring of positive characteristic is already a Cohen ring.*

*Proof.* In a non-strict pre-Cohen ring the topology is discrete. Thus it is complete. □

The first task is to show that Cohen rings exist, for any residue field and any characteristic. This foundational existence result goes back to the work of Hasse and Schmidt.

**Theorem 2.10** (Existence Theorem, [HS34, Theorem 20, p63])**.** *Let $k$ be a field of characteristic $p$. There exists a strict Cohen ring $(A, k)$. Moreover, for each $m \in \mathbb{N}_{>0}$, there exists a Cohen ring $(A_m, k)$ of characteristic $p^m$.*

## 3. Representatives

3.1. **Teichmüller's multiplicative representatives.** The notion of 'representatives' plays a key role in this subject.

**Definition 3.1** (cf [Tei36b, §4.])**.** Let $(A, k)$ be a pre-Cohen ring, and let $\alpha \in k$. A **representative** of $\alpha$ is some $a \in A$ with $\mathrm{res}(a) = \alpha$. A **multiplicative representative** of $\alpha$ is a representative $a \in A$ of $\alpha$ such that $a$ is a $p^n$-th power in $A$, for all $n \in \mathbb{N}$. A **choice of representatives** is a partial function

$$s : k \dashrightarrow A$$

such that $s(\alpha)$ is a representative of $\alpha$. To say that such a choice is **for** $P$ means that $P$ is the domain of $s$, i.e. $s : P \longrightarrow A$. Obviously, such a map is a **choice of multiplicative representatives** if $s(\alpha)$ is a multiplicative representative of $\alpha$, for all $\alpha$ in the domain of $s$.

We observe that, for any pre-Cohen ring $(A, k)$, there exist many choices of representatives for $k$, and of course for any subset $P$ of $k$. It is obvious that the largest subfield of $k$ for which multiplicative representatives may be chosen is $k^{p^\infty}$, which is by definition the subfield of elements which are $p^n$-th powers, for all $n \in \mathbb{N}$. Note that $k^{p^\infty}$ is the largest perfect subfield of $k$. The following straightforward lemma is the starting point for the study of multiplicative representatives. It can be proved directly by showing that $p^n$ divides the binomial coefficient $\binom{p^n}{k}$, for $k \in \{1, \dots, p^n - 1\}$.

**Lemma 3.2** ([Tei37, cf Hilfssatz 8]). *Let $(A, k)$ be a pre-Cohen ring, let $a, b \in A$, and let $m, n \in \mathbb{N}$. Then $a^{p^n} + b^{p^n} \equiv (a+b)^{p^n} \pmod{\mathfrak{m}^n}$. It follows that, if $a \equiv b \pmod{\mathfrak{m}^m}$, then $a^{p^n} \equiv b^{p^n} \pmod{\mathfrak{m}^{m+n}}$.*

Perhaps the most important result about multiplicative representatives is Theorem 3.3, which is due to Teichmüller.

**Theorem 3.3** (cf [Tei36b, §4. Satz]). *Let $(A, k)$ be a Cohen ring. There exists a unique choice of multiplicative representatives for $k^{p^\infty}$:*

$$s_A : k^{p^\infty} \longrightarrow A.$$

The proof can be found in many places, for example [Coh46, Lemma 7]. In fact, such a map $s_A$ is also multiplicative in a stronger sense, namely that $s_A(\alpha)s_A(\beta) = s_A(\alpha\beta)$, for all $\alpha, \beta \in k^{p^\infty}$.

3.2. **$\lambda$-maps.** A subset $\boldsymbol{\beta} \subseteq k$ is **$p$-independent** if $[k^p(\beta_1, ..., \beta_r) : k^p] = p^r$, for all pairwise distinct $r$-elements $\beta_1, \dots, \beta_r \in \boldsymbol{\beta}$, and for all $r \in \mathbb{N}$; and $\boldsymbol{\beta}$ is a **$p$-basis** if furthermore $k = k^p(\boldsymbol{\beta})$. Equivalently, a $p$-basis is a maximal $p$-independent subset. The cardinality of a $p$-basis of $k$ does not depend on the choice of any particular $p$-basis, and it is called the **imperfection degree**[3] of $k$. See [Tei36a], [Mac39a], and [Mac39b], for more information on $p$-independence and $p$-bases.

Our next task is to develop the theory of $\lambda$-maps and $\lambda$-representatives with respect to arbitrary $p$-independent subsets $\boldsymbol{\beta}$, which certainly may be infinite, since in general the imperfection degree of a field may be any cardinal number. Nevertheless, we note that, for our applications in Part 2, it will suffice to consider finite $p$-independent tuples.

For a cardinal $\nu$, and $m \in \mathbb{N}$, $m \in \mathbb{N}$ we define

$$P_{\nu,m} := \left\{ (i_\mu)_{\mu<\nu} \;\middle|\; |\{\mu < \nu \mid i_\mu \neq 0\}| < \infty \text{ and } \forall \mu < \nu,\ 0 \leq i_\mu < p^m \right\}$$

to be the set of the multi-indices of finite support, in $\nu$-many elements, and in which each index is a non-negative integer strictly less that $p^m$. In this context, 'finite support' means that any such multi-index contains only finitely many non-zero indices. We emphasise that this set is just a technical device to facilitate our analysis of $p$-independence. Note that the family $(P_{\nu,m})_{m\in\mathbb{N}}$ of these sets forms an inverse system, with maps

$$\begin{aligned} P_{\nu,m} &\longrightarrow P_{\nu,l} \\ I &\longmapsto \mathrm{red}_{m,l}(I), \end{aligned}$$

given by coordinate-wise reduction modulo $p^l$, for $l \leq m$. The relationships between the various multi-indices can be understood using the 'addition' map[4] $\oplus : P_{\nu,l} \times P_{\nu,m-l} \longrightarrow P_{\nu,m}$, for $l \leq m$, which we define by writing

$$I \oplus J := (i_\mu + p^l j_\mu)_{\mu<\nu}.$$

for $I = (i_\mu)_\mu \in P_{\nu,l}$ and $J = (j_\mu)_\mu \in P_{\nu,m-l}$. Indeed, $\oplus$ is a bijection.

We denote by $\Omega_\nu$ the set of $p$-independent subsets of $k$, which are indexed by $\nu$. Let $\boldsymbol{\beta} = (\beta_\mu)_{\mu<\nu} \in \Omega_\nu$ and let $I \in P_{\nu,m}$. We write

$$\beta^I := \prod_{\mu<\nu} \beta_\mu^{i_\mu}$$

[3]Imperfection degree is sometimes called *Ershov degree*.

[4]We caution that, although this 'addition' is written additively, it is not commutative.

for the $I$-th monomial of $\boldsymbol{\beta}$. For each $\boldsymbol{\beta} \in \Omega_\nu$ and each $\alpha \in k^{p^m}(\boldsymbol{\beta})$, there is a unique family $(\lambda_I^{\boldsymbol{\beta}}(\alpha))_{I \in P_{\nu,m}}$ of elements of $k$ such that

$$\alpha = \sum_{I \in P_{\nu,m}} \beta^I \lambda_I^{\boldsymbol{\beta}}(\alpha)^{p^m}.$$

Note that this sum is finite since $\lambda_I^{\boldsymbol{\beta}}(\alpha)$ is zero for cofinitely many $I \in P_{\nu,m}$. We refer to

$$\begin{aligned} \lambda_I^{\boldsymbol{\beta}} : k^{p^m}(\boldsymbol{\beta}) &\longrightarrow k \\ \alpha &\longmapsto \lambda_I^{\boldsymbol{\beta}}(\alpha) \end{aligned}$$

as the $I$-th **$\lambda$-map** with respect to $\boldsymbol{\beta}$. The compatibility between the $\lambda$-maps may be expressed as follows. For $I \in P_{\nu,l}$ and $J \in P_{\nu,m-l}$, we have

$$\lambda_J^{\boldsymbol{\beta}} \circ \lambda_I^{\boldsymbol{\beta}} = \lambda_{I\oplus J}^{\boldsymbol{\beta}},$$

where the left-hand side is restricted to a map $k^{p^m}(\boldsymbol{\beta}) \longrightarrow k$. This relationship is proved by noting that $\beta^I(\beta^J)^{p^l} = \beta^{I\oplus J}$ and

$$\lambda_I^{\boldsymbol{\beta}}(\alpha) = \sum_{J \in P_{\nu,m-l}} \beta^J \lambda_{I\oplus J}^{\boldsymbol{\beta}}(\alpha)^{p^{m-l}}.$$

See also section 10.

3.3. **$\lambda$-representatives.** We work with a pre-Cohen ring $(A,k)$, a $p$-independent subset $\boldsymbol{\beta} \in \Omega_\nu$, and representatives $s : \boldsymbol{\beta} \longrightarrow A$. For $I \in P_{\nu,m}$, we write

$$s(\beta^I) := \prod_{\mu<\nu} (s(\beta_\mu))^{i_\mu}.$$

Also, for each $\alpha \in k^{p^m}(\boldsymbol{\beta})$, we write $L_I^{\boldsymbol{\beta}}(\alpha) := \mathrm{res}^{-1}(\lambda_I^{\boldsymbol{\beta}}(\alpha))$ and $L_I^{\boldsymbol{\beta}}(\alpha)^{(p^n)} := \{a^{p^n} \mid a \in L_I^{\boldsymbol{\beta}}(\alpha)\}$.

**Definition 3.4.** A **$\lambda(s,m)$-representative** of $\alpha \in k^{p^m}(\boldsymbol{\beta})$ is any element of

$$U_m^s(\alpha) := \sum_{I \in P_{\nu,m}} s(\beta^I) L_I^{\boldsymbol{\beta}}(\alpha)^{(p^m)} + \mathfrak{m}^m.$$

In the proof of the following lemma, we follow the pattern of argument in [Ser79, Proposition 8] quite closely.

**Lemma 3.5.** *Let $\boldsymbol{\beta} \in \Omega_\nu$ with representatives $s : \boldsymbol{\beta} \longrightarrow A$, and let $\alpha \in k^{p^m}(\boldsymbol{\beta})$.*

- **(i)** *The set $U_m^s(\alpha)$ of $\lambda(s,m)$-representatives of $\alpha$ is non-empty and closed.*
- **(ii)** *If $l \leq m$ then $U_l^s(\alpha) \supseteq U_m^s(\alpha)$.*
- **(iii)** *For $a_1, a_2 \in U_m^s(\alpha)$, we have $a_1 - a_2 \in \mathfrak{m}^m$.*
- **(iv)** *If $(A,k)$ is non-strict of characteristic $\leq p^m$ then $|U_m^s(\alpha)| = 1$, so there is a unique $\lambda(s,m)$-representative of $\alpha$.*

*Proof.* For **(i)**, since $\alpha \in k^{p^m}(\boldsymbol{\beta})$, each $L_I^{\boldsymbol{\beta}}(\alpha)$ is non-empty. It follows that $U_m^s(\alpha)$ is non-empty. To see that $U_m^s(\alpha)$ is closed, note that the residue map is continuous when we endow $k$ with the discrete topology.

For **(ii)**, we begin by noting that $k^{p^l}(\boldsymbol{\beta}) \supseteq k^{p^m}(\boldsymbol{\beta})$, so that $L_J^{\boldsymbol{\beta}}(\alpha)$ is defined, for each $J \in P_{\nu,l}$. Let $a \in U_m^s(\alpha)$. Then there exists $l_I \in L_I^{\boldsymbol{\beta}}(\alpha)$, for $I \in P_{\nu,m}$, such that

$$a \equiv \sum_{I \in P_{\nu,m}} s(\beta^I) l_I^{p^m} \pmod{\mathfrak{m}^m}.$$

Moreover $\mathrm{res}(l_I) = \lambda_I^{\boldsymbol{\beta}}(\alpha)$. We freely write $I = J_1 \oplus J_2$ to mean not only that this equation holds but implicitly that $I \in P_{\nu,m}$, $J_1 \in P_{\nu,l}$, and $J_2 \in P_{\nu,m-l}$. We rearrange the expression for $a$:

$$\begin{aligned} a &\equiv \sum_{I \in P_{\nu,m}} s(\beta^I) l_I^{p^m} \pmod{\mathfrak{m}^m} \\ &\equiv \sum_{J_1 \in P_{\nu,l}} s(\beta^{J_1}) \Bigg( \sum_{J_2 \in P_{\nu,m-l}} s(\beta^{J_2}) l_{J_1 \oplus J_2}^{p^{m-l}} \Bigg)^{p^l} \pmod{\mathfrak{m}^l}, \end{aligned}$$

by Lemma 3.2. This leads us to denote

$$l_{J_1} := \sum_{J_2 \in P_{\nu,m-l}} s(\beta^{J_2}) l_{J_1 \oplus J_2}^{p^{m-l}},$$

for $J_1 \in P_l$, from which we have

$$a \equiv \sum_{J_1 \in P_{\nu,l}} s(\beta^{J_1}) l_{J_1}^{p^l} \pmod{\mathfrak{m}^l}.$$

It only remains to show that $l_{J_1} \in L_{J_1}^{\boldsymbol{\beta}}(\alpha)$, i.e. $\mathrm{res}(l_{J_1}) = \lambda_{J_1}^{\boldsymbol{\beta}}(\alpha)$. This follows from taking residues:

$$\mathrm{res}(a) = \sum_{J_1 \in P_{\nu,l}} \beta^{J_1} \mathrm{res}(l_{J_1})^{p^l},$$

and applying the linear independence of $\{\beta^{J_1} \mid J_1 \in P_{\nu,l}\}$ over $k^{p^l}$.

For **(iii)**, and for $i \in \{1,2\}$, since $\mathfrak{m}^m - \mathfrak{m}^m \subseteq \mathfrak{m}^m$, we may as well suppose that $a_i$ is of the form

$$a_i = \sum_{I \in P_{\nu,m}} s(\beta^I) l_{I,i}^{p^m},$$

for $l_{I,i} \in L_I^{\boldsymbol{\beta}}(\alpha)$. Thus $\mathrm{res}(l_{I,1}) = \mathrm{res}(l_{I,2}) = \lambda_I^{\boldsymbol{\beta}}(\alpha)$. In particular $l_{I,1} - l_{I,2} \in \mathfrak{m}$. Applying Lemma 3.2, we have $l_{I,1}^{p^m} - l_{I,2}^{p^m} \in \mathfrak{m}^{m+1}$. The result now follows from a simple calculation:

$$\begin{aligned} a_1 - a_2 &= \sum_{I \in P_{\nu,m}} s(\beta^I)(l_{I,1}^{p^m} - l_{I,2}^{p^m}) \\ &\in \sum_{I \in P_{\nu,m}} s(\beta^I) \mathfrak{m}^{m+1} \\ &\subseteq \mathfrak{m}^{m+1} \\ &\subseteq \mathfrak{m}^m. \end{aligned}$$

For **(iv)**, if $\mathrm{char}(A) \leq p^m$ then $\mathfrak{m}^m = 0$. It follows from **(iii)** that there is at most one element of $U_m^s(\alpha)$, and by **(i)** there is at least one element of $U_m^s(\alpha)$. □

**Definition 3.6.** If $(A,k)$ has characteristic $p^m$, then the map of **$\lambda(s)$-representatives** is the map $S : k^{p^m}(\boldsymbol{\beta}) \longrightarrow A$ which maps each $\alpha$ to its unique $\lambda(s,m)$-representative. On the other hand, if $(A,k)$ is a strict pre-Cohen ring, then the map of **$\lambda(s)$-representatives** is the map $S : \bigcap_{m \in \mathbb{N}} k^{p^m}(\boldsymbol{\beta}) \longrightarrow A$ such that each $\alpha$ is mapped to $S(\alpha)$ which is a $\lambda(s,m)$-representative of $\alpha$, for all $m \in \mathbb{N}$.

**Theorem 3.7.** *Let $\boldsymbol{\beta} \in \Omega_\nu$ with representatives $s : \boldsymbol{\beta} \longrightarrow A$. If $(A,k)$ is a Cohen ring then there is a unique map of $\lambda(s)$-representatives:*

- **(i)** *$S : k^{p^m}(\boldsymbol{\beta}) \longrightarrow A$ if $A$ is of characteristic $p^m$, and*
- **(ii)** *$S : \bigcap_{m \in \mathbb{N}} k^{p^m}(\boldsymbol{\beta}) \longrightarrow A$ if $A$ is of characteristic zero.*

*Proof.* Part **(i)** follows immediately from Lemma 3.5**(iv)**. For **(ii)**, let $\alpha \in \bigcap_m k^{p^m}(\boldsymbol{\beta})$. and consider the sequence $(U^s_m(\alpha))_{m\in\mathbb{N}}$. The elements of $A$ that are $\lambda(s,m)$-representatives of $\alpha$, for all $m$, are precisely the elements of the intersection $\bigcap_m U^s_m(\alpha)$. By Lemma 3.5**(iii)**, the intersection has at most one element. In fact, since this is the intersection of a descending chain of non-empty closed subsets of a complete metric space, the intersection is non-empty. Thus a map of $\lambda(s)$-representatives exists and is unique. □

*Remark* 3.8. The above statements admit several variations. First, if $\boldsymbol{\beta}$ is a $p$-basis of $k$ then $k^{p^m}(\boldsymbol{\beta}) = k$, for all $m \in \mathbb{N}$. Therefore in both the non-strict and strict cases, the domain of the map of $\lambda(s)$-representatives is $k$.

Second, if $A$ is of characteristic $p^m$, and if $\alpha \in k^{p^n}(\boldsymbol{\beta})$, for some $n \geq m$, then $U^s_n(\alpha)$ is defined, and it equals $U^s_m(\alpha)$. Therefore the unique $\lambda(s,m)$-representative of $\alpha$ is already a $\lambda(s,n)$-representative of $\alpha$.

Third, denote by $\mathbf{0} = (0,\ldots,0) \in P_{\nu,m}$ the multi-index consisting of a tuple of zeroes. Then, for any $\alpha \in k$, we have $\lambda^{\boldsymbol{\beta}}_{\mathbf{0}}(\alpha^{p^m}) = \alpha$, and so $s(\alpha^{p^m}) = s(\alpha)^{p^m}$. In particular, the restriction of the map of $\lambda(s)$-representatives $S$ to $k^{p^\infty}$ coincides with the unique choice of multiplicative representatives.

For a cardinal $\nu$, we denote by $\Theta_\nu$ the set of $\mathbf{b} = (b_\mu)_{\mu<\nu} \subseteq A$, which are indexed by $\nu$, such that $\mathrm{res}(\mathbf{b}) := (\mathrm{res}(b_\mu))_{\mu<\nu} \in \Omega_\nu$. Thus, $\Theta_\nu$ is the coordinate-wise pre-image of $\Omega_\nu$ under the residue map.

**Definition 3.9.** Suppose that the characteristic of $A$ is $p^m$. The map of **$\lambda$-representatives** is the partial map

$$S_{\nu,m} : \Theta_\nu \times k \dashrightarrow A,$$

which is defined for $\mathbf{b} \in \Theta_\nu$ and $\alpha \in k^{p^m}(\mathrm{res}(\mathbf{b}))$, and $S_{\nu,m}(\mathbf{b},\alpha)$ is the unique $\lambda(s,m)$-representative of $\alpha$, where $s$ is defined by $s(\mathrm{res}(b_\mu)) = \beta_\mu$, for $\mu < \nu$.

## 4. The Teichmüller Embedding Process

At the heart of all the structural arguments about Cohen rings is Teichmüller's embedding process, which we discuss in this section. The original formulation can be found in [Tei36b, §7]. Indeed, Mac Lane attributes this technique to Teichmüller, and describes it as the 'Teichmüller embedding process'. See [Mac39c, Theorem 6] for Mac Lane's version. In [Coh46, Lemma 12], Cohen rewrote Teichmüller's embedding process for an arbitrary complete local ring.

**Theorem 4.1** (Teichmüller Embedding Process)**.** *Let $(A,k)$ be a Cohen ring, let $\boldsymbol{\beta} \subseteq k$ be $p$-independent with representatives $s : \boldsymbol{\beta} \longrightarrow A$. There exists a Cohen ring $(A^T,k^T) \supseteq (A,k)$ such that*

- **(i)** $k^T = k(\boldsymbol{\beta}^{p^{-\infty}})$,
- **(ii)** *$s$ coincides with the restriction to $\boldsymbol{\beta}$ of the unique choice of multiplicative representatives $(k^T)^{p^\infty} \longrightarrow A^T$.*

*Proof.* This proof is closely based on those of Teichmüller ([Tei36b, §7]) and Cohen ([Coh46, Lemma 12]). It is a recursive construction. We begin by formally adjoining a $p$-th root of each $s(\beta)$, for each $\beta \in \boldsymbol{\beta}$. More constructively, we introduce a family of new variables $(X_\beta : \beta \in \boldsymbol{\beta})$, and let

$$A' := A[X_\beta : \beta \in \boldsymbol{\beta}]/\big(X^p_\beta - s(\beta) : \beta \in \boldsymbol{\beta}\big).$$

That is, $A'$ is the quotient of the ring $A[X_\beta : \beta \in \boldsymbol{\beta}]$ by the ideal generated by the polynomials $X^p_\beta - s(\beta)$, for $\beta \in \boldsymbol{\beta}$. The natural map $A \longrightarrow A'$ is injective.

Then $A'$ is a local ring with residue field $k' := k(\beta^{p^{-1}} : \beta \in \boldsymbol{\beta})$. That is, $(A',k')$ is pre-Cohen ring and, by identifying $A$ with its image in $A'$, we have $(A,k) \subseteq (A',k')$. We write $s'(\beta)$ for the image of $X_\beta$ in the quotient ring $A'$. If we write $\boldsymbol{\beta}' := \{\beta^{p^{-1}} \mid \beta \in \boldsymbol{\beta}\}$, then $s' : \boldsymbol{\beta}' \longrightarrow k'$ is a choice of representatives, and

$$s'(\beta^{p^{-1}})^p = s(\beta),$$

for all $\beta \in \boldsymbol{\beta}$.

Beginning with $(A, k)$, we continue this process recursively, with recursive step $(A, k) \longmapsto (A', k')$. In this way, we construct a chain $(A_n, k_n)_{n\in\mathbb{N}}$ of pre-Cohen rings, such that $\boldsymbol{\beta}_n = \{\beta^{p^{-n}} \mid \beta \in \boldsymbol{\beta}\}$ is $p$-independent in $k_n$ and $s_n : \boldsymbol{\beta}_n \longrightarrow A_n$ is a choice of representatives, such that

$$s_n(\beta^{p^{-n}})^{p^n} = s(\beta),$$

for all $n \in \mathbb{N}$ and all $\beta \in \boldsymbol{\beta}$.

The morphisms in this chain are embeddings, which we may even view as inclusions, by identifying of each $(A_n, k_n)$ with its image in $(A_{n+1}, k_{n+1})$. The direct limit is a pre-Cohen ring $(A_\infty, k_\infty) \supseteq (A, k)$. Taking the completion, we obtain a Cohen ring $(A^T, k^T) \supseteq (A, k)$. The union $s^T := \bigcup_n s_n$ is a choice of representatives for $\boldsymbol{\beta}^T := \bigcup_n \boldsymbol{\beta}_n$ which commutes with the Frobenius map. Also, by construction, we have $k^T = k(\boldsymbol{\beta}^{p^{-\infty}})$, and so $\boldsymbol{\beta}^T \subseteq (k^T)^{p^\infty}$. Therefore $s^T$ coincides with the restriction to $\boldsymbol{\beta}^T$ of the unique choice of multiplicative representatives $(k^T)^{p^\infty} \longrightarrow A^T$, as required. □

## 5. Mac Lane's Identity Theorem

In this section we consider Cohen subrings of Cohen rings. We study the 'identity' of such subrings inside their overrings: in Theorem 5.1, which was first clearly articulated by Mac Lane, we show that such a subring is determined by a choice of representatives of a $p$-basis of its residue field.

Teichmüller's discussion of this issue can be found in [Tei36b, §8]. Developing these ideas, Mac Lane's theorems [Mac39c, Theorem 7] and [Mac39c, Theorem 12] show that a complete subfield of a $\mathfrak{p}$-adic field, in his language, is determined by choice of representatives for a $p$-basis of the residue field. Indeed, in our view, Mac Lane is the first to have clearly articulated this portion of the overall argument. Nevertheless, we closely follow Cohen's exposition, particularly relevant parts of his proof of [Coh46, Theorem 11], which is in fact the theorem we will discuss in the next section.

**Theorem 5.1** (Mac Lane's Identity Theorem)**.** *Let $(B, l)$ be a pre-Cohen ring, with Cohen subrings $(A_1, k_1)$ and $(A_2, k_2)$. Suppose that $k_1 \subseteq k_2$ and that $k_2/k_1$ is separable. Let $\boldsymbol{\beta}$ be a $p$-basis of $k_1$ with representatives $s : \boldsymbol{\beta} \longrightarrow A_1$. If $s(\boldsymbol{\beta}) \subseteq A_2$ then $(A_1, k_1) \subseteq (A_2, k_2)$.*

*Proof.* Since $k_2/k_1$ is separable, there is a $p$-basis $\boldsymbol{\beta}_2 \subseteq k_2$ which contains $\boldsymbol{\beta}$. Let $s_{2,0} : \boldsymbol{\beta}_2 \longrightarrow A_2$ be any choice of representatives extending $s$.

We claim that, for all $a_1 \in A_1$ and all $r \in \mathbb{N}_{>0}$, there exists $a_2 \in A_2$ with $a_1 - a_2 \in \mathfrak{m}_B^r$. Topologically, this amounts to claiming that $A_1$ is contained in the closure of $A_2$, with respect to the local topology on $B$. We prove this by induction on $r$. The base case, i.e. $r = 1$, follows from the fact that $k_1 \subseteq k_2$. Assume, as an inductive hypothesis, that for some $r \in \mathbb{N}_{>0}$ that for all $a_1 \in A_1$ there exists $a_2 \in A_2$ with $a_1 - a_2 \in \mathfrak{m}_B^r$. Let $a_1 \in A_1$ and denote $\alpha := \mathrm{res}(a_1) \in k_1$.

Applying Theorem 3.7, we let $S_1 : k_1 \longrightarrow A_1$ be the unique map of $\lambda(s)$-representatives. Let $S_2 : k_2 \longrightarrow A_2$ be the unique map of $\lambda(s_{2,0})$-representatives. Then $S_2$ extends $S_1$. Let $\hat{a}_1 \in A_1$ be such that $a_1 = S_1(\alpha) + p\hat{a}_1$. By inductive hypothesis, choose $\hat{a}_2 \in A_2$ such that $\hat{a}_1 - \hat{a}_2 \in \mathfrak{m}_B^r$. Write $a_2 := S_2(\alpha) + p\hat{a}_2 \in A_2$. Therefore

$$a_1 - a_2 = S_1(\alpha) - S_2(\alpha) + p(\hat{a}_1 - \hat{a}_2) = p(\hat{a}_1 - \hat{a}_2) \in \mathfrak{m}_B^{r+1},$$

as required. Since $(A_2, k_2)$ is complete, we have $(A_1, k_1) \subseteq (A_2, k_2)$, as required. □

## 6. Cohen's Homomorphism Theorem and Structure Theorem

The remaining ingredient of a structure theorem is the relationship between two arbitrary Cohen rings with the same residue field. Such a relationship exists, in the form of a morphism, and such a morphism is uniquely determined by specifying the image of a set of representatives of a $p$-base of the residue field.

Cohen's paper [Coh46] appears to be the first to study the case of characteristic $p^k$, $k > 0$. In this section we state and prove a version of Cohen's Theorem, [Coh46, Theorem 11], suitable for our setting.

**Definition 6.1.** Let $(A, k)$ and $(B, l)$ be pre-Cohen rings, and let $\phi = (\phi_A, \phi_k) : (A, k) \longrightarrow (B, l)$ be a morphism. Also, let $\boldsymbol{\beta} \subseteq k$ be a $p$-basis of $k$, and let $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \phi_k(\boldsymbol{\beta}) \longrightarrow B$ be representatives. We say that $\phi$ **respects** $s_A$ and $s_B$ if $\phi_A \circ s_A = s_B \circ \phi_k$. See Figure 1.

$$\begin{tikzcd}
B \arrow[rr, dotted, "\mathrm{res}_B"] & & l \arrow[ll, dotted, bend left, "s_B"] \\
A \arrow[u, "\phi_A"] \arrow[rr, dotted, "\mathrm{res}_A"] & & k \arrow[u, "\phi_k"'] \arrow[ll, dotted, bend left, "s_A"]
\end{tikzcd}$$

Figure 1. Illustration of Definition 6.1

**Theorem 6.2** (Cohen's Homomorphism Theorem)**.** *Let $(A, k)$ and $(B, l)$ be Cohen rings, and let $\phi_k : k \longrightarrow l$ be an embedding of fields such that $l/\phi_k(k)$ is a separable extension. Let $\boldsymbol{\beta}$ be a $p$-basis of $k$ and let $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \phi_k(\boldsymbol{\beta}) \longrightarrow B$ be representatives. Suppose that $(A, k)$ is strict. Then there exists a unique ring homomorphism $\phi_A : A \longrightarrow B$ such that*

$$\phi = (\phi_A, \phi_k) : (A, k) \longrightarrow (B, l)$$

*is a morphism which respects $s_A$ and $s_B$.*

*Moreover, if $(B, l)$ is also strict then $\phi$ is an embedding. Otherwise, if the characteristic of $(B, l)$ is $p^m$ then $\phi$ factors through the natural morphism*

$$(A, k) \longrightarrow (A/\mathfrak{m}_A^m, k).$$

*Proof.* This proof is closely based on that of Cohen ([Coh46, Theorem 11]). For notational simplicity, we identify $k$ with its image in $l$ under the embedding $\phi_k$. Then $\phi_k$ is the inclusion map id, and $l/k$ is a separable extension.

To begin with, we suppose that $k$ is perfect. Thus $\boldsymbol{\beta}$ is empty, and we dispense with both of the maps $s_A$ and $s_B$. Since $(A, k)$ is a strict Cohen ring, we have $(\mathbb{Z}_p, \mathbb{F}_p) \subseteq (A, k)$, and there is the following natural morphism:

$$\phi_0 : (\mathbb{Z}_p, \mathbb{F}_p) \longrightarrow (B, l).$$

Let $T$ be a transcendence basis of $k/\mathbb{F}_p$. Since $k$ is perfect, we have $T \subseteq k^{p^\infty} = k \subseteq l$. By Theorem 3.3, there are unique choices of multiplicative representatives:

$$s_{A,0} : T \longrightarrow A$$

and

$$s_{B,0} : T \longrightarrow B.$$

Since $(A, k)$ is strict, the set $s_{A,0}(T)$ is algebraically independent over $\mathbb{Z}_p$, and we may extend $\phi_0$ to a morphism

$$\phi_{1,0} : (\mathbb{Z}_p[s_{A,0}(T)], \mathbb{F}_p(T)) \longrightarrow (B, l)$$

by writing $\phi_{1,0}(s_{A,0}(t)) = s_{B,0}(t)$, for each $t \in T$. Note that $\phi_{1,0}$ is the inclusion map on the residue field. In fact, for each $n \in \mathbb{N}$, we may construct in the same way a morphism

$$\phi_{1,n} : (\mathbb{Z}_p[s_{A,0}(T^{p^{-n}}), \mathbb{F}_p(T^{p^{-n}})) \longrightarrow (B, l),$$

by writing $\phi_{1,n}(s_{A,0}(t^{p^{-n}})) = s_{B,0}(t^{p^{-n}})$, for each $t \in T$. Again, $\phi_{1,n}$ is the inclusion map on the residue field. Since $s_{A,0}$ and $s_{B,0}$ are multiplicative, the family $(\phi_{1,n})_{n\in\mathbb{N}}$ of morphisms is consistent, and so forms a chain. Taking the direct limit (i.e. union), and denoting $A_0 := \mathbb{Z}_p[s_{A,0}(T^{p^{-n}}) \mid n \in \mathbb{N}]$ and $k_0 := \mathbb{F}_p(T)^{\mathrm{perf}}$, we have constructed a morphism

$$\phi_2 : (A_0, k_0) \longrightarrow (B, l),$$

which again is the inclusion map on the residue field.

The final part of this construction is to extend $\phi_2$ to have domain $(A, k)$. Since strict Cohen rings are henselian valuation rings, and $k/k_0$ is separable algebraic, this extension can be accomplished by a direct application of Hensel's Lemma, as in e.g. [Kuh11, Lemma 9.30]. More precisely, for a separable irreducible polynomial $f \in A_0[X]$ and $\alpha \in k$ with $\mathrm{res}(f)(\alpha) = 0$, by Hensel's Lemma we obtain $a \in A$ such that $f(a) = 0$. Likewise, we obtain $b \in B$ with $\phi_2(f)(b) = 0$. We now extend $\phi_2$ to a morphism

$$\phi_3 : (A_0[a], k_0(\alpha)) \longrightarrow (B, l)$$

by sending $a \longmapsto b$. Taking the directed limit of morphisms constructed in this way, we obtain

$$\phi : (A, k) \longrightarrow (B, l)$$

as required. It remains to show that $\phi$ is the unique such morphism, but this follows from Theorem 5.1.

We turn to the case that $k$ is imperfect. We are given a $p$-basis $\boldsymbol{\beta}$ of $k$ with representatives $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \boldsymbol{\beta} \longrightarrow B$. Note that $\boldsymbol{\beta}$ is $p$-independent in $l$, by our assumption that $l/k$ is separable. By Theorem 4.1, there exists a Cohen ring $(A^T, k^T) \supseteq (A, k)$ such that

- **(i)** $k^T = k^{\mathrm{perf}}$, and
- **(ii)** $s_A$ coincides with the unique choice of multiplicative representatives $\boldsymbol{\beta} \longrightarrow A^T$.

By another application of Theorem 4.1, there exists a Cohen ring $(B^T, l^T) \supseteq (B, l)$ such that

- **(iii)** $l^T = l(\boldsymbol{\beta}^{p^{-\infty}})$, and
- **(iv)** $s_B$ coincides with the unique choice of multiplicative representatives $\boldsymbol{\beta} \longrightarrow B^T$.

Since $k^T \subseteq l^T$ and $k^T$ is perfect, by the first part of this proof there exists a unique morphism

$$\phi : (A^T, k^T) \longrightarrow (B^T, l^T).$$

As Cohen notes, it now suffices to argue that the image of $(A, k)$ under $\phi$ is a subring of $(B, l)$. We show that $\phi_A(s_A(\boldsymbol{\beta})) \subseteq s_B(\boldsymbol{\beta})$. The composition $\phi_A \circ s_A : \boldsymbol{\beta} \longrightarrow B^T$ is a choice of multiplicative representatives, so it must coincide with $s_B$. Thus both the image of $(A, k)$ under $\phi$, and $(B, l)$ contain $s_B(\boldsymbol{\beta})$. Since $l/k$ is separable, by Theorem 5.1, we have $(\phi_A(A), k) \subseteq (B, l)$. The 'uniqueness' part of the statement follows immediately from another application of Theorem 5.1. This completes the proof in the case that $k$ is imperfect.

Finally, we note that either $\phi$ is an embedding, or it has non-trivial kernel. The only non-trivial proper ideals in $(A, k)$ are of the form $\mathfrak{m}_A^m$, for $m \in \mathbb{N}_{>0}$. □

The following corollary is a general version of Cohen's Structure Theorem which includes the non-strict case.

**Corollary 6.3** (Cohen Structure Theorem, v.1). *Let $k$ be a field of characteristic $p > 0$, and let $\boldsymbol{\beta} \subseteq k$ be a $p$-basis. Let $(A_1, k)$ and $(A_2, k)$ be two Cohen rings of the same characteristic, with representatives $s_i : \boldsymbol{\beta} \longrightarrow A_i$, for $i = 1, 2$. There exists a unique isomorphism of Cohen rings*

$$\phi = (\phi_A, \mathrm{id}_k) : (A_1, k) \longrightarrow (A_2, k),$$

*which respects $s_1$ and $s_2$, and which is the identity on the residue fields.*

*Proof.* If both $(A_1, k)$ and $(A_2, k)$ are strict then both existence and uniqueness follow from Theorem 6.2. Suppose next that both $(A_1, k)$ and $(A_2, k)$ are of characteristic $p^m$. Let $(B, k)$ be a strict Cohen ring. By Theorem 6.2 there are unique morphisms $\phi_i = (\phi_{i,A}, \mathrm{id}) : (B, k) \longrightarrow (A_i, k)$, for $i = 1, 2$. Moreover, both $\phi_{i,A}$ are surjective and both factor through the quotient map $B \longrightarrow B/\mathfrak{m}^m$. Thus, by the Isomorphism Theorem, both $(A_i, k)$ are isomorphic to $(B/\mathfrak{m}^m, k)$. Any isomorphism which respects $s_1$ and $s_2$ must come about in the same way, and so uniqueness also follows from Theorem 6.2. □

**Corollary 6.4** (Cohen Structure Theorem, v.2). *Let $(A_1, k)$ and $(A_2, k)$ be Cohen rings of the same characteristic, and with the same residue field. There exists an isomorphism of Cohen rings*

$$\phi = (\phi_A, \mathrm{id}_k) : (A_1, k) \longrightarrow (A_2, k),$$

*which is the identity on the residue fields.*

*Proof.* Immediate from Corollary 6.3. □

## 7. The Relative Structure Theorem

Our aim is to combine the results of the previous sections to give a clear statement of the relative structure of Cohen rings. That is, we will describe the morphisms between Cohen rings which fix a common subring, or which extend a given morphism between subrings. It should be noted that, this is closely based on the work of Teichmüller, Mac Lane, Cohen, and others. See for example [Tei36b], [Mac39c], and [Coh46].

**Theorem 7.1** (Embedding Lemma). *Let $(A_1, k_1) \subseteq (A_2, k_2)$ and $(B_1, l_1) \subseteq (B_2, l_2)$ be two extensions of Cohen rings, and suppose that $k_2/k_1$ is separable. Let*

$$\phi = (\phi_A, \phi_k) : (A_1, k_1) \longrightarrow (B_1, l_1)$$

*be a morphism, and let $\Phi_k : k_2 \longrightarrow l_2$ be an embedding of fields which extends $\phi_k$, and is such that $l_2/\Phi_k(k_2)$ is separable. Let $\boldsymbol{\beta}$ be a $p$-base of $k_2$ over $k_1$, and let $s_A : \boldsymbol{\beta} \longrightarrow A_2$ and $s_B : \Phi_k(\boldsymbol{\beta}) \longrightarrow B_2$ be choices of representatives.*

*There exists a unique morphism of Cohen rings*

$$\Phi := (\Phi_A, \Phi_k) : (A_2, k_2) \longrightarrow (B_2, l_2),$$

*which respects $s_A$ and $s_B$, which is $\Phi_k$ on the residue fields, and which extends $\phi$.*

*Proof.* We are given a $p$-basis $\boldsymbol{\beta}$ of $k_2$ over $k_1$. Choose any $p$-basis $\boldsymbol{\beta}_{A,1}$ of $k_1$ and any representatives $s_1 : \boldsymbol{\beta}_1 \longrightarrow A_1$. Since $k_2/k_1$ is separable, $\boldsymbol{\beta}_{A,2} := \boldsymbol{\beta} \sqcup \boldsymbol{\beta}_{A,1}$ is a $p$-basis of $k_2$. We define

$$\begin{aligned} s_{A,2} : \boldsymbol{\beta}_{A,2} &\longrightarrow A_2 \\ b &\longmapsto \begin{cases} s_{A,1}(b) & b \in \boldsymbol{\beta}_{A,1} \\ s_A(b) & b \in \boldsymbol{\beta}, \end{cases} \end{aligned}$$

which is a choice of representatives for $\boldsymbol{\beta}_{A,2}$. Next we let $\boldsymbol{\beta}_{B,2} := \phi_k(\boldsymbol{\beta}_{A,1}) \sqcup \Phi_k(\boldsymbol{\beta})$. We define

$$\begin{aligned} s_{B,2} : \boldsymbol{\beta}_{B,2} &\longrightarrow B_2 \\ b &\longmapsto \begin{cases} \phi_A(s_{A,1}(b)) & b \in \phi_k(\boldsymbol{\beta}_{A,1}) \\ s_B(b) & b \in \Phi_k(\boldsymbol{\beta}), \end{cases} \end{aligned}$$

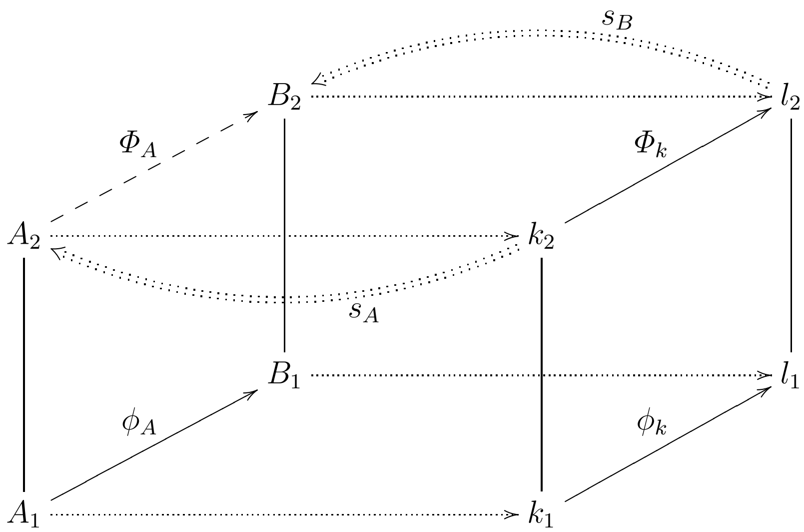


FIGURE 2. Illustration of Theorem 7.1

which is a choice of representatives for $\boldsymbol{\beta}_{B,2}$. It follows from Theorem 6.2 that there is a unique morphism

$$\Phi = (\Phi_A, \Phi_k) : (A_2, k_2) \longrightarrow (B_2, k_2),$$

which respects $s_{A,2}$ and $s_{B,2}$, and which is $\Phi_k$ on the residue fields. Observe that $\Phi$ extends $\phi$ since in particular $\Phi$ respects $s_{A,1}$ and $\phi_A \circ s_{A,1}$ (the latter being a choice of representatives for $\phi_k(\boldsymbol{\beta}_{A,1})$). This proves the existence part of our claim. For uniqueness, if $\Psi$ is any other morphism which extends $\phi$ and respects $s_A$ and $s_B$ then we may argue that it also respects $s_{A,2}$ and $s_{B,2}$, just as for $\Phi$. Uniqueness then follows from Theorem 6.2. □

**Corollary 7.2** (Relative Structure Theorem, cf [Mac39c, Theorem 8])**.** *Let $(A_1, k_1)$ and $(A_2, k_2)$ be two Cohen rings, and let $(A_0, k_0)$ be a pre-Cohen subring common to both. Suppose that both $k_1/k_0$ and $k_2/k$ are separable extensions. Then*

$$(A_1, k_1) \cong_{(A_0,k_0)} (A_2, k_2) \qquad \Longleftrightarrow \qquad k_1 \cong_{k_0} k_2.$$

*Proof.* This is immediate from Theorem 7.1. □

In the following theorem, we deal with **enriched Cohen rings**, which are 3-tuples $(A, k, S)$ consisting of a Cohen ring $(A, k)$ equipped with family $S = (S_r)$, where $S_r$ is the partial map of $\lambda$-representatives, as in Definition 3.9. Morphisms between enriched Cohen rings are morphisms of Cohen rings which commutes with the $\lambda$-representatives in the obvious way.

**Corollary 7.3** (Relative Structure Theorem, enriched with representatives)**.** *Let $(A_1, k_1, S_1)$ and $(A_2, k_2, S_2)$ be two enriched Cohen rings, and let $(A_0, k_0, S_0)$ be a common enriched Cohen subring. In particular, this implies that all three rings are of the same characteristic and that both extensions $k_i/k_0$ are separable. Then*

$$(A_1, k_1, S_1) \cong_{(A_0,k_0,S_0)} (A_2, k_2, S_2) \qquad \Longleftrightarrow \qquad k_1 \cong_{k_0} k_2.$$

*Proof.* This is also immediate from Theorem 7.1. □

## 8. Inverse limit

Let $k$ be a field of characteristic $p$. For each $m > 0$, let $(A_m, k, S_m)$ be an enriched Cohen ring of characteristic $p^m$. If $(A, k)$ is a strict Cohen ring, by Theorem 6.2, there is a unique morphism $\mathrm{res}_m : (A, k) \longrightarrow (A_m, k)$ which is the identity on $k$. In fact, for $m \leq n$, the morphism $\mathrm{res}_m$ factors through $\mathrm{res}_n$; and there is an induced morphism $\mathrm{res}_{n,m} : (A_n, k) \longrightarrow (A_m, k)$, which in fact automatically respects the $\lambda$-representatives. In other words, $\mathrm{res}_{n,m} : (A_n, k, S_n) \longrightarrow (A_m, k, S_m)$ is a morphism of enriched Cohen rings which is the identity on $k$.

**Proposition 8.1.** *The family*

$$\big((A_m, k, S_m)_m, (\mathrm{res}_{n,m})_{n\geq m}\big)$$

*forms an inverse system. The inverse limit of this system exists, and is unique up to isomorphism, and it is a strict enriched Cohen ring* $(A_\infty, k, S_\infty)$.

*Proof.* The 'sub-inverse system' consisting simply of the rings $A_m$ and the morphisms $\mathrm{res}_{n,m}$ is an inverse system of rings, and we may verify that the inverse limit is a strict Cohen ring $(A_\infty, k)$, together with morphisms $\mathrm{res}_m : (A_\infty, k) \longrightarrow (A_m, k)$ given by taking the quotient $A_\infty \longrightarrow A_\infty/\mathfrak{m}^m_{A_\infty}$. Let $\boldsymbol{\beta} \subseteq k$ be a $p$-independent $r$-tuple. If $s : \boldsymbol{\beta} \longrightarrow A_\infty$ is a choice of representatives then we define the system of $\lambda(s)$-representatives

$$S : \bigcap_r k^{p^r}(\boldsymbol{\beta}) \longrightarrow A_\infty$$
$$\alpha \longmapsto (S_n(\alpha))_n.$$

Since all the morphisms $\mathrm{res}_{n,m}$ respect the representatives, the morphisms $\mathrm{res}_m$ also respect the representatives. $\square$

## 9. Cohen–Witt rings

Let $k$ denote a field of characteristic $p > 0$. For each natural number $n \in \mathbb{N}$, we denote the $n$**-th Witt ring** over $k$ by $W_{n+1}(k)$, and the **infinite Witt ring** we denote by $W[k]$, as described, for example, in [vdD14] and in many other places.

If $k$ is perfect, then $W[k]$ is a complete discrete valuation ring of characteristic zero with residue field $k$. That is, $(W[k], k)$ is a strict Cohen ring. By Theorem 6.2, $(W[k], k)$ may be viewed as providing the canonical example of a Cohen ring with residue field $k$, canonical in the sense that for perfect $k$ there is a canonical isomorphism between any two strict Cohen rings with residue field $k$. Likewise, $(W_n(k), k)$ is the canonical example of a Cohen ring with residue field $k$, of characteristic $p^n$.

On the other hand, if $k$ is imperfect, then $W[k]$ fails to be a valuation ring. There is a less well-known construction, appropriate for the case of imperfect residue fields, which constructs Cohen rings as subrings of Witt rings. To mitigate the conflict with our own terminology, we will refer to these more concrete rings as 'Cohen–Witt rings'. We fix a $p$-basis $\boldsymbol{\beta}$ of $k$. For each $n \in \mathbb{N}$, the $n$**-th Cohen–Witt ring** over $k$, which we denote by $C_{n+1}(k)$, is the subring of $W_{n+1}(k)$ generated by $W_{n+1}(k^{p^n})$ and the elements $[\beta] = (\beta, 0, ...)$, for $\beta \in \boldsymbol{\beta}$. That is

$$C_{n+1}(k) := W_{n+1}(k^{p^n})\big([\beta] | \beta \in \boldsymbol{\beta}\big).$$

We note that $C_{n+1}(k)$ is a local ring, with maximal ideal $(p)$ and residue field $k$. Thus $(C_{n+1}(k), k)$ is indeed a Cohen ring. There are representatives $s_n : \boldsymbol{\beta} \longrightarrow C_{n+1}(k)$, given by $s_n(\beta) = [\beta]$, for $\beta \in \boldsymbol{\beta}$. The maps $\pi_n : W_{n+1}(k) \longrightarrow W_n(k)$, which are given by the truncation of the Witt vectors, restrict to surjections

$$\pi_n|_{C_{n+1}(k)} : C_{n+1}(k) \longrightarrow C_n(k).$$

Just as with the Witt rings, the Cohen–Witt rings equipped with these maps form an inverse system, as in Proposition 8.1, the inverse limit of which is the **strict Cohen–Witt ring** over $k$:

$$C[k] := \varprojlim C_{n+1}(k).$$

This comes equipped with representatives $s : \boldsymbol{\beta} \longrightarrow C_{n+1}(k)$. It is a consequence of Corollary 6.4 that any strict Cohen ring $(A, k)$ is isomorphic to the strict Cohen–Witt ring $C[k]$, though the isomorphism is not canonical in the sense that it depends on our choices of $\boldsymbol{\beta}$ and $s$.

# Part 2. The Model Theory

Having developed the algebraic theory of Cohen rings, we are now in a position to describe their first-order theories. Let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ denote the first-order language of rings.

## 10. The language of the residue field

Recall the notation introduced in section 3.3 for $p$-independent tuples and component maps. Let $\mathcal{L}_\lambda$ be $\mathcal{L}_{\text{ring}}$ expanded by a family

$$\big(\boldsymbol{\theta}_n(x_1, ..., x_n)\big)_{n\in\mathbb{N}},$$

where $\boldsymbol{\theta}_n$ is an $n$-ary relation symbol, and a family

$$\big(\boldsymbol{\lambda}_I(x_1, ..., x_n, y)\big)_{n\in\mathbb{N}, I\in P_n},$$

where $\boldsymbol{\lambda}_I$ is an $(n+1)$-ary function symbol. We take $T_\lambda$ to be the $\mathcal{L}_\lambda$-theory which extends the theory of fields and which asserts of a model $k$ that the interpretation of $\boldsymbol{\theta}_n$ defines in $k^n$ the set of $p$-independent $n$-tuples, and the interpretation of $\boldsymbol{\lambda}_I$ in $k$ is the function

$$\lambda_I : k^{n+1} \longrightarrow k$$

such that if $k \models \boldsymbol{\theta}_n(b_1, \dots, b_n)$ and $a \in k^p(b_1, \dots, b_n)$ then

$$a = \sum_{I\in P_n} m_I(b_1, \dots, b_n)\lambda_I(b_1, \dots, b_n, a)^p.$$

Otherwise, if either $k \models \neg\boldsymbol{\theta}_n(b_1, \dots, b_n)$ or $a \notin k^p(b_1, \dots, b_n)$, then

$$\lambda_I(b_1, \dots, b_n, a) = 0.$$

Note that we are abusing notation by using $\lambda_I$ for the function symbol and for its interpretation in $k$.

**Lemma 10.1.** *Every field $k$ expands uniquely to a model $k_\lambda$ of $T_\lambda$. Moreover, a field extension $l/k$ is separable if and only if $l_\lambda/k_\lambda$ is an extension of $\mathcal{L}_\lambda$-structures.*

*Proof.* Clear. □

From now on we will not often distinguish notationally between a field $k$ and its canonical expansion $k_\lambda$ to a model of $T_\lambda$. We may refer to this convention as our *separability assumption.*

## 11. Quantifier elimination in the non-strict case

Consider the two-sorted language $\mathcal{L}_2$, which has sorts $\mathbf{A}$ and $\mathbf{k}$. The sort $\mathbf{A}$ is equipped with $\mathcal{L}_{\text{ring}}$, and the sort $\mathbf{k}$ is equipped with $\mathcal{L}_\lambda$. There is an additional function symbol $\mathbf{res} : \mathbf{A} \longrightarrow \mathbf{k}$. Let $T_2$ be the $\mathcal{L}_2$-theory that requires of $(A, k, \text{res})$ the following:

- **(i)** $A$ is a local ring, with maximal ideal $\mathfrak{m} = (p)$,
- **(ii)** with residue field $k$, which is a model of $T_\lambda$, and
- **(iii)** with residue map $\text{res} : A \longrightarrow k$, which is a ring epimorphism.

For $k_0$ any field, we define $T_2(k_0)$ to extend $T_2$ by axioms which assert of $(A, k, \text{res})$ that

- **(iv)**$_{k_0}$ the residue field $k$ is $\mathcal{L}_\lambda$-elementarily equivalent to $k_0$.

Consider the expansion $\mathcal{L}_{2,S}$ of $\mathcal{L}_2$ in which, for each $r \in \mathbb{N}$, there is an additional $r$-ary relation symbol $\boldsymbol{\Theta}_r$ on $\mathbf{A}$ and a function symbol $\mathbf{S}_r : \mathbf{A}^r \times \mathbf{k} \longrightarrow \mathbf{A}$. For $n \in \mathbb{N}$, we define $T_2(n)$ to be an $\mathcal{L}_{2,S}$-theory extending $T_2$ by axioms which assert of a model $(A, k, \text{res}, (\Theta_r), (S_r))$ that

- **(v)**$_n$ the characteristic of $A$ is $p^n$,
- **(vi)**$_n$ for each $r \in \mathbb{N}$, $\Theta_r$ is the pre-image under the residue map $\text{res} : A \longrightarrow k$ of the set of $p$-independent $r$-tuples in $k$, and
- **(vii)**$_n$ for each $r \in \mathbb{N}$, $S_r$ is the map of $\lambda$-representatives $S_{r,n}$, as in Definition 3.9.

Let $T_2(k_0, n)$ denote the union $T_2(k_0) \cup T_2(n)$. For brevity, we often write $(A, k)$ instead of $(A, k, \mathrm{res})$ or $(A, k, \mathrm{res}, (\Theta_r), (S_r))$ if the extra structure is clear from the context.

*Remark* 11.1. Of course, by design, each pre-Cohen ring $(A, k)$ may be viewed naturally as an $\mathcal{L}_2$-structure which is a model of $T_2$. Each non-strict Cohen ring admits exactly one expansion to a model of $T_{2,S}$, since the extra structure is definable in the language $\mathcal{L}_2$.

**Proposition 11.2.** *Each of the properties* **(i)**-**(iii)**, *and* $\mathbf{(iv)}_{k_0}$, *is* $\mathcal{L}_2$*-axiomatisable. Each of the properties* $\mathbf{(v)}_n$, $\mathbf{(vi)}_n$, *and* $\mathbf{(vii)}_n$*-is* $\mathcal{L}_{2,S}$*-axiomatisable.*

**Theorem 11.3.** *Let $k$ be a field of characteristic $p$ and let $n \in \mathbb{N}$. The theory $T_2(k, n)$ eliminates quantifiers relative to the sort* $\mathbf{k}$.

*Proof.* For background on relative quantifier elimination, see [Rid17, Appendix A]. We denote by $T_2^M(k, n)$ the Morleyization of $T_2(k, n)$ with respect to the sort $\mathbf{k}$. We show that $T_2^M(k, n)$ admits elimination of quantifiers, by applying Shoenfield's Criterion, [Sho71]. Suppose that we are in the following situation. Let $\mathcal{A}_1 = (A_1, k_1)$ and $\mathcal{A}_2 = (A_2, k_2)$ be two models of $T_2^M(k, n)$, with common $\mathcal{L}_2^M$-substructure $\mathcal{A}_0 = (A_0, k_0)$. We may also suppose that $\mathcal{A}_0$ and $\mathcal{A}_1$ are countable, and that $\mathcal{A}_2$ is $\aleph_1$-saturated. Since we have Morleyized the $\mathbf{k}$-sort, there is an elementary embedding $\phi_k : k_1 \longrightarrow k_2$ which is the identity on $k_0$. Next we note that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are Cohen rings. Moreover, the three rings $A_0, A_1, A_2$ have the same characteristic $p^n$. The ring $A_0$ may not be a local ring, nor is the map $\mathrm{res}_0 : A_0 \longrightarrow k_0$ necessarily surjective. We denote the image by $R_A := \mathrm{res}_0(A_0)$. Since $R_A \subseteq k_0$, $R_A$ is an integral domain, the kernel $\mathfrak{p}_0$ of $\mathrm{res}_0$ is a prime ideal. Localising $A_0$ at $\mathfrak{p}_0$, we obtain a local ring $\bar{A}_0$, with residue field $k_A \subseteq k_0$, which embeds uniquely into each of $(A_i, k_i)$, for $i = 1, 2$, by the universal property of localisations.

Let $\boldsymbol{\beta} \subseteq k_A$ be a $p$-basis. By the technique of clearing denominators, we may suppose that $\boldsymbol{\beta} \subseteq R_A$. Then we may choose representatives $s : \boldsymbol{\beta} \longrightarrow A_0$. Denote by $\mathbf{b}$ the tuple corresponding to the image $s(\boldsymbol{\beta})$. For each $r \in \mathbb{N}$, denote by $\Theta_{i,r}$ the $r$-ary relation on $A_i$ which is the interpretation of $\boldsymbol{\Theta}_r$ in $(A_i, k_i)$. Then each sub-$r$-tuple $\mathbf{b}' \subseteq \mathbf{b}$ satisfies

$$\mathbf{b}' \in \Theta_{0,r} \subseteq \Theta_{i,r},$$

for $i = 1, 2$. Therefore $\mathrm{res}_i(\mathbf{b}')$ is a $p$-independent $r$-tuple in $k_i$, for $i = 1, 2$. This shows that $k_i/k_A$ is separable. Applying Theorem 7.1, there exists an embedding $\phi = (\phi_A, \phi_k) : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ which induces $\phi_k$ on the residue fields and which extends the identity map on $\mathcal{A}_0$. □

## 12. Quantifier elimination in the strict case

12.1. **Angular components.** Let $(K, v)$ be a valued field, unramified of mixed characteristic $(0, p)$. An **$n$-th angular component** is a map

$$\mathrm{ac}_n : K \longrightarrow \mathcal{O}_v/p^n\mathfrak{m}_v$$

such that

- **(i)** $\mathrm{ac}_n(x) = 0$ if and only if $x = 0$,
- **(ii)** $\mathrm{ac}_n$ restricts to a group homomorphism $K^\times \longrightarrow (\mathcal{O}_v/p^n\mathfrak{m}_v)^\times$, and
- **(iii)** the restriction of $\mathrm{ac}_n$ to $\mathcal{O}_v \setminus p^n\mathfrak{m}_v$ agrees with the corresponding restriction of the canonical surjection $\mathcal{O}_v \longrightarrow \mathcal{O}_v/p^n\mathfrak{m}_v$.

A **system** of angular component maps is a family $\mathrm{ac}_\bullet := (\mathrm{ac}_n)_{n\in\mathbb{N}}$ which commutes with the residue maps, i.e. for $m < n$ we have

$$\mathrm{ac}_m = \mathrm{res}_{n,m} \circ \mathrm{ac}_n.$$

A triple $(K, v, \mathrm{ac}_\bullet)$ is called an **ac-valued field**. Since angular component maps exist whenever a cross section exists, any sufficiently saturated valued field admits a system of angular component maps $\mathrm{ac}_\bullet$.

12.2. **The languages $\mathcal{L}_{\mathrm{ac}}$ and $\mathcal{L}_{\mathrm{ac},S}$.** To study the theories of strict Cohen rings, we introduce the multi-sorted language $\mathcal{L}_{\mathrm{ac}}$, with sorts $\mathbf{K}$, $\mathbf{A}$, $\mathbf{\Gamma}$, and $\mathbf{R}_n$, for $n \in \mathbb{N}_{\geq 1}$. The sort $\mathbf{K}$ is equipped with $\mathcal{L}_{\mathrm{ring}}$, the sort $\mathbf{\Gamma}$ is equipped with $\mathcal{L}_{\mathrm{oag}}$, which is the language of ordered abelian groups (written additively) with extra constant symbol $\infty$, and each sort $\mathbf{R}_n$ is equipped with $\mathcal{L}_{\mathrm{ring}}$. There is a function symbol $\mathbf{v} : \mathbf{K} \longrightarrow \mathbf{\Gamma}$, and for each $n \in \mathbb{N}_{\geq 1}$ there are function symbols $\mathbf{r}_n : \mathbf{A} \longrightarrow \mathbf{R}_n$ and $\mathbf{ac}_n : \mathbf{K} \longrightarrow \mathbf{R}_n$. We write $\mathbf{R}_\bullet$ to denote the collection of sorts $(\mathbf{R}_n)_{n\geq 1}$, and $\mathbf{r}_\bullet$ to denote the collection $(\mathbf{r}_n)_{n\geq 1}$, and $\mathbf{ac}_\bullet$ to denote the collection $(\mathbf{ac}_n)_{n\geq 1}$. We extend this convention to interpretations of these symbols in $\mathcal{L}_{\mathrm{ac}}$-structures.

Let $T_{\mathrm{ac}}$ be the $\mathcal{L}_{\mathrm{ac}}$-theory that requires of

$$(K, A_v, \varGamma \sqcup \{\infty\}, R_\bullet, v, r_\bullet, \mathrm{ac}_\bullet)$$

that

**(i)** $K$ is a field of characteristic zero,

**(ii)** $\varGamma \sqcup \{\infty\}$ is the disjoint union of an ordered abelian group $\varGamma$ with a singleton consisting of an extra element $\infty$, and the union is endowed with the $\mathcal{L}_{\mathrm{oag}}$-structure in the usual way,

**(iii)** $v : K \longrightarrow \varGamma \cup \{\infty\}$ is a (surjective) valuation,

**(iv)** the corresponding valuation ring is $A_v$ and its maximal ideal is $\mathfrak{m}_v = (p)$,

**(v)** for each $n$, the ring $R_n$ is the quotient $\mathcal{O}_v/(p^n)$, and is endowed with the $\mathcal{L}_{\mathrm{ring}}$-structure in the usual way (in particular, $R_1$ is identified with the residue field $k$);

**(vi)** $r_\bullet$ is the system of residue maps, and

**(vii)** $\mathrm{ac}_\bullet$ is a system of angular component maps.

For brevity, we write $(K, A_v, \varGamma, R_\bullet)$ instead of $(K, A_v, \varGamma \sqcup \{\infty\}, R_\bullet, v, r_\bullet, \mathrm{ac}_\bullet)$ for a model of $T_{\mathrm{ac}}$. Given a field $k_0$ of characteristic $p$, the theory $T_{\mathrm{ac}}(k_0)$ is the extension of $T_{\mathrm{ac}}$ by axioms that assert of $(K, A_v, \varGamma, R_\bullet)$ that

**(viii)**$_{k_0}$ the residue field $k$ is $\mathcal{L}_\lambda$-elementarily equivalent to $k_0$.

Similarly, given an ordered abelian group $\varGamma_0$, the theory $T_{\mathrm{ac}}(\varGamma_0)$ is the extension of $T_{\mathrm{ac}}$ by axioms that assert of $(K, A_v, \varGamma, R_\bullet)$ that

**(ix)**$_{\varGamma_0}$ $\varGamma$ is elementarily equivalent to $\varGamma_0$ as $\mathcal{L}_{\mathrm{oag}}$-structures.

Let $T_{\mathrm{ac}}(k_0, \varGamma)$ denote the union $T_{\mathrm{ac}}(k_0) \cup T_{\mathrm{ac}}(\varGamma)$.

*Remark* 12.1. Note that our language $\mathcal{L}_{\mathrm{ac}}$ is essentially the same as Bèlair's language $\mathcal{L}_{\mathrm{co}}$, except that we include the sort $\mathbf{A}$, as a convenient domain for the function symbols $\mathbf{r}_n$. The intended interpretation of $\mathbf{A}$ is as the valuation ring, which anyway is quantifier-free definable in models (in the language without the extra sort).

Finally, we consider the expansion $\mathcal{L}_{\mathrm{ac},S}$ of $\mathcal{L}_{\mathrm{ac}}$ as follows. We include the $\mathcal{L}_{2,S}$-structure on each $(\mathbf{R}_n, \mathbf{R}_1)$. More precisely, for each $n, r \in \mathbb{N}$, we include: the residue map $\mathbf{res}_n : \mathbf{R}_n \longrightarrow \mathbf{R}_1$; the $r$-ary predicate symbol $\boldsymbol{\varTheta}_{r,n}$ on the sort $\mathbf{R}_n$, and the $(r+1)$-ary function symbol $\mathbf{S}_{r,n} : \mathbf{R}_n \times \mathbf{R}_1 \longrightarrow \mathbf{R}_n$. We define the $\mathcal{L}_{\mathrm{ac},S}$-theory $T_{\mathrm{ac},S}$ to extend $T_{\mathrm{ac}}$ by axioms which assert of $(K, \varGamma, R_\bullet)$ that

**(x)** Each pair $(R_n, R_1, \mathrm{res}_n, (\varTheta_{r,n})_r, (S_{r,n})_r)$ is a model of $T_2(n)$.

The $\mathcal{L}_{\mathrm{ac},S}$-theories $T_{\mathrm{ac},S}(k_0)$, $T_{\mathrm{ac},S}(\varGamma_0)$, and $T_{\mathrm{ac},S}(k_0, \varGamma_n)$ are defined in the obvious way. Finally, we consider the language $\mathcal{L}^P_{\mathrm{ac},S}$ which is an expansion of $\mathcal{L}_{\mathrm{ac},S}$ by equipping the $\mathbf{\Gamma}$ sort with the Presburger language, in which the theory of $\mathbb{Z}$-groups has quantifier elimination.

*Remark* 12.2. Again, by design, each pre-Cohen ring $(A, k)$ of characteristic zero may be viewed as an $\mathcal{L}_{\mathrm{ac},S}$-structure

$$(A, \mathbb{Z}, (A/(p^n))),$$

which is a model of $T_{\mathrm{ac},S}(k)$. In fact, a strict Cohen ring admits a unique expansion to a model of $T_{\mathrm{ac},S}$.

**Theorem 12.3** ([Bél99, Théorème 5.1])**.** *The theory of unramified henselian ac-valued fields of mixed characteristic $(0,p)$ admits elimination of quantifiers over the base field in the language $\mathcal{L}_R$.*

We extend Bélair's Theorem by finding a language in which unramified henselian ac-valued fields admit elimination of quantifiers over the sorts $\mathbf{K}$ and $\mathbf{R}_n$, for $n > 1$.

**Theorem 12.4** ([Bél99, Théorème 5.3])**.** *Let $(K, v, (\mathrm{ac}_n))$ be a valued field of characteristic $(0,p)$ and $(L, v, (\mathrm{ac}_n))$, $(F, v, (\mathrm{ac}_n))$ are henselian unramified extensions of $(K, v, (\mathrm{ac}_n))$, such that $(F, v, (\mathrm{ac}_n))$ is $|L|^+$-saturated. Let $\alpha : vL \longrightarrow vF$ be an embedding of ordered abelian groups such that $\alpha|_{vK} = \mathrm{id}$, let $\beta_n : A_n(L) \longrightarrow A_n(F)$ be an embedding of rings such that $\beta_n|_{A_n(K)} = \mathrm{id}$ and such that the $\beta_n$ are compatible with the natural projective system $(A_n)$, that is to say $\beta_n\pi_n = \pi_n\beta_{n+1}$, where $\pi_n : A_{n+1} \longrightarrow A_n$ is the canonical surjection. Then there exists an embedding $f : (L, v, (\mathrm{ac}_n)) \longrightarrow (K, v, (\mathrm{ac}_n))$ such that $f|_K = \mathrm{id}$, $f_v = \alpha$, and $f_{\mathrm{res}_n} = \beta_n$, for all $n$.*

**Theorem 12.5.** *Let $k$ be a field of characteristic $p$ and let $\Gamma$ be an ordered abelian group with minimum positive element. The theory $T_{\mathrm{ac},S}(k,\Gamma)$ eliminates quantifiers relative to the sort $\mathbf{\Gamma}$ of the value group and the sort $\mathbf{R}_1$ of the residue field.*

*Proof.* We denote by $T^M_{\mathrm{ac},S}(k,\Gamma)$ the Morleyization of $T_{\mathrm{ac},S}(k,\Gamma)$ with respect to the sort $\mathbf{R}_1$. We show that $T^M_{\mathrm{ac},S}(k,\Gamma)$ admits elimination of quantifiers, by applying Shoenfield's Criterion, [Sho71]. Suppose that we are in the following situation. Let $\mathcal{K}_1 = (K_1, A_1, \Gamma_1, R_{1,\bullet})$ and $\mathcal{K}_2 = (K_2, A_2, \Gamma_2, R_{2,\bullet})$ be two models of $T^M_{\mathrm{ac},S}(k,\Gamma)$, with common $\mathcal{L}^M_{\mathrm{ac},S}$-substructure $\mathcal{B} = (B, A_B, \Gamma_B, R_{B,\bullet})$. We suppose that $\mathcal{B}$ and $\mathcal{K}_1$ are countable, and that $\mathcal{K}_2$ is $\aleph_1$-saturated. Thus, there exists an $\mathcal{L}_{\mathrm{oag}}$-elementary embedding $\alpha_\Gamma : \Gamma_1 \longrightarrow \Gamma_2$, which is the identity on $\Gamma_B$, and also an $\mathcal{L}_{\mathrm{ring}}$-elementary embedding $\alpha_k : R_{1,1} \longrightarrow R_{2,1}$, which is the identity on $R_{B,1}$. For each $n \in \mathbb{N}$, the extensions of $n$-th residue fields $R_{B,n} \subseteq R_{1,n}$ and $R_{B,n} \subseteq R_{2,n}$ are elementary by the quantifier elimination for non-strict Cohen rings, Theorem 11.3. In particular, we may also assume that each $R_{B,n}$ is a Cohen ring, by replacing each if necessary with its localisation, canonically embedded in both $R_{1,n}$ and $R_{2,n}$, and compatible with $\alpha_n$. Next, by replacing $B$ with its field of fractions, we may assume that $B$ is a subfield of $K_1$ and $K_2$.

Thus, we are in the situation of Theorem 12.4, and we have verified the hypotheses. Therefore there exists an $\mathcal{L}_{\mathrm{ac},S}$-embedding $f : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$, which induces $\alpha_\Gamma$ on the sort $\mathbf{\Gamma}$, and $\alpha_k$ on the sort $\mathbf{R}_1$, and moreover $f$ is the identity on $\mathcal{B}$. This verifies Shoenfield's Criterion. □

**Corollary 12.6.** *Let $k$ be a field of characteristic $p$. The theory $T^P_{\mathrm{ac},S}(k,\mathbb{Z})$ eliminates quantifiers over the sorts $\mathbf{K}$, $\mathbf{\Gamma}$, and $\mathbf{R}_n$, for $n > 1$.*

*Proof.* Immediate from Theorem 12.5 and [Rid17, Remark A.8]. □

## 13. Complete theories, Ax–Kochen/Ershov, and Stable Embeddedness

A valued field $(K, v)$ of mixed characteristic $(0,p)$ is **unramified** if $v(p)$ is the minimum positive element of $\Gamma_v$.

**Theorem 13.1** (Completeness)**.** *Let $k$ be a field of characteristic $p$.*

- **(i)** *The theory $T_2(k,n)$ is complete, for each $n \in \mathbb{N}_{>0}$.*
- **(ii)** *The theory $T_{\mathrm{ac}}(k,\mathbb{Z})$ is complete.*

*Proof.* For **(i)**, we fix $n \in \mathbb{N}_{>0}$ and let $(A_1,k_1),(A_2,k_2) \models T_2(k,n)$. By the Keisler–Shelah Theorem, [She71], taking suitable ultrapowers, we may assume that $k_1 = k_2 = k$. Then $(A_1,k_1)$ and $(A_2,k_2)$ are two Cohen rings of the same characteristic $p^n$ and same residue field $k$.

By Corollary 6.4, $(A_1, k_1)$ and $(A_2, k_2)$ are isomorphic by an isomorphism inducing the identity on $k$.

For **(ii)**, first note that each model $(K, v)$ of $T_{\mathrm{ac}}(k, \mathbb{Z})$ has a unique expansion to a model of $T^P_{\mathrm{ac}}(k, \mathbb{Z})$, and an expansion (not unique in general) to a model of $T^P_{\mathrm{ac},s}(k, \mathbb{Z})$. The analogous expansion of $(\mathbb{Q}, v_p)$ (unique since $\mathbb{F}_p$ is perfect) is a substructure. Completeness now follows from quantifier-elimination, Theorem 12.5. □

**Definition 13.2.** The language of valued fields, denoted $\mathcal{L}_{\mathrm{val}}$, is the language of rings augmented by a unary predicate $\mathcal{O}$. A valued field $(K, v)$ is viewed as an $\mathcal{L}_{\mathrm{val}}$-structure by interpreting $\mathcal{O}$ as the valuation ring $\mathcal{O}_v$ of $v$.

*Remark* 13.3. The Ax–Kochen/Ershov Principle for unramified henselian valued fields of mixed characteristic (i.e. part **(ii)** of the above theorem) is stated in [Bél99, Corollaire 5.2]. Bélair's proof goes through in the case of a perfect residue field since it uses the rings of Witt vectors. His proof may be adapted to account for imperfect residue fields by the use of the structure theorem for Cohen rings.

**Corollary 13.4** (Ax–Kochen/Ershov Principle)**.** *Let $(K_1, v_1)$ and $(K_2, v_2)$ be unramified henselian valued fields of mixed characteristic $(0, p)$. The following are equivalent.*

- **(i)** *$(K_1, v_1)$ and $(K_2, v_2)$ are $\mathcal{L}_{\mathrm{ac}}$-elementarily equivalent,*
- **(ii)** *$(K_1, v_1)$ and $(K_2, v_2)$ are $\mathcal{L}_{\mathrm{val}}$-elementarily equivalent,*
- **(iii)** *$\Gamma_{v_1} \equiv \Gamma_{v_2}$ and $k_1 \equiv k_2$.*

*Proof.* Clearly **(i)** $\implies$ **(ii)** $\implies$ **(iii)**. It remains to show **(iii)** $\implies$ **(i)**, so we suppose $\Gamma_{v_1} \equiv \Gamma_{v_2}$ and $k_1 \equiv k_2$. By the Keisler–Shelah Theorem, [She71], and by taking suitable ultrapowers, we may assume that $\Gamma_{v_1} \cong \Gamma_{v_2}$, that $k_1 \cong k_2$ (the latter as $\mathcal{L}_\lambda$-structures), and that both $(K_1, v_1)$ and $(K_2, v_2)$ are $\aleph_1$-saturated. Let $w_i$ denote the finest proper coarsening of $v_i$, and let $\bar{v}_i$ denote the valuation induced on $Kw_i$ by $v_i$. Then both $(Kw_i, \bar{v}_i)$ are complete unramified valued fields of mixed characteristic, so are $\mathcal{L}_{\mathrm{ac}}$-isomorphic by Corollary 6.4. More precisely, there is an $\mathcal{L}_{\mathrm{ac}}$-isomorphism

$$\phi : (K_1 w_1, \bar{v}_1) \longrightarrow (K_2 w_2, \bar{v}_2).$$

Since also $w_1 K_1 = v_1 K_1/\mathbb{Z}$ and $w_2 K_2 = v_2 K_2/\mathbb{Z}$ are isomorphic, $(K_1, w_1)$ and $(K_2, w_2)$ are elementarily equivalent, as valued fields, by the Theorem of Ax–Kochen/Ershov in equal characteristic zero. We need something slightly more: we need that $(K_1, v_1)$ and $(K_2, v_2)$ are $\mathcal{L}_{\mathrm{ac}}$-elementarily equivalent. For a final time, we pass to a suitable ultrapower, so that $(K_1, w_1)$ and $(K_2, w_2)$ are isomorphic. Since in the theory of henselian valued fields of equal characteristic zero, residue fields are stably embedded, we may find an isomorphism between $(K_1, w_1)$ and $(K_2, w_2)$ which induces the map $\phi$ between the residue fields. Therefore there is an $\mathcal{L}_{\mathrm{ac}}$-isomorphism $(K_1, v_1) \longrightarrow (K_2, v_2)$. □

The Ax–Kochen/Ershov Principle, above, immediately gives an axiomatisation of the complete theories of unramified henselian valued fields, as follows.

**Corollary 13.5.** *Let $(K, v)$ be an unramified henselian valued field of mixed characteristic. The complete $\mathcal{L}_{\mathrm{val}}$-theory of $(K, v)$ is axiomatised by*

- **(i)** *$(K, v)$ is an henselian valued field of mixed characteristic $(0, p)$,*
- **(ii)** *the value group is elementarily equivalent to $vK$ and $v(p)$ is minimum positive, and*
- **(iii)** *the residue field is elementarily equivalent to $Kv$.*

Another form of Ax–Kochen/Ershov principle is the following relative model completeness theorem.

**Corollary 13.6** (Relative Model Completeness)**.** *Let $(K_1, v_1) \subseteq (K_2, v_2)$ be an extension of unramified henselian valued fields of mixed characteristic $(0, p)$, viewed as $\mathcal{L}_{\mathrm{ac},S}$-structures. The following are equivalent.*

**(i)** $(K_1, v_1) \preceq (K_2, v_2)$ *as $\mathcal{L}_{\mathrm{ac},S}$-structures,*

**(ii)** $\Gamma_{v_1} \preceq \Gamma_{v_2}$ *and* $k_1 \preceq k_2$.

*Proof.* That **(i)** implies **(ii)** is immediate. For the converse, the assumptions together with the quantifier elimination for non-strict Cohen rings, namely Theorem 11.3, give that for each $n \in \mathbb{N}$, the extension of the $n$-th residue ring $R_{1,n} \subseteq R_{2,n}$ is elementary. This verifies the assumptions of [Bél99, Corollaire 5.2(2)], and therefore **(i)** follows. □

Finally, we address the issue of the stable embeddedness of the residue field. Let $\mathcal{M}$ be an uncountable saturated model of the countable first-order theory $T$, and let $P$ be a definable set in $\mathcal{M}$. Recall, for example from [CH99, Appendix], that $P$ is *stably embedded* if whenever $D$ is a $M$-definable subset of $P^n$, then $D$ is definable with parameters from $P$.

**Theorem 13.7.** *Let $\mathcal{K} = (K, A, \Gamma, R_\bullet)$ be a model of $T^P_{\mathrm{ac},S}(k, \mathbb{Z})$. Then the residue field $k$ is stably embedded, as a pure field.*

*Proof.* By [CH99, Appendix, Lemma 1], it suffices to show that every automorphism of $k$ lifts to an automorphism of $\mathcal{K}$. Let $\phi_k : k \longrightarrow k$ be any an automorphism of $k$. Let $A$ be the valuation ring on $K$ corresponding to the finest proper coarsening of the given $p$-valuation. Then $(A, k)$ is a strict Cohen ring. By Corollary 6.4, there exists an automorphism $\phi_A : A \longrightarrow A$ such that $\phi = (\phi_A, \phi_k)$ is an automorphism of $(A, k)$. Finally, we apply that fact that $A$ is stably embedded in $\mathcal{K}$ to find an automorphism of $\mathcal{K}$ which extends $\phi_k$. □

**Corollary 13.8.** *Let $(K, v)$ be an unramified henselian valued field of mixed characteristic $(0, p)$, viewed as an $\mathcal{L}_{\mathrm{val}}$-structure. Then the residue field $k$ is stably embedded, as a pure field.*


## Acknowledgements

We would like to extend our sincere thanks to Martin Hils, Florian Pop, Silvain Rideau, and Tom Scanlon, for numerous very helpful conversations.

This work was begun while we were participating in the *Model Theory, Combinatorics and Valued fields* trimester at the Institut Henri Poincaré and the Centre Émile Borel, and we would like to extend our thanks to the organisers. Sylvy Anscombe was also partially supported by The Leverhulme Trust under grant RPG-2017-179. Franziska Jahnke was also funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure and the CRC878, as well as by a Fellowship from the Daimler and Benz Foundation.


## References


[AJ18] Sylvy Anscombe and Franziska Jahnke. *Characterizing NIP henselian fields.* Manuscript, 2019.

[Bél99] Luc Bélair. *Types dans les corps valués munis d'applications coefficients.* Illinois J. Math., 43(2):410–425, 1999.

[CH99] Zoé Chatzidakis and Ehud Hrushovski. *Model theory of difference fields.* Trans. Amer. Math. Soc., 351(8):2997–3071, 1999.

[Che43] Claude Chevalley. *On the theory of local rings.* Ann. Math., 44(4):690–708, 1943.

[Coh46] I. S. Cohen. *On the structure and ideal theory of complete local rings.* Trans. Amer. Math. Soc., 59:54–106, 1946.

[DM15] Jamshid Derakhshan and Angus Macintyre. *Model completeness for henselian fields with finite ramification valued in a $\mathbb{Z}$-group.* Manuscript, 2015.

[HS34] H. Hasse and F. K. Schmidt. *Die Struktur diskret bewerteter Körper.* J. reine angew. Math., 170:4–63, 1934.

[Kru37] Wolfgang Krull. *Beiträge zur Arithmetik kommutativer Integritätsberiche. III Zum Dimensionsbegriff der Idealtheorie.* Math. Zeit., 42:745-766, 1937.

[Kru38] Wolfgang Krull. *Dimensionstheorie in Stellenringen.* J. reine angew. Math., 179:204–226, 1938.

[Kuh11] Franz-Viktor Kuhlmann. *Valuation theory*, book in progress. Available online: http://math.usask.ca/~fvk/Fvkbook.htm, 2011.

[Mac39a] Saunders Mac Lane. *Modular fields. I. Separating transcendence bases.* Duke Math. J., 5:372–393, 1939.

[Mac39b] Saunders Mac Lane. *Steinitz field towers for modular fields.* Trans. Amer. Math. Soc., 46:23–45, 1939.

[Mac39c] Saunders Mac Lane. *Subfields and Automorphisms Groups of $p$-Adic Fields.* Ann. Math., 40(2):423–442, 1939.

[Mac40] Saunders Mac Lane. *Modular fields.* Amer. Math. Monthly, 47:259–274, 1940.

[PR84] Alexander Prestel and Peter Roquette. *Formally $p$-adic fields.* Lecture Notes in Logic, Springer, 1984.

[Rid17] Silvain Rideau. *Some properties of analytic difference valued fields.* Journal of the Institute of Mathematics of Jussieu, 16(3):447–499, 2017.

[Roq03] Peter Roquette. *History of Valuation Theory, Part I.* Available online: https://www.mathi.uni-heidelberg.de/~roquette/hist_val.pdf, 2003.

[Sch72] Colette Schoeller *Groupes affines, commutatifs, unipotents sur un corps non parfait.* Bull. de la S. M. F., 100:241–300, 1972.

[Ser79] Jean-Pierre Serre. *Local Fields* (English translation of *Corps locaux*). Springer, 1979.

[Ser00] Jean-Pierre Serre. *Local Algebra* (English translation of *Algèbre Locale – Multiplicités*). Springer, 2000.

[She71] S. Shelah. *Every two elementarily equivalent models have isomorphic ultrapowers.* Israel J. Math., 10:224–233, 1971.

[Sho71] J.R. Shoenfield. *A theorem on quantifier elimination.* Symposia Math., 5:173–176, 1971.

[Tei36a] Oswald Teichmüller. *$p$-Algebren.* Deutsche Math., 1:362–388, 1936.

[Tei36b] Oswald Teichmüller. *Über die Struktur diskret bewerteter perfekter Körper.* In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalishe Klasse: Fachgruppe 1, Nachrichten aus der Mathematik*, 1(10):152–161, 1936.

[Tei37] Oswald Teichmüller. *Diskret bewerteter perfekte Körper mit unvollkommenem Restklassenkörper.* J. reine angew. Math., 176:141–152, 1937.

[vdD14] Lou van den Dries. *Lectures on the Model Theory of Valued Fields.* In Lou van den Dries, Jochen Koenigsmann, H. Dugald Macpherson, Anand Pillay, Carlo Toffalori, Alex J. Wilkie, *Model Theory in Algebra, Analysis and Arithmetic*, 55–157. Lecture Notes in Mathematics, vol. 2111. Springer, Berlin, Heidelberg, 2014.

[War93] S. Warner. *Topological Rings.* North-Holland Mathematics Studies, Elsevier, 1993.

[Wei84] Volker Weispfenning. *Quantifier elimination and decision procedures for valued fields.* In: Müller G.H., Richter M.M. (eds) *Models and Sets.* Lecture Notes in Mathematics, vol 1103. Springer, Berlin, Heidelberg, 1984.

[Wit37] E. Witt. *Zyklische Körper und Algebren der Charakteristik $p$ vom Grad $p^n$.* J. reine angew. Math., 176:126–140, 1937.



Jeremiah Horrocks Institute, Leighton Building Le7, University of Central Lancashire, Preston PR1 2HE, United Kingdom

*E-mail address*: sanscombe@uclan.ac.uk

Institut für Mathematische Logik und Grundlagenforschung, University of Münster, Einsteinstr. 62, 48149 Münster, Germany

*E-mail address*: franziska.jahnke@wwu.de