HOMOTOPY FIXED POINTS FOR LUBIN-TATE SPECTRA

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Abstract. We construct a stable model structure on profinite symmetric spectra with a continuous action of an arbitrary profinite group. This provides a natural framework for the construction of homotopy fixed point spectra and of homotopy fixed point spectral sequences for the action of the extended Morava stabilizer group on Lubin-Tate spectra. These continuous homotopy fixed points are canonically equivalent to the homotopy fixed points of Devinatz and Hopkins but have a drastically simplified construction.

1. Introduction

For the action of a discrete group G on a spectrum X, there are well known constructions for the homotopy fixed point spectrum X^{hG} and homotopy fixed point spectral sequence. For fibrant X, the spectrum X^{hG} is given by the G-fixed points of the function spectrum $\hom(EG, X)$, where EG is a contractible free G-space, i.e. $X^{hG} = \hom_G(EG, X)$. For each spectrum Z, the spectral sequence

$$H^*(G; X^*Z) \Rightarrow [Z, X^{hG}]^* \tag{1}$$

is induced by the usual filtration of the bar construction for EG. But in some cases of interest, the group G and the spectrum X carry additional structures that one would like to remember when applying the above constructions. This is for example the case for a prominent group action in chromatic homotopy theory. Let us quickly describe this example.

Let p be a fixed prime, $n \geq 1$ an integer and $\mathbb{F}_{p^n}$ the field with p^n elements. Let S_n be the nth Morava stabilizer group, i.e. the automorphism group of the height n Honda formal group law Γ_n over $\mathbb{F}_{p^n}$. We denote by $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ and let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the semidirect product. Note that S_n, $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ and hence also G_n are profinite groups.

By the work of Lubin and Tate [19], there is a universal ring of deformations $E(\mathbb{F}_{p^n}, \Gamma_n) = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$ of $(\mathbb{F}_{p^n}, \Gamma_n)$, where $W(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors of $\mathbb{F}_{p^n}$. The MU_*-module $E(\mathbb{F}_{p^n}, \Gamma_n)[u, u^{-1}]$ induces via the Landweber exact functor theorem a homology theory and hence a spectrum, denoted by E_n and called Lubin-Tate spectrum, with $E_{n*} = E(\mathbb{F}_{p^n}, \Gamma_n)[u, u^{-1}]$. The profinite group G_n acts on the ring E_{n*}, cf. [8]. By Brown representability, this induces an action of G_n by ring maps in the stable homotopy category. Furthermore, Goerss, Hopkins and Miller have shown that there is a G_n-action on the spectrum-level on E_n that induces the action in the stable category, see [13] and [26].

Each homotopy group $\pi_t E_n$ has the structure of a continuous profinite G_n-module. The continuity of the action is an important property for stable homotopy theory. In fact, by Morava's change of rings theorem, the $K(n)_*$-local E_n-Adams spectral sequence for the sphere spectrum S^0 has the form

$$H_c^*(G_n; E_{n*}) \Rightarrow \pi_* L_{K(n)} S^0 \tag{2}$$

where the E_2-term is the continuous cohomology of G_n with profinite coefficients, $K(n)$ denotes the nth Morava K-theory and $L_{K(n)}$ denotes $K(n)_*$-localization, cf. [20] and [6]. So $L_{K(n)}S^0$ looks like a continuous G_n-homotopy fixed point spectrum of E_n and one would like to interpret the above spectral sequence as a descent spectral sequence of the G_n-action, see [9]. But the drawback for applying the classical construction of homotopy fixed points and its spectral sequence (1) is that it does not reflect the topology on G_n. The function spectrum $\hom_G(EG, E_n)$ should consist of continuous maps in some sense and the E_2-term of the spectral sequence (1) should be the continuous cohomology of G.

Devinatz and Hopkins [9] have circumvented this problem and given an ad hoc argument for the construction of homotopy fixed points of E_n using specific properties of G_n as a p-adic analytic group. Moreover, for every closed subgroup G of G_n, they construct a descent spectral sequence for the homotopy fixed point spectrum E_n^{hG} with the correct E_2-term.

But the question remained if there is a way to view E_n as an actual continuous spectrum and to find a natural framework for the continuous homotopy fixed point spectral sequence. The purpose of this paper is to give an answer to this question. Since the homotopy groups $\pi_t E_n$ are profinite groups, a first guess would be to ask if there is a profinite structure on E_n. And, in fact, there is one. Consequently, the natural setting to study the action of G_n on E_n is the category of continuous profinite G_n-spectra.

A different answer has been given by Davis in [4] and by Behrens and Davis in [1]. Davis uses the idea of Devinatz and Hopkins to start with the homotopy fixed point spectrum of [9] for an open subgroup $U \subset G_n$, denoted by E_n^{dhU}, and defines a new spectrum $F_n := \operatorname{colim}_U E_n^{dhU}$ where U runs through the open subgroups of G_n. The $K(n)_*$-localization of F_n is equivalent to E_n. One can regard F_n as a continuous discrete G_n-spectrum. Furthermore, Davis develops a stable homotopy theory for continuous discrete G-spectra, for an arbitrary profinite group G, based on the work of Goerss [12] and Jardine [18]. Then he can define systematically the homotopy fixed points for closed subgroups of G_n and construct a continuous homotopy fixed point spectral sequence.

Another approach has been developed by Fausk. In [11], he constructs a model structure for pro-G-spectra, where G denotes a compact Hausdorff topological group. If G is a profinite group, there is a completion functor from pro-G-spectra to profinite G-spectra described below, but this is not an equivalence. Fausk also obtained results on homotopy fixed points, descent spectral sequences and iterated homotopy fixed points. These results are equivalent to those of [4] if G has finite virtual cohomological dimension. If one wants to apply them to Lubin-Tate spectra, one also has to use the results of [9] for open subgroups.

Hence the above question still remained open how to view E_n as a continuous spectrum without using [9] for open subgroups U of G_n.

The approach of the present paper gives an answer to this question as well and is in this sense more general than the one of [4]. In particular, we give a new unified construction of homotopy fixed points for any closed subgroup and hence, in particular, also a new construction for open subgroups of G_n. Moreover, as the homotopy groups of E_n are not discrete, but profinite G_n-modules, it seems to be natural to use the profinite setting.

We will study in general continuous actions on profinite spectra of an arbitrary profinite group G. Therefore, we show that the category $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$ of symmetric continuous profinite G-spectra is equipped with a natural stable model structure. The resulting fibrant replacement functor R_G in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$ will allow us to give a natural definition for homotopy fixed point spectra. In fact, the homotopy fixed points of a continuous profinite G-spectrum X are defined as the G-fixed points of the function spectrum $\hom(EG, R_G X)$ of continuous maps in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$. The homotopy fixed point spectral sequence is then obtained from the usual bar construction for EG. The striking advantage of studying profinite actions in the category of profinite spectra is that G and its classifying space EG yield natural objects in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$.

One should note that although EG and X are profinite, the function spectrum $\hom_G(EG, X)$ does not in general inherit a profinite structure, since roughly speaking, the limit of EG is turned into a colimit. But if one is interested in a profinite version of the homotopy fixed points, one can use the homotopy equivalent profinite spectrum $\hom_G(*, R_G X)$.

Applying these techniques to the above described action of G_n on the Lubin-Tate spectrum E_n, we will prove the following results.

Theorem 1.1. *E_n is a continuous profinite G_n-spectrum.*

This is just an observation that the decomposition of E_n as a limit of finite spectra $E_n \wedge M_I$, where M_I denotes the generalized Moore spectra of [15], can be refined further to get a G_n-compatible decomposition as a profinite G_n-spectrum. This allows us to construct a natural descent spectral sequence for any closed subgroup of G_n.

Theorem 1.2. *Let G be any closed subgroup of G_n.*

(i) *There is a homotopy fixed point spectrum E_n^{hG} which is equivalent to the fixed point spectrum E_n^{dhG} of* [9]. *In particular, $E_n^{hG_n} \simeq E_n^{dhG_n} \simeq L_{K(n)} S^0$.*

(ii) *For any spectrum Z, there is a natural strongly convergent descent spectral sequence starting from continuous cohomology*

$$H_c^*(G; E_n^* Z) \Rightarrow (E_n^{hG})^* Z$$

which is isomorphic to the spectral sequence of [9].

(iii) *Let K be a closed normal subgroup of G. Then E_n^{hG} is naturally equivalent to $(E_n^{hK})^{hG/K}$. For any spectrum Z, there is a strongly convergent spectral sequence for iterated homotopy fixed points*

$$H_c^*(G/K; (E_n^{hK})^* Z) \Rightarrow (E_n^{hG})^* Z.$$

This theorem restates the main result of [9], but has a drastically simpler proof. In loc. cit. different methods are used to define first homotopy fixed points for open subgroups and then for closed subgroups of G_n using special properties of G_n. We recall that, since G_n is profinite, open subgroups in G_n are exactly the closed subgroups of finite index. One of the significant improvements of Theorem 1.2 compared to [9] and [4] is that there is one unified construction for all closed (or open)subgroups. This advantage becomes even more apparent when we study iterated homotopy fixed points. Part (iii) of Theorem 1.2 is more general than the result in [9], since it holds for closed subgroups K of arbitrary index in G, whereas in [9] one had to assume that G/K is finite. This generalization had also been obtained by Davis in [5], but again the proof given here is much shorter especially in the case of subgroups of infinite index and fits in a general picture of continuous homotopy fixed points providing a new construction for iterated homotopy fixed points for open subgroups.

Finally, the methods of this paper might also be of interest for studying Galois extensions of ring spectra in the sense [28]. In loc. cit. Rognes develops a Galois theory for commutative S^0-algebras in analogy to the algebraic Galois theory. The step from finite to infinite extensions involves the action of a profinite group on such spectra. Such infinite Galois extensions have been studied by Behrens and Davis in [1]. The case of continuous profinite G-spectra might shed some light on the picture from a different point of view.

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2. Homotopy theory of continuous profinite spaces

2.1. **Summary on profinite spaces.** We start with basic definitions and invariants for profinite spaces that will be necessary to consctruct a homotopy category for profinite spectra.

For a category $\mathcal{C}$ with small limits, the pro-category of $\mathcal{C}$, denoted pro-$\mathcal{C}$, has as objects all cofiltering diagrams $X : I \to \mathcal{C}$. Its sets of morphisms are defined as

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{C}}(X,Y) := \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

A constant pro-object is one indexed by the category with one object and one identity map. The functor sending an object X of $\mathcal{C}$ to the constant pro-object with value X makes $\mathcal{C}$ a full subcategory of pro-$\mathcal{C}$. The right adjoint of this embedding is the limit functor lim: pro-$\mathcal{C} \to \mathcal{C}$, which sends a pro-object X to the limit in $\mathcal{C}$ of the diagram corresponding to X.

Let $\mathcal{E}$ denote the category of sets and let $\mathcal{F}$ be the full subcategory of finite sets. Let $\hat{\mathcal{E}}$ be the category of compact Hausdorff and totally disconnected topological spaces. We may identify $\mathcal{F}$ with a full subcategory of $\hat{\mathcal{E}}$ in the obvious way. The limit functor lim: pro-$\mathcal{F} \to \hat{\mathcal{E}}$ is an equivalence of categories.

We denote by $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) the category of simplicial profinite sets (resp. simplicial sets). The objects of $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) will be called *profinite spaces* (resp. *spaces*). The forgetful functor $\hat{\mathcal{E}} \to \mathcal{E}$ admits a left adjoint $\hat{(\cdot)} : \mathcal{E} \to \hat{\mathcal{E}}$. It induces a functor $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$, which is called *profinite completion*. It is left adjoint to the forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ which sends a profinite space to its underlying simplicial set.

For a profinite space X, let $\mathcal{R}(X)$ be the set of simplicial open equivalence relations on X. An element R of $\mathcal{R}(X)$ is a simplicial profinite subset of the product $X \times X$ such that, in each degree n, R_n is an equivalence relation on X_n and an open subset of $X_n \times X_n$. It is ordered by inclusion. For every element R of $\mathcal{R}(X)$, the quotient X/R is a simplicial finite set and the map $X \to X/R$ is a map of profinite spaces.

The canonical map $X \to \lim_{R\in\mathcal{R}(X)} X/R$ is an isomorphism in $\hat{\mathcal{S}}$, cf. [21], Lemme 1.

Let X be a profinite space. The continuous cohomology $H^*_c(X;\pi)$ of X with coefficients in the topological abelian group π is defined as the cohomology of the complex $C^*(X;\pi)$ of continuous cochains of X with values in π, i.e. $C^n(X;\pi)$ denotes the set $\mathrm{Hom}_{\hat{\mathcal{E}}}(X_n,\pi)$ of continuous maps $\alpha : X_n \to \pi$ and the differentials $\delta^n : C^n(X;\pi) \to C^{n+1}(X;\pi)$ are the morphisms associating to α the map $\sum_{i=0}^{n+1} \alpha\circ d_i$, where d_i denotes the ith face map of X. If π is a finite abelian group and Z a simplicial set, then the cohomologies $H^*(Z;\pi)$ and $H^*_c(\hat{Z};\pi)$ are canonically isomorphic.

If G is an arbitrary profinite group, we may still define the first cohomology of X with coefficients in G as done by Morel in [21] p. 355. The functor $X \mapsto \mathrm{Hom}_{\hat{\mathcal{E}}}(X_0,G)$ is represented in $\hat{\mathcal{S}}$ by a profinite space EG. We define the 1-cocycles $Z^1(X;G)$ to be the set of continuous maps $f : X_1 \to G$ such that $f(d_0x)f(d_2x) = f(d_1x)$ for every $x\in X_1$. The functor $X \mapsto Z^1(X;G)$ is represented by a profinite space BG. Explicit constructions of EG and BG may be given in the standard way. Furthermore, there is a map $\delta : \mathrm{Hom}_{\hat{\mathcal{S}}}(X,EG) \to Z^1(X;G) \cong \mathrm{Hom}_{\hat{\mathcal{S}}}(X,BG)$ which sends $f : X_0 \to G$ to the 1-cocycle $x \mapsto \delta f(x) = f(d_0x)f(d_1x)^{-1}$. We denote by $B^1(X;G)$ the image of δ in $Z^1(X;G)$ and we define the pointed set $H^1(X;G)$ to be the quotient $Z^1(X;G)/B^1(X;G)$. Finally, if X is a profinite space, we define π_0X to be the coequalizer in $\hat{\mathcal{E}}$ of the diagram $d_0,d_1 : X_1 \rightrightarrows X_0$.

The profinite fundamental group of X is defined via covering spaces, see [23]. There is a universal profinite covering space $(\tilde{X},x)$ of X at a vertex $x\in X_0$. Then $\pi_1(X,x)$ is defined to be the group of automorphisms of $(\tilde{X},x)$ over (X,x). It has a natural structure of a profinite group as the limit of the finite automorphism groups of the finite Galois coverings of (X,x). For any details, we refer the reader to [23]. Its relation to the usual fundamental group of a simplicial set is described by the following.

Proposition 2.1. *For a pointed simplicial set X, the canonical map from the profinite group completion of $\pi_1(X)$ to $\pi_1(\hat{X})$ is an isomorphism, i.e. $\widehat{\pi_1(X)} \cong \pi_1(\hat{X})$ as profinite groups.*

Definition 2.2. *A morphism $f : X \to Y$ in $\hat{\mathcal{S}}$ is called,*
(1) *a* weak equivalence *if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets, $f_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ is an isomorphism of profinite groups for every vertex $x\in X_0$ and $f^* : H^q(Y;\mathcal{M}) \to H^q(X;f^*\mathcal{M})$ is an isomorphism for every local coefficient system $\mathcal{M}$ of finite abelian groups on Y for every $q\geq 0$;*
(2) *a* cofibration *if f is a levelwise monomorphism;*
(3) *a* fibration *if it has the right lifting property with respect to every cofibration that is also a weak equivalence.*

Theorem 2.3. *The above defined classes of weak equivalences, cofibrations and fibrations provide $\hat{\mathcal{S}}$ with the structure of a fibrantly generated left proper model category. We denote the homotopy category by $\hat{\mathcal{H}}$.*

Proof. See [23]. □

We consider the category $\mathcal{S}$ of simplicial sets with the usual model structure of [25]. We denote its homotopy category by $\mathcal{H}$. Then the following result follows immediately.

Proposition 2.4. 1. *The levelwise completion functor* $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ *preserves weak equivalences and cofibrations.*
2. *The forgetful functor* $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ *preserves fibrations and weak equivalences between fibrant objects.*
3. *The induced completion functor* $\hat{(\cdot)} : \mathcal{H} \to \hat{\mathcal{H}}$ *and the right derived functor* $R|\cdot| : \hat{\mathcal{H}} \to \mathcal{H}$ *form a pair of adjoint functors.*

Definition 2.5. *Let X be a pointed profinite space and let RX be a fibrant replacement of X in the above model structure on $\hat{\mathcal{S}}_*$. Then we define the* nth profinite homotopy group of X *for $n \geq 2$ to be the profinite group*

$$\pi_n(X) := \pi_0(\Omega^n(RX)).$$

In other words, in order to get the homotopy groups of a profinite space X, we first take a fibrant replacement of X in $\hat{\mathcal{S}}$, and then take the usual homotopy groups of the fibrant simplicial set RX.

Remark 2.6. Morel proved that there is a model structure on $\hat{\mathcal{S}}$ for each prime number p in which the weak equivalences are maps that induce isomorphisms on $\mathbb{Z}/p$-cohomology. The fibrant replacement functor R^p yields a rigid version of Bousfield-Kan $\mathbb{Z}/p$-completion. The homotopy groups for this structure are pro-p-groups being defined as above using R^p. The generating fibrations and trivial fibrations are given by the canonical maps $L(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+1)$, $K(\mathbb{Z}/p, n) \to *$, respectively by the maps $L(\mathbb{Z}/p, n) \to *$ for every $n \geq 0$, see [21] and [23].

2.2. **Profinite G-spaces.** Let G be a fixed profinite group. Let X be a profinite set on which G acts continuously, i.e. G is acting on X and the map $\mu : G \times X \to X$ is continuous. In this situation we say that X is a profinite G-set. If X is a profinite space and G acts continuously on each X_n such that the action is compatible with the structure maps, then we call X a profinite G-space. We denote by $\hat{\mathcal{S}}_G$ the category of profinite G-spaces. If X is a pointed profinite space with a continuous G-action, then we call X a pointed profinite G-space if the basepoint is fixed under the action of G. We denote the corresponding category by $\hat{\mathcal{S}}_{*G}$.

While a discrete G-space Y is characterized as the colimit over the fixed point spaces Y^U over all open subgroups, a profinite G-space X is the limit over its orbit spaces X_U. More explicitly, for an open and hence closed normal subgroup U of G, let X_U be the quotient space under the action by U, i.e. the quotient $X/\sim$ with $x \sim y$ in X if both are in the same orbit under U. It is easy to show that the canonical map $\phi : X \to \lim_U X_U$ is a homeomorphism, where U runs through the open normal subgroups of G.

Now, for $Z, X \in \hat{\mathcal{S}}$ let $\hom(Z, X)$ be the space of continuous maps. If Z is a simplicial finite set and $X \in \hat{\mathcal{S}}$, then $\hom(Z, X)$ has a natural structure as a profinite space. In order to get a model structure on $\hat{\mathcal{S}}_G$, one uses the fact that the model struture on $\hat{\mathcal{S}}$ is fibrantly generated in analogy to [12]. In a first step we enlarge the generating sets for fibrations P and trivial fibrations Q by applying $\hom(G/U, -)$ for any open normal subgroup U of G to P and Q respectively. This induces an intermediate strict model structure. A further Bousfield localization then yields the following result whose proof is given in [24].

Theorem 2.7. *There is a left proper fibrantly generated simplicial model structure on the category of profinite G-spaces such that a map f is a weak equivalence (respectively cofibration) in $\hat{\mathcal{S}}_{*G}$ if and only if its underlying map is a weak equivalence (respectively cofibration) in $\hat{\mathcal{S}}_*$. We denote its homotopy category by $\hat{\mathcal{H}}_{*G}$. The underlying map of a fibration in $\hat{\mathcal{S}}_{*G}$ is also a fibration in $\hat{\mathcal{S}}_*$ (and in $\mathcal{S}_*$).*

3. Symmetric profinite G-spectra

Let us now turn to a stable version. A profinite spectrum X consists of a sequence $X_n \in \hat{\mathcal{S}}_*$ of pointed profinite spaces and maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ in $\hat{\mathcal{S}}_*$ for $n \geq 0$. A morphism $f : X \to Y$ of spectra consists of maps $f_n : X_n \to Y_n$ in $\hat{\mathcal{S}}_*$ for $n \geq 0$ such that $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$. We denote by $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ the corresponding category of profinite spectra. By Theorem 2.36 of [23] there is a stable homotopy category $\hat{\mathcal{SH}}$ of profinite spectra. In this model structure, a map $f : X \to Y$ is a stable equivalence if it induces a weak equivalence of mapping spaces $\mathrm{Map}(Y, E) \to \mathrm{Map}(X, E)$ for all Ω-spectra E; and f is a cofibration if $X_0 \to Y_0$ and the induced maps $X_n \amalg_{S^1 \wedge X_{n-1}} S^1 \wedge Y_{n-1} \to Y_n$ are monomorphisms for all n.
Now let G be as always a profinite group. We consider the simplicial finite set S^1 as a profinite G-space with trivial action. We call X a profinite G-spectrum if, for $n \geq 0$, each X_n is a pointed profinite G-space and each $S^1 \wedge X_n \to X_{n+1}$ is a G-equivariant map. We denote the category of profinite G-spectra by $\mathrm{Sp}(\hat{\mathcal{S}}_*)_G$.
There is a model structure on profinite G-spectra such that a map is a stable weak equivalence (resp. cofibration) if and only if it is a stable weak equivalence (resp. cofibration) in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$. It can be obtained as the stabilization of the model structure on $\hat{\mathcal{S}}_{*G}$ of Theorem 2.7 using the techniques of [16] and the localization results of [22], Theorems 6 and 14, for fibrantly generated model categories.
Furthermore, there is also a category of profinite spectra with better properties. Based on the work of Hovey, Shipley and Smith and their category of symmetric spectra $\mathrm{Sp}^{\Sigma}(\mathcal{S}_*)$, [16] and [17], we consider the category $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$ of continuous profinite symmetric G-spectra.
Let Σ_n be the symmetric group of permutations of the set $\{1, 2, \ldots, n\}$.

Definition 3.1. *A symmetric profinite spectrum X is a sequence of pointed Σ_n-profinite spaces $\{X_n\}_n$, a pointed Σ_n-equivariant map $S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$ and such that the composite*

$$S^p \wedge X_n \to S^{p-1} \wedge X_{n+1} \to \ldots \to X_{n+p}$$

is $\Sigma_p \times \Sigma_n$-equivariant. A map of symmetric profinite spectra $X \to Y$ is a collection of Σ_n-equivariant maps $X_n \to Y_n$ in $\hat{\mathcal{S}}_*$ compatible with the structure maps of X and Y. We denote the category of symmetric profinite spectra by $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$.

We want to construct a stable monoidal model structure on $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$. We call a map $f : A \to B$ in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ a cofibration if the map $A_0 \to B_0$ is a cofibration in $\hat{\mathcal{S}}_*$ and the induced maps $A_n \amalg_{L_n A} L_n B \to B_n$ are Σ_n-equivariant cofibrations in $\hat{\mathcal{S}}_*$ for every $n \geq 1$, where L_n is the latching object of [16], Definition 8.4. A symmetric spectrum $X \in \mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ is called an Ω-spectrum if X is level fibrant and the adjoints $X_n \to \Omega X_{n+1} = \hom_*(S^1, X_{n+1})$ of the structure maps of X are weak equivalences for all n. A symmetric spectrum E is called injective if it has the extension property with respect to every monomorphism f of symmetric spectra that is also a level equivalence, i.e. for every diagram in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & E \\ Y & \end{tikzcd}$$

where f is a level equivalence and a monomorphism there is a map $h : Y \to E$ such that $g = hf$. A map $f : X \to Y$ of symmetric profinite spectra is a stable equivalence if $\mathrm{Map}_{\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)}(f, E)$ is a weak equivalence of simplicial sets for every injective profinite Ω-spectrum E. Finally, we call a map in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ a fibration if it has the right lifting property with respect to all maps that are both stable equivalences and cofibrations.

Theorem 3.2. *The above classes of stable equivalences, cofibrations and fibrations define a stable monoidal model structure on symmetric profinite spectra. The profinite completion functor* $\hat{(\cdot)}$ *is a monoidal left Quillen functor from symmetric spectra of* [17] *to profinite symmetric spectra.*

Proof. By Theorem 2.3, $\hat{\mathcal{S}}_*$ is a left proper fibrantly generated simplicial model structure. This allows to deduce the result from the dual machinery and ideas of [16] and [17] using the localization of [22], Theorem 6. □

Remark 3.3. One should note that if the stable homotopy groups $\pi_* X$ of an arbitrary symmetric spectrum X happen to have the structure of profinite groups, then they do not change if we consider $\hat{X}$ as an object in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ and take the homotopy groups of $\hat{X}$ as a profinite spectrum. For, $\pi_* \hat{X}$ is just the profinite completion of $\pi_* X$, see [23].

Now let again G be a profinite group. We want to stabilize the homotopy category of $\hat{\mathcal{S}}_{*G}$ using a monoidal model structure as above. This leads to the following notion of (naive) continuous G-spectra.

Definition 3.4. *A symmetric continuous profinite G-spectrum X is a sequence of pointed Σ_n-profinite G-spaces $\{X_n\}_n$, a pointed $G \times \Sigma_n$-equivariant map $S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$ and such that the composite*

$$S^p \wedge X_n \to S^{p-1} \wedge X_{n+1} \to \ldots \to X_{n+p}$$

is $G \times \Sigma_p \times \Sigma_n$-equivariant. A map of symmetric profinite spectra $X \to Y$ is a collection of Σ_n-equivariant maps $X_n \to Y_n$ in $\hat{\mathcal{S}}_*$ compatible with the structure maps of X and Y. We denote the category of symmetric profinite G-spectra by $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$.

The general procedure to stabilize a model structure of [16] applied to Theorem 2.7 implies the following result.

Theorem 3.5. *There is a left proper model structure on* $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$ *in which the stable equivalences (respectively cofibrations) are the stable equivalences (respectively cofibrations) of underlying profinite symmetric spectra in* $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$ *and the fibrations are the maps having the right lifting property with respect to all maps that are both stable equivalences and cofibrations. We denote its homotopy category by* $\hat{\mathcal{SH}}_G$.

Let $t : \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*) \to \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$ be the functor that equips a spectrum X with the trivial G-action. Its right adjoint is the levelwise defined fixed point functor $(-)^G$: $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G}) \to \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$. Since t clearly preserves weak equivalences and cofibrations, we deduce the following result.

Corollary 3.6. *The pair* $(t, (-)^G)$ *forms a Quillen pair of functors.*

Corollary 3.7. *If a map* f *is a fibration in* $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_G$*, then its underlying map is a fibration in* $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$.

Proof. This follows immediately from the fact that the underlying map of a trivial cofibration in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$ is a trivial cofibration in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$. □

Stable profinite homotopy groups of a profinite spectrum X are defined as the stable homotopy groups of the underlying spectrum of a fibrant replacement RX in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$. If X is equipped with the structure of a profinite G-spectrum, let $R_G X$ be a fibrant replacement of X in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$. We get an induced action of G on each stable homotopy group $\pi_k X = \mathrm{Hom}_{\hat{\mathcal{SH}}}(S^n, R_G X)$. Moreover, since S^n is a finite spectrum, $\pi_k X$ inherits a profinite structure and the action of G on each $\pi_k X$ is continuous. This may be rephrased in the following statement.

Corollary 3.8. *If* X *is a symmetric profinite* G*-spectrum then each stable profinite homotopy group* $\pi_k X$ *is a continuous profinite* G*-module.*

4. Homotopy fixed point spectra

4.1. **Homotopy fixed point spectra.** As for profinite spaces we define the homotopy fixed point spectrum as the homotopy limit of the G-diagram of a fibrant replacement.

Definition 4.1. *Let* $X \in \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$ *be a profinite* G*-spectrum and let* R_G *be a fixed functorial fibrant replacement in* $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$*. Then we define the homotopy fixed point spectrum* X^{hG} *of* X *to be* $X^{hG} := \mathrm{hom}_G(EG_+, R_G X)$ *in* $\mathrm{Sp}^\Sigma(\mathcal{S}_*)$.

Remark 4.2. Since the set of maps out of a profinite set does not carry a profinite structure in general, X^{hG} is only symmetric spectrum. If we need a profinite version, we can consider the profinite symmetric spectrum $\hat{X}^{hG} := \mathrm{hom}_G(*, R_G X)$. Since $EG \to *$ is a homotopy equivalence, the induced map $\hat{X}^{hG} \to X^{hG}$ is a natural equivalence of underlying spectra. But one should note that although X^{hG} and $\hat{X}^{hG}$ are fibrant in $\mathrm{Sp}^\Sigma(\mathcal{S}_*)$, $\hat{X}^{hG}$ might not be fibrant as a profinite spectrum. Hence if we want to calclulate the stable homotopy groups of $\hat{X}^{hG}$ as a profinite spectrum, we have to take a fibrant replacement in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)$. The resulting profinite homotopy groups might differ from the stable homotopy groups of X^{hG} as a spectrum. But if the stable homotopy groups $\pi_* X^{hG}$ happen to have a profinite structure, they are already isomorphic to the stable homotopy groups of $\hat{X}^{hG}$ as a profinite spectrum.

By Corollary 3.6, the functor $(-)^G$ preserves stable equivalences between fibrant profinite G-spectra. Since EG is cofibrant and contractible, this implies the following result, which shows that $\hat{(\cdot)}^{hG} : \hat{\mathcal{SH}}_G \to \mathcal{SH}$ is in fact the total right derived functor of taking fixed points.

Lemma 4.3. *If $X \to Y$ is a stable equivalence in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$, then $\hat{X}^{hG} \to \hat{Y}^{hG}$ is a stable equivalence in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$.*

4.2. **Homotopy fixed point spectral sequence.** Let E be a symmetric profinite G-spectrum. Its stable homotopy groups are continuous profinite G-modules. Let $\{I_k\}$ be a system of ideals in $E_* = \pi_* E$ such that

$$\pi_t E = \lim_k \pi_t E / I_k \pi_t E$$

is a compatible decomposition of $\pi_t E$ as an inverse limit of finite G-modules. Now if Z is a an arbitrary spectrum, let $\{Z_\alpha\}$ be the directed system of its finite subspectra. Then we can equip the E-cohomology groups of Z with the structure of a profinite G-module by setting

$$E^t Z = \lim_{k,\alpha} E^t Z / I_k E^t Z_\alpha.$$

Theorem 4.4. *Let G be a profinite group and let E be a symmetric profinite G-spectrum. Let Z be any spectrum. There is a descent spectral sequence whose $E_2^{s,t}$-term is the sth continuous cohomology of G with coefficients the profinite G-module $E^t Z$:*

$$E_2^{s,t} = H_c^s(G; E^t Z) \Rightarrow (E^{hG})^{s+t} Z.$$

This spectral sequence is strongly convergent if either that G has finite cohomological dimension or if $E^t Z$ is nonzero only for finitely many t.

Proof. This is a version of the homotopy limit spectral sequence of Bousfield and Kan [3] for profinite spectra. We can assume that E is fibrant in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$. A symmetric profinite G-spectrum may be viewed as a functor $E : G \to \mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$. We consider the category $c\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ of cosimplicial profinite spectra equipped with the model structure of [3] X, §4. As remarked in [3] XI, 5.7, there is a cosimplicial replacement functor $\Pi^* E \in c\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$ for a diagram of profinite spectra since there exist products in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$. This cosimplicial replacement $\Pi^* E$ is in codimension n given by $\Pi^n E = \hom_G(G^{n+1}, E) \in \mathrm{Sp}^{\Sigma}(\mathcal{S}_*)$. Since E is fibrant in $\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_{*G})$, its cosimplicial resolution is a fibrant object in $c\mathrm{Sp}^{\Sigma}(\hat{\mathcal{S}}_*)$. Let $\Delta[*]$ be the cosimplicial simplicial finite set which in dimension n is the standard n-simplex with the usual coface and codegeneracy maps, and let $\mathrm{sk}_s \Delta[*]$ be the cosimplicial space which in dimension n is the s-skeleton of $\Delta[n]$. Recall that, given a cosimplicial spectrum Y, the total profinite spectrum of Y is defined to be the limit

$$\mathrm{Tot}\, Y := \lim_s \mathrm{Tot}_s Y$$

where $\mathrm{Tot}_s Y := \hom(\mathrm{sk}_s \Delta[*], Y)$ is the profinite function spectrum of continuous cosimplicial maps from $\Delta[*]$ to Y. By mapping a spectrum Z into the tower

$\{\mathrm{Tot}_s Y\}$, there is a spectral sequence [3] X, §6:

$$E_2^{s,t} = \lim_{\Delta}{}^s([Z,Y]^t) \Rightarrow [Z, \mathrm{Tot}\, Y]^{t+s}$$

where $[Z,Y]^t$ denotes maps in the stable homotopy category $\mathcal{SH}$ that lower dimension by t and $\lim^s$ the sth derived functor of the inverse limit. Now we apply this to $Y = \Pi^* E$. We have to check that the E_2-term is continuous cohomology of G. By [3] X, 7.2, there are natural isomorphisms $E_2^{s,t} \cong \pi^s[Z, \Pi^* E]^t$ for $t \geq s \geq 0$, where π^s denotes the cohomotopy of a cosimplicial group. Since $\Pi^n E$ is fibrant, there are natural isomorphisms $[Z, \Pi^* E]^t \cong \Pi^*[Z,E]^t$ by [3] XI, 5.7. This implies that the above cohomotopy groups are cohomology groups of the complex $C^*(G;[Z,E]^t)$ given in degree s by the set of continuous maps from $G^s \to [Z,E]^t$, where $[Z,E]^t = E^t Z$ is equipped with the above profinite structure. Hence we have identified the E_2-term with the continuous cohomology groups of the statement.
It follows immediately from the definition of $\Pi^* E$ that the total spectrum of this cosimplicial object is equal to $\hom_G(EG_+, E) \in \mathrm{Sp}^{\Sigma}(\mathcal{S}_*)$, i.e. the abutment of the spectral sequence is $(E^{hG})^{s+t} Z$. Finally, if one of the assumptions is satisfied, we get $\lim_r^1 E_r^{s,t} = 0$ and the spectral sequence is strongly convergent. □

Corollary 4.5. *Let G be a profinite group and let X be a symmetric profinite G-spectrum. There is a descent spectral sequence, whose $E_2^{s,t}$-term is the sth continuous cohomology of G with coefficients the profinite G-module $\pi_t(X)$:*

$$E_2^{s,t} = H_c^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}).$$

This spectral sequence is strongly convergent if either that G has finite cohomological dimension or if $\pi_t X$ is nonzero only for finitely many t.

4.3. **Iterated homotopy fixed point spectra.** Let H be a closed subgroup of G. It is a natural question how the homotopy fixed points under the action of G, H and, if H is a normal subgroup, G/H are related to each other. The answer is given by the following theorem, which is an analogue of [1], Proposition 3.3.1. When H is not also open, it turns out that the picture for a profinite group acting on a profinite spectrum is much more convenient than for discrete spectra.

Theorem 4.6. *Let X be a symmetric profinite G-spectrum and let H be a closed normal subgroup of G. Then the following statements hold.*
(1) *The fixed point spectrum $(R_G X)^H$ is fibrant as a profinite G/H-spectrum.*
(2) *The fibrant profinite G-spectrum $R_G X$ is fibrant as a profinite H-spectrum.*
(3) *The profinite homotopy fixed point spectrum $\hat{X}^{hH}$ is a continuous G/H-spectrum.*
(4) *There is a stable equivalence $\hat{X}^{hG} \simeq (\hat{X}^{hH})^{hG/H}$.*
(5) *For any spectrum Z, there is a spectral sequence for iterated homotopy fixed points*

$$H_c^*(G/H; (\hat{X}^{hH})^* Z) \Rightarrow (\hat{X}^{hG})^* Z.$$

Proof. The proof is essentially the same as of [1], Proposition 3.3.1. We repeat it on the one hand for the convenience of the reader and on the other hand to point out that there is an essential advantage when proving assertion (2).
(1) It is clear that $(R_G X)^H$ is a G/H-spectrum. It remains to prove that it is

fibrant as a G/H-spectrum. The restriction along the quotient map $G \to G/H$ induces a functor

$$\mathrm{Res}^G_{G/H} : \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_{G/H} \to \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G})$$

which is the left adjoint of the fixed point functor $(-)^H : \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_{*G}) \to \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_{G/H}$. Since $\mathrm{Res}^G_{G/H}$ obviously preserves cofibrations and stable equivalences, the functor $(-)^H$ preserves fibrant objects.
(2) For a profinite H-spectrum Y, the Borel construction

$$\mathrm{Ind}^G_H Y = G_+ \wedge_H Y$$

defines an induction functor $\mathrm{Ind}^G_H : \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_H \to \mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_G$. It is the left adjoint to the restriction Res^H_G along the inclusion $H \hookrightarrow G$. Since there is a non-equivariant isomorphism $\mathrm{Ind}^G_H Y = G/H_+ \wedge Y$, Ind^G_H preserves cofibrations and stable equivalences. Hence Res^H_G preserves fibrant objects. The difference to [1], is that $G_+ \wedge_H Y$ is a profinite spectrum for any closed subgroup H, since G, H and G/H are profinite, whereas for discrete spectra one needs that H is open in G, i.e. that G/H is finite.
(3) By (2), $R_G X$ is a fibrant profinite H-spectrum. Since $X \to R_G X$ is an H-equivariant trivial cofibration, it is a trivial cofibration in $\mathrm{Sp}^\Sigma(\hat{\mathcal{S}}_*)_H$. Hence we may consider $\hat{X}^{hH} = (R_G X)^H$ as the H-fixed point spectrum of $R_G X$. As remarked above, this is a G/H-spectrum.
(4) Combining the three previous assertions yields the following sequence of stable equivalences of profinite symmetric spectra which proves the last assertion:

$$\hat{X}^{hG} \simeq (R_G X)^G = ((R_G X)^H)^{G/H} \simeq (\hat{X}^{hH})^{hG/H}.$$

(5) This follows immediately from (4) and Theorem 4.4. □

5. Morava stabilizer groups and Lubin-Tate spectra

We return to our main example of the introduction. So for a fixed prime number p and an integer $n \geq 1$, G_n denotes the extended Morava stabilizer group. Its arithmetical origin and importance has been mentioned in the introduction. We focus on its impact for homotopy theory. The profinite group G_n can also be described as the group of automorphisms of the Lubin-Tate spectrum E_n in the stable homotopy category. Hopkins and Miller have shown that G_n is in fact the automorphism group of E_n by $\mathcal{A}_\infty$-maps, cf. [26]. Later on, Goerss and Hopkins extended this result to the E_∞-setting. Hence G_n acts on the spectrum-level on E_n by E_∞-ring maps, cf. [13]. We will show now that this action is in fact a continuous action of the profinite group G_n on the profinite spectrum E_n.
Let BP be the Brown-Peterson spectrum for the fixed prime p and let I be an ideal in BP_* of the form $(p^{i_0}, u_1^{i_1}, \ldots, u_{n-1}^{i_{n-1}})$. These ideals form a cofiltered system. By Remark 1.3 of [9], for any finite spectrum X, the group

$$\pi_t(E_n \wedge X) \cong \lim_I \pi_t(E_n \wedge X)/I\pi_t(E_n \wedge X)$$

is a continuous profinite G_n-module. In fact, for t odd these groups vanish, for t even the quotients $\pi_t(E_n \wedge X)/I\pi_t(E_n \wedge X)$ are finite discrete groups and the decomposition as an inverse limit of these finite discrete G_n-modules is G_n-compatible.

For a cofinal subsystem of such ideals I, there is a finite generalized Moore spectrum M_I with trivial G_n-action such that $\pi_t(E_n)/I \cong \pi_t(E_n \wedge M_I)$. Moreover, not only the homotopy groups of E_n may be written as such an inverse limit but also E_n itself, i.e. $E_n \cong \lim_I E_n \wedge M_I$, cf. [15].

Although $\pi_t(E_n \wedge M_I)$ is a finite and hence discrete G_n-module, G_n does not act discretely on the spectra $E_n \wedge M_I$, i.e. there is no open subgroup U of G_n such that the G_n-action factors through G_n/U. A proof of this fact may be found in [4] Lemma 6.2. Instead of being discrete, we show that E_n may in fact be viewed as a continuous profinite G_n-spectrum in the sense of the previous sections. The point is that one has to decompose E_n further.

For two integers $a < b$, we define an endo-functor P_a^b on profinite spectra as the composite of taking connective covers and the usual Postnikov sections. Since both constructions are functorial, P_a^b is compatible with the group action on a spectrum. So for any G_n-spectrum X, $P_a^b X$ is a G_n-spectrum, the homotopy groups $\pi_t P_a^b X$ are isomorphic as G_n-modules to the homotopy groups of X when $a \leq t \leq b$ and vanish otherwise. By the construction, there are natural maps $P_a^b X \to P_a^{b-1} X$ and $P_{a-1}^b X \to P_a^b X$. Then E_n has the following G_n-equivariant decomposition

$$E_n = \operatorname*{colim}_a \lim_b \lim_I P_a^b(E_n \wedge M_I). \tag{3}$$

Since each spectrum $P_a^b(E_n \wedge M_I)$ is finite for every ideal I and every pair of integers $a < b$, the limit $\lim_b \lim_I P_a^b(E_n \wedge M_I)$ is naturally a profinite spectrum for every a. Moreover, as the homotopy groups of E_n are all profinite groups, we can take the colimit over a in the category of profinite G_n-spectra, which is just the profinite completion of the colimit of the underlying spectra. Hence E_n is in fact a continuous profinite G_n-spectrum. This proves Theorem 1.1 and allows us to apply the machinery of the previous section.

Definition 5.1. *Let G be any closed subgroup of G_n. The homotopy fixed point spectrum E_n^{hG} of the Lubin-Tate spectrum is the homotopy fixed point spectrum of Definition 4.1 of E_n considered via restriction as a profinite G-spectrum, i.e.*

$$E_n^{hG} := \hom_G(EG_+, R_G E_n).$$

Another example of a profinite spectrum is given by Morava K-theory. Let $K(n)$ be the nth p-primary Morava K-theory. Its homotopy groups are profinite in each dimension. The same argument as for E_n shows that we may consider $K(n)$ as a profinite spectrum. Just as for ordinary spectra, applying the localization results of [14] and [22] shows that there is a Bousfield localization for profinite spectra and hence a localization functor also denoted by $\hat{L} = L_{K(n)}$.

Now we prove Theorem 1.2 of the Introduction. If Z is a spectrum, let $\{Z_\alpha\}$ be the directed system of its finite subspectra and let $I_n = (p, v_1, \ldots, v_{n-1}) \subset E_{n*}$. Then we can regard

$$E_n^* Z = \lim_{\alpha,k} E_n^* Z_\alpha / I_n^k E_n^* Z_\alpha$$

as a profinite G_n-module, see [9], Remark 1.3. The Bousfield-Kan spectral sequence of Theorem 4.4 then yields a spectral sequence

$$H_c^*(G, E_n^* Z) \Rightarrow (E_n^{hG})^* Z$$

for any closed subgroup G of G_n. Since G_n is a p-adic analytic group, so is G and its continuous cohomology groups with profinite coefficients are also profinite groups, cf. [30]. Hence the spectral sequence above converges strongly. This proves the first assertion of part (ii) of Theorem 1.2. Part (iii) of Theorem 1.2 on iterated homotopy fixed point spectra is an immediate consequence of Theorem 4.6. It remains to show that E_n^{hG} is equivalent to the construction of Devinatz and Hopkins in [9].

Let us quickly recall the construction of loc. cit. We will use the notation of [1] and [4] to denote the Devinatz-Hopkins fixed point spectra of E_n by E_n^{dhG}. First let $G = U$ be an open subgroup of G_n. In order to construct E_n^{dhU}, Devinatz and Hopkins define a contravariant functor $\mathbf{F}$ from the category of continuous finite left G_n-sets together with the left G_n-set G_n to the category of spectra. Then every element $g \in G_n$ induces via right multiplication r_g on G_n a map $\mathbf{F}(r_g) : E_n \to E_n$, $\mathbf{F}(G_n) = E_n$ and they define $E_n^{dhU} := \mathbf{F}(G_n/U)$, see [9], Theorem 1.

The functor $\mathbf{F}$ defines a natural map of spectra $\mathbf{F}(G_n/U) = E_n^{dhU} \to E_n = \mathbf{F}(G_n)$ corresponding to the canonical quotient map $G_n \to G_n/U$. Since $\mathbf{F}$ is a functor, this map is G_n-equivariant and thus its image lies in the actual fixed point spectrum E_n^U of E_n. Moreover, there is a canonical map from the fixed points $E_n^U = \hom_U(*, E_n)$ to the homotopy fixed points E_n^{hU} of Definition 5.1. We conclude that there exists a canonical map of spectra

$$E_n^{dhU} \to E_n^{hU}. \tag{4}$$

In a second step, the homotopy fixed points for a closed subgroup of G_n are defined. Since G_n is a p-adic analytic profinite group, it is possible to find sequence of open normal subgroups of G_n

$$G_n = U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_i \supsetneq \cdots$$

with $\bigcap_i U_i = \{e\}$. After choosing such a sequence, for a closed subgroup G of G_n, Devinatz and Hopkins set

$$E_n^{dhG} := \hat{L}(\operatorname*{hocolim}_i E_n^{dh(U_iG)})$$

where the colimit is the homotopy colimit in the category $\mathcal{E}$ of commutative S^0-algebras in the category of S^0-modules of [10].

As remarked in [9], p. 5, the maps in the colimit to define E_n^{dhG} are compatible with the G_n-action and induce a canonical map $E_n^{dhG} \to E_n^G$ into the G-fixed points of E_n. Composition with the map $E_n^G \to E_n^{hG}$ yields again a canonical map $E_n^{dhG} \to E_n^{hG}$.

Hence for any closed (or open) subgroup G of G_n we have constructed a comparison map between the homotopy fixed points such that the diagram

$$\begin{tikzcd} E_n^{dhG} \arrow[rr] \arrow[dr] & & E_n^{hG} \arrow[dl] \\ & E_n & \end{tikzcd} \tag{5}$$

commutes.

In order to show that this map $E_n^{dhG} \to E_n^{hG}$ is a weak equivalence for any closed G, we apply the homotopy fixed point spectral sequences of Corollary 4.5 and of [9], Theorems 1 and 2. By comparing both constructions of the descent spectral sequences, we will see that the commutative diagram (5) comes equipped with a natural map of spectral sequences. But to see this, we need a few more details of the constructions of [9].

First let $G = U$ be an open subgroup of G_n such that G_n/U is a finite set. Then the spectral sequence of [9], Theorem 1, is the Bousfield-Kan spectral sequence associated to the cosimplicial resolution $\Pi^*(\bar{C}_{G_n/U})$, where $\bar{C}_{G_n/U}$ is the E_∞-lift [9], Theorem 3.2, of the cosimplicial diagram of spectra $C_{G_n/U}$ given in degree j by

$$C^j_{G_n/U} := \hat{L}(\Pi_{G_n/U}E_n \wedge E_n^{(j)})$$

where $\Pi_{G_n/U}E_n$ denotes the finite product of copies of E_n, one for each element of G_n/U, and the coface and codegeneraccy maps are defined as in [9], 4.11. This construction is functorial on G_n-sets such that the quotient map $G_n \to G_n/U$ induces a map

$$\Pi^*(\bar{C}_{G_n/U}) \to \Pi^*(\bar{C}_{G_n})$$

where C_{G_n} is defined by $C^j_{G_n} = \hat{L}(E_n \wedge E_n^{(j+1)})$. Moreover, there is a canonical map from $C^*_{G_n}$ to the constant cosimplicial diagram E_n which is a weak equivalence by [9], Lemma 5.4. Since all these maps are G_n-equivariant, the image of this map lies in fact in the U-fixed points $E_n^U = \mathrm{hom}_U(*, E_n)$ of E_n considered as a constant cosimplicial diagram. Now let $\Pi^*_U E_n$ be the cosimplicial resolution of the diagram defined by the action of U on E_n. There is a canonical map $\Pi^* E_n^U \to \Pi^*_U E_n$ given in degree j by $\mathrm{hom}_U(*, E_n) \to \mathrm{hom}_U(U^{j+1}, E_n)$. Composing all these maps yields a map of cosimplicial diagrams

$$\Pi^*\bar{C}_{G_n/U} \to \Pi^* E_n^U \to \Pi^*_U E_n. \tag{6}$$

The homotopy fixed points E_n^{dhU} are defined as the total space of the cosimplicial object $\Pi^*\bar{C}_{G_n/U}$, i.e. $E_n^{dhU} := \mathrm{Tot}\,(\Pi^*\bar{C}_{G_n/U})$. As it becomes apparent now, the comparison map (4) is induced by taking total spaces of the map (6). Hence we have defined a map of strongly convergent spectral sequences compatible with (5). Using the identifications of the E_2-terms of Theorem 4.4 and of [9], Lemma 4.20, 4.22 and Proposition 4.25, we conclude that (6) induces an isomorphism of spectral sequences from the E_2-terms on. Thus the comparison map $E_n^{dhU} \to E_n^{hU}$ is a weak equivalence of spectra.

If G is an arbitrary closed subgroup, the descent spectral sequence of [9], Theorem 2, computing the homotopy groups of E_n^{dhG} is defined as the $K(n)_*$-local E_n-Adams spectral sequence of E_n^{dhG}. Devinatz and Hopkins show that the Adams spectral sequence obtained from mapping S^0 into a $K(n)_*$-local E_n-Adams resolution is isomorphic to the Bousfield-Kan spectral sequence obtained from mapping S^0 into the tower of the total space of the cosimplicial resolution of $C_{G_n/G}$, see [9], Appendix, especially Remark A.9, where the cosimplicial diagram $C_{G_n/G}$ is defined in degree j by

$$C^j_{G_n/G} := \hat{L}(E_n^{dhG} \wedge E_n^{(j+1)}).$$

We deduce from this, that the comparison map $E_n^{dhG} \to E_n^{hG}$ is in fact also induced by a map of cosimplicial diagrams as above

$$\Pi^* C_{G_n/G} \to \Pi^* E_n^G \to \Pi^*_G E_n. \tag{7}$$

This shows again that we get a map of strongly convergent spectral sequences which is an isomorphism from the E_2-terms onward. This follows again by Theorem 4.4 and by [9], Lemma 4.20, Proposition 6.3 and the natural transformation (6.5) in [9]. We conlcude that the map $E_n^{dhG} \to E_n^{hG}$ in (5) induces an isomorphism $\pi_* E_n^{dhG} \to \pi_* E_n^{hG}$ on homotopy groups for every closed subgroup G of G_n. Hence there is a canonical equivalence between both homotopy fixed point constructions. Finally, the equivalences $E_n^{dhG_n} \simeq L_{K(n)}S^0$ and $E_n^{dhG_n} \simeq E_n^{hG_n}$ give an equivalence

$$E_n^{hG_n} \simeq L_{K(n)}S^0.$$

This completes the proof of Theorem 1.2.

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