An introduction to étale cobordism

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Abstract

Since étale cobordism is a relatively new cohomology theory for schemes, we give a brief outline of its motivation and known properties and a short cut to its construction. For any details or proofs please see [24], [25] and soon [26].

1 Motivation

For the proof of the Milnor Conjecture Voevodsky invented algebraic cobordism, a new cohomology theory for schemes represented by the Thom spectrum in Morel's and Voevodsky's framework of $\mathbb{A}^1$-homotopy theory. Independently, Levine and Morel constructed in a geometric way an algebraic cobordism theory as the universal oriented cohomology theory on smooth schemes and used it to prove Rost's Degree Formula in characteristic zero. Already these first two examples show that algebraic cobordism is a powerful and interesting new cohomology theory.
Now let us look at the situation for algebraic K-theory. Since it is very difficult to compute algebraic K-theory, Friedlander and Dwyer invented étale K-theory. It turned out to be a useful tool for the study of algebraic K-theory of number fields and other examples. Because of its relationship with étale cohomology by an Atiyah-Hirzebruch spectral sequence, one could in particular prove conjectures on the relation between the values of zeta functions and ratios of orders of K-groups. A detailed survey was given by Thomason in the 1980s [32]. Another application of étale K-theory was given by Thomason [30] who proved absolute cohomological purity conjectured by Grothendieck for regular schemes and $\mathbb{Z}/\ell^{\nu}$-coefficients for ℓ large using localization sequences in étale K-theory, before Gabber proved it in general. Finally, Thomason showed in his celebrated paper [31] that there is a strong relationship between algebraic and étale K-theory by showing that both agree with $\mathbb{Z}/\ell^{\nu}$-coefficients agree after inverting a Bott element in algebraic K-theory which is naturally invertible in étale K-theory. A detailed survey was given by Thomason in the 1980s [32].
This led to an étale cobordism theory that might help to understand algebraic cobordism and that hopefully finds applications in number theory. The idea of the construction is due to Artin-Mazur and Friedlander-Dwyer. In analogy to the complex realization of a scheme over the complex numbers, we use the

étale topological type of schemes over arbitrary fields. Then we apply the usual machinery of topological cohomology theories. Friedlander has done this for K-theory, we do it for cobordism. As an early striking application of étale homotopy theory, Quillen and Friedlander had proved the Adams conjecture [6]. Another remarkable application of étale homotopy theory of schemes has been given recently by Dugger and Isaksen proving the sums-of-squares formulas in positive characteristics [4].
There had already been studies of étale cobordism as Snaith [28] constructed a p-adic cobordism for schemes in a different way in the 1970s and Joshua used Friedlander's approach in the 1980s. The most important new insight of [24] is the construction of a stable étale realization functor from motivic spectra to profinite spaces, a stable homotopy theory of profinite spaces and a resulting systematic approach to étale topological cohomology theories.
We will study two versions of étale cobordism, where we restrict to the case of $\mathbb{Z}/\ell^{\nu}$-coefficients for a prime ℓ different from the characteristic of a fixed base field. One which is sufficient if we are mainly interested in mere topological questions over a separably closed field, denoted by $\hat{MU}^*_{\text{ét}}(-;\mathbb{Z}/\ell^{\nu})$, see [24]. The other one gives a good definition over arbitrary fields but is more difficult to analyze, denoted by $\hat{MGL}^{*,*}_{\text{ét}}(-;\mathbb{Z}/\ell^{\nu})$, see [26].
As the most important result, let us state a comparison theorem for algebraic and étale cobordism which makes it clear that étale cobordism is a useful tool for the calculation of algebraic cobordism. We denote by $\beta \in MGL^{0,1}(k;\mathbb{Z}/\ell^{\nu})$ a Bott element corresponding to a primitive ℓ^{ν}th root of unity. See the end of the next section for more details.

Conditional Theorem 1.1 *Let X be a smooth scheme of finite type over an algebraically closed field k with* char $k \neq \ell$. *If we assume the existence and convergence of the above Atiyah-Hirzebruch spectral sequence from motivic cohomology to algebraic cobordism, then ϕ is an isomorphism*

$$\phi : (\oplus_{p,q} MGL^{p,q}(X;\mathbb{Z}/\ell^{\nu}))[\beta^{-1}] \xrightarrow{\cong} \oplus_{p,q} \hat{MU}^p_{\text{ét}}(X;\mathbb{Z}/\ell^{\nu}).$$

The following general conjecture would yield in particular a description of algebraic cobordism over fields of positive characteristic and in arbitrary degrees.

Conjecture 1.2 *Let X be a smooth scheme of finite type over a field k with* char $k \neq \ell$ *and of finite ℓ-cohomological dimension. Then the map ϕ becomes an isomorphism between étale and algebraic cobordism with $\mathbb{Z}/\ell^{\nu}$-coefficients after inverting a Bott element:*

$$\phi : MGL^{*,*}(X;\mathbb{Z}/\ell^{\nu})[\beta^{-1}] \xrightarrow{\cong} \hat{MGL}^{*,*}_{\text{ét}}(X;\mathbb{Z}/\ell^{\nu}).$$

Although $\hat{MGL}^{*,*}_{\text{ét}}(-;\mathbb{Z}/\ell^{\nu})$ is more general and seems to be the right theory for arbitrary fields, the former has its own right of existence. First of all, it is easier to compute and secondly, it generalizes directly topological cobordism of complex varieties. In particular, as B. Kahn suggested, we hope to use $\hat{MU}^*_{\text{ét}}(-;\mathbb{Z}/\ell^{\nu})$, as Totaro did for complex varieties [33], for applications to the study of the cycle class map from Chow groups to étale cohomology, $CH^*(X)/\mathbb{Z}/\ell^{\nu} \to H^{2*}_{\text{ét}}(X;\mathbb{Z}/\ell^{\nu})$ for schemes over fields of positive characteristic $p \neq \ell$ containing a primitive ℓ^{ν}th root of unity.

2 Results

Let k be a base field. We denote by Sm/k the category of smooth quasi-projective schemes of finite type over k. Except where otherwise stated, we assume $X \in \mathrm{Sm}/k$.

2.1 Properties

1. **Coefficients**: Let k be separably closed, then the étale homotopy type of k is a contractible space and we get

$$\hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu) \cong \mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^\nu,$$

where $\mathbb{L}^*$ is the Lazard ring, universal object in the category of formal group laws. It is isomorphic to a polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$ in infinitely many indeterminants where the degree of x_i is $-2i$, see [1].

2. **Comparison with complex cobordism**: Let X be a complex algebraic variety and let $X(\mathbb{C})$ be the topological space of complex points. It follows from the Riemann Existence Theorem of [2] that, for every ν, there is an isomorphism

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \cong MU^*(X(\mathbb{C}); \mathbb{Z}/\ell^\nu).$$

3. $\mathbb{A}^1$**-homotopy invariance**: The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$p^* : \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \xrightarrow{\cong} \hat{MU}^*_{\text{ét}}(X \times \mathbb{A}^1; \mathbb{Z}/\ell^\nu).$$

If the characteristic of the base field is $p > 0$, it is crucial for the $\mathbb{A}^1$-invariance above, that we consider $\mathbb{Z}/\ell^\nu$-coefficients with $\ell \neq p$. Since $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ is non-trivial in this case, we could not get an $\mathbb{A}^1$-invariant theory with arbitrary coefficients. The definition of $\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ does not depend on the characteristic of k. But since almost all other cohomology theories are $\mathbb{A}^1$-invariant and we are looking for comparison results, we want étale cobordism to be $\mathbb{A}^1$-invariant.

4. **Atiyah-Hirzebruch spectral sequence**: As in the topological situation, one constructs the following spectral sequence using the skeletal filtration. One gets that for every locally noetherian scheme X, there is a convergent spectral sequence

$$E_2^{p,q} = H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q) \Longrightarrow \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu),$$

which is convergent if the $E_2^{p,q}$-term is finite for every p and q or vanishes for p large enough. This is in particular the case for smooth schemes over k. Note that $\mathbb{Z}/\ell^\nu \otimes MU^q$ is a finite abelian group, since MU^* is finitely generated abelian in every degree. So the E_2-term is a well known object. This makes the Atiyah-Hirzebruch spectral sequence so valuable, since étale cohomology can be calculated in many cases. We will consider below some very easy consequences of the spectral sequence. Again, if ℓ is equal to the characteristic $p > 0$ of k, the spectral sequence above could be constructed in the same way, but the E_2-term would not be as interesting, since étale cohomology with p-torsion coefficients does not behave well. Unfortunately, we do not yet know a good answer for the case $\ell = p$.

5. **Mayer-Vietoris sequence**: Let $U \stackrel{i}{\hookrightarrow} X \stackrel{p}{\leftarrow} V$ be an elementary distinguished square in Sm/k. Then it follows from the excision theorem of [12] that there is a long exact sequence of graded groups

$$\cdots \to \hat{MU}^n_{\text{ét}}(X;\mathbb{Z}/\ell^\nu) \to \hat{MU}^n_{\text{ét}}(U;\mathbb{Z}/\ell^\nu) \oplus \hat{MU}^n_{\text{ét}}(V;\mathbb{Z}/\ell^\nu) \to$$
$$\to \hat{MU}^n_{\text{ét}}(U \times_X V;\mathbb{Z}/\ell^\nu) \to \hat{MU}^{n+1}_{\text{ét}}(X;\mathbb{Z}/\ell^\nu) \to \cdots .$$

6. **Orientability**: If k is a field of finite ℓ-cohomological dimension and without ℓ-torsion, cf. [24], of characteristic different from ℓ, then $\hat{MU}^*_{\text{ét}}(-;\mathbb{Z}/\ell)$ and $M\hat{GL}^{2*,*}_{\text{ét}}(-;\mathbb{Z}/\ell^\nu)$ satisfy the axioms of an oriented cohomology theory on Sm/k of [22]; in particular it has Chern classes, it satisfies a projective bundle formula and there are push-forward maps for projective morphisms.

7. **Map to étale cohomology**: The morphism of profinite spectra $\hat{MU} \to H\mathbb{Z}/\ell^\nu$ induced by the orientation yields a unique map of profinite étale cohomology theories for every X in Sm/k: $\hat{MU}^*_{\text{ét}}(X;\mathbb{Z}/\ell^\nu) \to H^*_{\text{ét}}(X;\mathbb{Z}/\ell^\nu)$.

8. **Base change**: Let R be a discrete valuation ring with separably closed closed residue field k, and let $f : X \to \operatorname{Spec} R$ be a smooth proper scheme with geometrically connected fibres X_0, X_1. Then the profinite étale homotopy types of X_0 and X_1 are canonically isomorphic by [2] and hence we get $\hat{MU}^*_{\text{ét}}(X_0;\mathbb{Z}/\ell^\nu) \cong \hat{MU}^*_{\text{ét}}(X_1;\mathbb{Z}/\ell^\nu)$.

9. **Galois action**: Let $\bar{k}$ be a separable closure of k and let $G := \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of k. Unfortunately, the definition of $\hat{MU}^*_{\text{ét}}(-;\mathbb{Z}/\ell^\nu)$ is not algebraic enough to allow an interesting action of G on $\hat{MU}^*_{\text{ét}}(\bar{k};\mathbb{Z}/\ell^\nu)$. In fact, the natural induced action is trivial. This is one more argument to give another definition of étale cobordism that allows interesting Galois actions. This will be provided by $M\hat{GL}^{*,*}_{\text{ét}}(-;\mathbb{Z}/\ell^\nu)$.

2.2 Examples

1. The (reduced) étale cobordism of a finite field k, char $k \neq \ell$, is given by the isomorphism $\hat{MU}^n_{\text{ét}}(k;\mathbb{Z}/\ell^\nu) = \mathbb{L}^{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^\nu$.

2. Let k be a local field (of characteristic $\neq \ell$) and let ℓ^{ν_0} be the greatest common divisor of ℓ^ν and $q-1$, where q denotes the number of elements in the residue field of k. Using the Atiyah-Hirzebruch spectral sequence and the fact that the Galois action is trivial, we are able to compute the étale cobordism of k. For all n, the (reduced) profinite étale cobordism groups with $\mathbb{Z}/\ell^\nu$-coefficients of k are given by

$$\hat{MU}^n_{\text{ét}}(k;\mathbb{Z}/\ell^\nu) = \begin{cases} \mathbb{Z}/\ell^{\nu_0} \otimes \mathbb{L}^{n-2} & : \ n \text{ even} \\ \mathbb{Z}/\ell^{\nu_0} \otimes \mathbb{L}^{n-1} & : \ n \text{ odd.} \end{cases}$$

3. Let k be a separably closed field of characteristic $p \neq \ell$ and let X be a connected projective smooth curve of genus g over k. Then the (reduced) étale cobordism of X with $\mathbb{Z}/\ell^\nu$-coefficients is given by

$$\hat{MU}^n_{\text{ét}}(X;\mathbb{Z}/\ell^\nu) \cong \begin{cases} \mathbb{Z}/\ell^{\nu} \otimes \mathbb{L}^{n-2} & : \ n \text{ even} \\ \oplus_{i=1}^{i=2g}(\mathbb{Z}/\ell^{\nu} \otimes \mathbb{L}^{n-1}) & : \ n \text{ odd.} \end{cases}$$

2.3 Comparison with algebraic cobordism

For $X \in \mathrm{Sm}/k$ let $\Omega^*(X)$ be the algebraic cobordism ring of Levine and Morel [17]. In analogy to Quillen's insight for complex cobordism, $\Omega^*(X)$ is constructed as the universal oriented cohomology theory for smooth schemes over k. In degree d it is generated by classes $[Y \to X]$ of projective morphisms from smooth irreducible schemes Y of relative dimension d to X. Since étale cobordism is orientable, there is a canonical morphism of oriented cohomology theories

$$\theta : \Omega^*(X) \to \hat{MU}_{\text{ét}}^{2*}(X; \mathbb{Z}/\ell)$$

defined by sending a generator $[f : Y \to X] \in \Omega^*(X)$, $f : Y \to X$ a projective morphism between smooth schemes, to the element $f_*(1_Y) \in \hat{MU}_{\text{ét}}^{2*}(X; \mathbb{Z}/\ell)$.
On the other hand, Voevodsky had constructed algebraic cobordism as the cohomology theory represented by the algebraic Thom spectrum MGL in the stable $\mathbb{A}^1$-homotopy theory, see [34]. Both theories should agree in the sense that $\Omega^*(X)$ is isomorphic to $MGL^{2*,*}(X)$. The stable étale realization functor $\hat{\mathrm{Et}} : \mathcal{SH}^{\mathbb{P}^1}(k) \to \hat{\mathcal{SH}}_2$ of [24] induces a natural map for every X in Sm/k

$$\phi : MGL^{p,q}(X; \mathbb{Z}/\ell) \to \hat{MU}_{\text{ét}}^{p}(X; \mathbb{Z}/\ell).$$

For every $\in \mathrm{Sm}/k$, there is a canonical commutative diagram of morphisms of oriented cohomology theories

$$\begin{array}{ccc} \Omega^*(X; \mathbb{Z}/\ell) & \xrightarrow{\theta_{MGL}} & MGL^{2*,*}(X; \mathbb{Z}/\ell) \\ \theta_{\hat{MU}} \searrow & & \swarrow \phi \\ & \hat{MU}_{\text{ét}}^{2*}(X; \mathbb{Z}/\ell). & \end{array}$$

2.4 Inverting a Bott element

Let $H^p(X; \mathbb{Z}/n(q))$ denote the motivic cohomology of a smooth scheme X over a field k. For $\mathrm{Spec}\, k$ there is an isomorphism $H^0(\mathrm{Spec}\, k; \mathbb{Z}/n(1)) \cong \mu_n(k)$ with the group of n-th roots of unity in k. Assuming that k contains an n-th root of unity ζ, we have a corresponding motivic Bott element $\zeta_n \in H^0(\mathrm{Spec}\, k; \mathbb{Z}/n(1))$.
Levine has shown in [16] that motivic $\mathbb{Z}/n$-cohomology of a smooth scheme over k agrees with étale $\mathbb{Z}/n$-cohomology after inverting the Bott element. I am grateful to Marc Levine for an explanation of his ideas.
Furthermore, Hopkins and Morel announced the construction of a motivic Atiyah-Hirzebruch spectral sequence. It is the slice filtration spectral sequence conjectured by Voevodsky in [36] from motivic cohomology with coefficients in the ring MU^* to algebraic cobordism

$$E_2^{*,*,*} = H^{*,*}(X, MU^*) \Longrightarrow MGL^{*,*}(X).$$

We consider the spectral sequence for $\mathrm{Spec}\, k$. Since $E_2^{p,1,q}(\mathrm{Spec}\, k)$ is concentrated in degrees $p = 0$ and $p = 1$, we deduce $MGL^{0,1}(k) \cong k^\times$. For $\mathbb{Z}/n$-coefficients, the exact sequence for coefficients implies that we get an isomorphism $MGL^{0,1}(k; \mathbb{Z}/n) \cong \mu_n(k)$ and, via the spectral sequence, the motivic Bott element defined above, is sent to an induced *Bott element* $\beta_n \in MGL^{0,1}(k; \mathbb{Z}/n)$.

Let us now suppose that k is algebraically closed, $n = \ell^\nu$ and char $k \neq \ell$. In particular, k contains an ℓ^ν-th root of unity ζ. It defines an element $\zeta \in H^0_{\text{ét}}(\operatorname{Spec} k; \mu_{\ell^\nu})$. The element $\zeta \cdot 1_{\hat{MU}/\ell^\nu}$ induces via the Atiyah-Hirzebruch spectral sequence an element $\beta_{\hat{MU}/\ell^\nu} \in \hat{MU}^0_{\text{ét}}(\operatorname{Spec} k; \mathbb{Z}/\ell^\nu)$. The multiplication with ζ yields an isomorphism of spectral sequences and hence multiplication with $\beta_{\hat{MU}/\ell^\nu}$ is an isomorphism on étale cobordism. Furthermore, the map ϕ sends β_{ℓ^ν} to $\beta_{\hat{MU}/\ell^\nu}$ and induces a localized map $\phi : MGL^{*,*}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \to \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$.
Finally, using the Dold-Thom theorem and the Künneth isomorphism for symmetric products in étale cohomology over an algebraically closed field, one shows that the étale realization defines a map from motivic $\mathbb{Z}/\ell^\nu$-cohomology to étale $\mathbb{Z}/\ell^\nu$-cohomology. With a fixed isomorphism $\mathbb{Z}/\ell^\nu(1) \cong \mathbb{Z}/\ell^\nu$, then this map is the unique map of oriented cohomology theories. Furthermore, localization at β is exact. Consequently, the étale realization yields a map of Atiyah-Hirzebruch spectral sequences, whose E_2-terms agree by Theorem 1.1 of [16]. This implies the following

Conditional Theorem 2.1 *Let X be a smooth scheme of finite type over an algebraically closed field k with* char $k \neq \ell$. *If we assume the existence and convergence of the above Atiyah-Hirzebruch spectral sequence from motivic cohomology to algebraic cobordism, then ϕ is an isomorphism*

$$\phi : (\oplus_{p,q} MGL^{p,q}(X; \mathbb{Z}/\ell^\nu))[\beta^{-1}] \xrightarrow{\cong} \oplus_{p,q} \hat{MU}^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu).$$

3 Constructions

3.1 Profinite Homotopy

Let $\hat{\mathcal{E}}$ be the category of profinite sets, i.e. compact and totally disconnected topological spaces. We denote by $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) the category of simplicial profinite sets (resp. simplicial sets). The objects of $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) will be called *profinite spaces* (resp. *spaces*). We denote by $X \mapsto \hat{X}$ the levelwise profinite completion functor $\mathcal{S} \to \hat{\mathcal{S}}$.
Let X be a profinite space. We denote by $H^*(X; \pi)$ the continuous cohomology of X with coefficients in the topological abelian group π [20]. Note that if π is a finite abelian group and Z a simplicial set, then the cohomologies $H^*(Z; \pi)$ and $H^*(\hat{Z}; \pi)$ are canonically isomorphic.
Let ℓ be a fixed prime number. Fabien Morel has shown in [20] that the category $\hat{\mathcal{S}}$ can be given the structure of a closed model category. The weak equivalences are the maps inducing isomorphisms in continuous cohomology with coefficients $\mathbb{Z}/\ell$; the cofibrations are the degreewise monomorphisms. The homotopy category is denoted by $\hat{\mathcal{H}}^\ell$. The corresponding homotopy groups are pro-ℓ-groups.
In [25] we show that there is another model structure on $\hat{\mathcal{S}}$, already indicated by Morel in [20], with the same cofibrations, but the weak equivalences are maps inducing isomorphisms on profinite fundamental groups and cohomology with finite local coefficients. We denote the homotopy category by $\hat{\mathcal{H}}$. The difficulty

in the proof lies in the problem that one cannot define homotopy groups directly without a fibrant replacement functor. The solution is to define fundamental groups via finite coverings as for the étale fundamental group of schemes, then to say what is a local coefficient system on a profinite space and then to show without using homotopy groups but only cohomology with possibly non-abelian constant coefficients or abelian local coefficient systems that we get a fibrantly generated model structure on $\hat{\mathcal{S}}$. Afterwards, we define also higher homotopy groups, which are all profinite groups.
As for simplicial sets we may consider spectra, i.e. sequences $E_0, E_1, \ldots$ and maps $S^1 \wedge E_n \to E_{n+1}$. In [24], [25] we show using the methods of [11] that there is a stable model structure on $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ for which the prolongation $S^1 \wedge \cdot : \mathrm{Sp}(\hat{\mathcal{S}}_*) \to \mathrm{Sp}(\hat{\mathcal{S}}_*)$ is a Quillen equivalence. In particular, the stable equivalences are the maps that induce an isomorphism on all generalized cohomology theories, represented by profinite $\hat{\Omega}$-spectra; the stable cofibrations are the maps $i : A \to B$ such that i_0 and the induced maps $j_n : A_n \amalg_{S^1 \wedge A_{n-1}} S^1 \wedge B_{n-1} \to B_n$ are monomorphisms for all n. We denote the corresponding homotopy categories of profinite spectra by $\hat{\mathcal{SH}}$, respectively by $\hat{\mathcal{SH}}^{\ell}$ which depends on ℓ and is used in [24].

3.2 Algebraic geometry

The étale topological type $\mathrm{Et}\, X$ of a noetherian scheme X is an object defined via étale hypercoverings that encodes the étale homotopy information on X. It was originally defined as a pro-simplicial set by Artin-Mazur [2] and Friedlander [9]. We study its profinite completion $\hat{\mathrm{Et}}\, X$ in $\hat{\mathcal{S}}$. Isaksen [12] and Schmidt [27] independently extended the definition to motivic spaces, and finally it induces a functor from motivic spectra to $\hat{\mathcal{SH}}$ and $\hat{\mathcal{SH}}^{\ell}$, see [24] and [25].
The crucial point on $\hat{\mathrm{Et}}\, X$ is that it associates to X an almost usual space such that we have $\pi_1(\hat{\mathrm{Et}}\, X) = \pi_1^{\text{ét}}(X)$ and $H^*(\hat{\mathrm{Et}}\, X; \mathcal{L}(F)) = H^*_{\text{ét}}(X; F)$ for every locally constant sheaf F on X where $\mathcal{L}(F)$ denotes the corresponding local coefficient system on $\hat{\mathrm{Et}}\, X$, see [2] and [9].

Examples 1. Let k be a field, then $\hat{\mathrm{Et}}\, k$ is a BG_k, i.e. a profinite classifying space for the absolute Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ of k.
2. In particular, $\hat{\mathrm{Et}}\, \mathbb{F}_q$ is isomorphic to S^1 in $\hat{\mathcal{H}}$.
3. Over a separably closed field, $\hat{\mathrm{Et}}\, \mathbb{G}_m$ is a $K(\hat{\mathbb{Z}}(1), 1)$.
4. Again if k is separably closed, we have $\hat{\mathrm{Et}}\, \mathbb{P}^1_k \cong K(\hat{\mathbb{Z}}(1), 2)$.
5. Let L/k be a Galois extension of a field k with Galois group $G_{L/k}$, X a locally noetherian scheme over k and $X_L := X \otimes_k L$ the corresponding extension. Then $\hat{\mathrm{Et}}\, X_L \to \hat{\mathrm{Et}}\, X$ is a principal $G_{L/k}$-fibration and $\hat{\mathrm{Et}}\, X$ is homotopy equivalent in $\hat{\mathcal{S}}$ to the fibre product $EG_{L/k} \times_{BG_{L/k}} \hat{\mathrm{Et}}\, X_L$.

Completing away from the characteristic The reason why we have to consider the $\mathbb{Z}/\ell$-model structures on $\hat{\mathcal{S}}$, is the following. If k is a field of characteristic $p > 0$, then $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ is non-trivial. Since algebraic cobordism and all other cohomology theories of interest here are in fact $\mathbb{A}^1$-invariant theories, i.e. the projection $\pi : X \times \mathbb{A}^1 \to X$ induces an isomorphism $\pi^* : E^*(X) \to E^*(X \times \mathbb{A}^1)$, we want to make étale cobordism also into an $\mathbb{A}^1$-invariant theory. For this purpose, we complete away from the characteristic such that $\hat{\mathrm{Et}}\,\pi : \hat{\mathrm{Et}}\,(X \times \mathbb{A}^1) \to \hat{\mathrm{Et}}\,X$ becomes an isomorphism. This is the case when we work in $\hat{\mathcal{H}}^\ell$, since $\pi^* : H^*_{\text{ét}}(X; F) \to H^*_{\text{ét}}(X \times \mathbb{A}^1; F)$ is an isomorphism for all torsion sheaves F whose torsion is prime to the base field characteristic p, cf. for example [19]. The same problem affects étale K-theory as well, of course. If $\mathrm{char}\, k = 0$, we may also consider $\hat{\mathcal{H}}$ and get $\mathbb{A}^1$-invariant theories. But since we are mainly interested in applications with $\mathbb{Z}/\ell^\nu$-coefficients, this annoyance is not that annoying after all.

Last step: $\hat{\mathrm{Et}}\,\mathbb{P}^1$-spectra As a final technical step, we want the smash product with the étale realization of the projective line over k to be an isomorphism; i.e. we consider the category of sequences $E_0, E_1, \ldots$ of pointed profinite spaces and structure maps $\hat{\mathrm{Et}}\,\mathbb{P}^1_k \wedge E_n \to E_{n+1}$ and, starting with the weak equivalences in $\hat{\mathcal{S}}$ that we are interested in, we formally invert the functor $\hat{\mathrm{Et}}\,\mathbb{P}^1_k \wedge -$ in this category using as above the procedures of [11] and [24]. Details will be given in [26]. We denote the resulting homotopy categories by $\hat{\mathcal{SH}}_{\mathbb{P}^1}$, respectively $\hat{\mathcal{SH}}^\ell_{\mathbb{P}^1}$. As above, $\hat{\mathrm{Et}}$ extends to a functor from the stable homotopy category of motivic $\mathbb{P}^1$-spectra to $\hat{\mathcal{SH}}_{\mathbb{P}^1}$, respectively $\hat{\mathcal{SH}}^\ell_{\mathbb{P}^1}$.

3.3 Definition of étale cobordism

1. We define the étale topological cobordism of a locally noetherian scheme X, to be the étale cohomology theory represented by the profinitely completed cobordism spectrum $\hat{MU}$, i.e.

$$\hat{MU}^n_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) := \mathrm{Hom}_{\hat{\mathcal{SH}}^\ell}(\Sigma^\infty(\hat{\mathrm{Et}}\,X), \hat{MU}/\ell^\nu[n])$$

where we add a base-point if X is not already pointed.
2. We define the étale cobordism of a locally noetherian scheme X over a field k, to be the étale cohomology theory represented by the profinite cobordism spectrum $\hat{MGL} := \hat{\mathrm{Et}}\,(MGL)$, i.e.

$$\hat{MGL}^{p,q}_{\text{ét}}(X) := \mathrm{Hom}_{\hat{\mathcal{SH}}_{\mathbb{P}^1}/\hat{\mathrm{Et}}\,k}(\Sigma^\infty(\hat{\mathrm{Et}}\,X), \hat{MGL}[p-2q] \wedge (\hat{\mathrm{Et}}\,\mathbb{P}^1_k)^q)$$

where Hom denotes maps in the stable homotopy category of $\hat{\mathrm{Et}}\,\mathbb{P}^1_k$-spectra over $\hat{\mathrm{Et}}\,k$.

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