Treeable equivalence relations and essential countability

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- Treeable analytic equivalence relations
- Embeddibility of $E_1$
An equivalence relation $E$ on $X$ is **Borel reducible** to an equivalence relation $F$ on $Y$, $E \leq_B F$, if there is a Borel $f : X \to Y$ such that $x_1 E x_2$ iff $f(x_1) F f(x_2)$. We write $E \leq_c F$ when $f$ is continuous, and $E \sqsubseteq_c F$ when $f$ is injective.
Potential Wadge class

Definition

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For a Wadge class $\Gamma$, we say that a relation $R$ on $(X, \tau)$ is *potentially in* $\Gamma$, $R \in \text{pot}(\Gamma)$, if there is a topology $\tau'$ on $X$ with the same Borel sets such that $R$ is in $\Gamma(X, \tau')$. 
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Fact: An equivalence relation $E$ is potentially $\Gamma$ iff it is *essentially* $\Gamma$, i.e., there is $F$ in $\Gamma$ such that $E \leq_B F$. 
Potential Wadge class

Basic facts

- The diagonal $\Delta(2^\mathbb{N}) = \{(x, x) : x \in 2^\mathbb{N}\}$ is not in $\text{pot}(\Sigma_1^0)$. 

Rephrasing of standard dichotomy theorems

For a Borel equivalence relation $E$:

- $E/\in\text{pot}(\Sigma_1^0)$ iff $\Delta(2^\mathbb{N}) \sqsubseteq^c E$ (Silver).
- $E/\in\text{pot}(\Pi_1^0)$ iff $E_0 \sqsubseteq^c E$ (Harrington-Kechris-Louveau).
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**Note**

Hjorth and Kechris showed that the $\Pi^0_3$ equivalence relation $E_3 = E_0^\omega$ is minimal above $E_0$, i.e., if $E \leq_B E_3$ then either $E \sim_B E_3$ or $E \leq_B E_0$. Hence $E_3$ would be the only possible candidate for a minimum non-pot($\Sigma^0_3$) BEQ.
A *treeing* of an equivalence relation $E$ on $X$ is an acyclic graph on $X$ whose connected components are the equivalence classes of $E$. An equivalence relation is *treeable* if there is an analytic treeing $\mathcal{T}$ of $E$. 

Treeings have been primarily studied for countable Borel equivalence relations. Basic facts:

- There are countable BEQs which are not treeable.
- There is a maximum countable treeable BEQ, $E^\infty_T$.
- A treeable BEQ admits a Borel treeing.
- There are non-hyperfinite countable treeable BEQs (e.g., $E^\infty_T$).
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The class of countable treeable BEQs is closed under $\subseteq$, $\leq_B$, and several other operations.

Hjorth showed that there are many countable treeable BEQs: The partial order of inclusion among Borel subsets of Baire space can be embedded into the quasi-order of $\leq_B$ among countable treeable BEQs.

Any orbit equivalence relation $E_{X\mathcal{G}}$ which is treeable is essentially countable (Hjorth).

But little else is known about uncountable treeable Borel equivalence relations.

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The equivalence relation $\mathcal{E}_1$ is defined on $(2^\mathbb{N})^\mathbb{N}$ by $\bar{x}\mathcal{E}_1\bar{y}$ iff \( \exists m \forall n \geq m(x_n = y_n) \).
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There is a dichotomy for hypersmooth equivalence relations:

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Kechris and Louveau also showed that \( E_{1} \) is not reducible to any *idealistic* BEQ; in particular it is not reducible to any orbit equivalence relation \( E \bigwedge^{X}_{G} \). They asked if the converse holds:
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**Question (Kechris-Louveau)**

If $E$ is a Borel equivalence relation, is it the case that either $E_1$ is reducible to $E$ or $E$ is idealistic?
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**Question (Hjorth)**

*If $E$ is a treeable Borel equivalence relation, is it the case that exactly one of:*

1. $E$ is essentially countable, or
2. $E_1 \sqsubseteq_c E$?
A particular class of treeable BEQs

We will describe a class of treeable Borel equivalence relations which we call *sparsely treeable*. By analyzing the sparsely treeable BEQs, we can show:

- The collection of treeable BEQs is unbounded in terms of $\leq_B$ and potential Wadge class.
- There is no minimum non-pot (Γ) BEQ for Γ a Borel Wadge class of rank at least 3.
- If $E$ is a BEQ which is not essentially hyperfinite, then there are BEQs of arbitrarily high Wadge class which are $\leq_B$-incomparable with $E$.
- Hjorth’s question is true for sparsely treeable BEQs.

A fundamental tool for these results will be a strengthening of the Kechris-Louveau dichotomy for embedding $E_1$. 
We will describe a class of treeable Borel equivalence relations which we call *sparsely treeable*. By analyzing the sparsely treeable BEQs, we can show:

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Closed treeings with acyclic levels

**Definition**

We say a treeing $\mathcal{T}$ of $E$ on $X$ is *closed* if $\mathcal{T}$ is a closed subset of $X^2$. When $X = 2^\mathbb{N}$ (or $\mathbb{N}^{\mathbb{N}}$) we may find a tree $T$ on $2 \times 2$ (or $\mathbb{N} \times \mathbb{N}$) such that $\mathcal{T} = [T]$. 
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A closed treeing on $2^\mathbb{N}$ (or $\mathbb{N}^\mathbb{N}$) has acyclic levels if the induced graphing of $2^n$ induced by $T \cap (2 \times 2)^n$ is acyclic for each $n \geq 1$. 
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**Lemma**

*Let \( \mathcal{T} \) be a closed treeing with acyclic levels. Then \( \mathcal{T}^n \) is closed for all \( n \), and \( \alpha E_{\mathcal{T}} \beta \) iff \( \lim_{n} d_n(\alpha \restriction n, \beta \restriction n) < \infty \), where \( d_n \) is the induced path distance on \( 2^n \).*
Example

Let \( T = \bigcup T_n \), where \( T_0 = \emptyset \) and

\[
T_{n+1} = \{(s \bowtie i, t \bowtie i) : (s, t) \in T_n \land i \in 2\}
\cup \{(0^{n+1}, 1^{n+1}), (1^{n+1}, 0^{n+1})\}.
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Then \( T = [T] \) induces the equivalence relation \( E \) given by

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x E y \iff \forall \infty n(x(n) = y(n) \lor (x(n) = x(n-1) \land y(n) = y(n-1)))
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Example

$E_1$ is bi-reducible with an $E$ which admits a closed treeing with acyclic levels.
We say a closed treeing $T$ with acyclic levels is \textit{densely splitting} if for each $(s, t) \in T$ we have:

1. $(s \triangleleft i, t \triangleleft i) \in T$ for all $i \in 2$.
2. $\exists r \exists i (s \triangleleft r \triangleleft i, t \triangleleft r \triangleleft (1 - i)) \in T$. 

Lemma

If $T_1$ and $T_2$ are closed treeings with acyclic levels and $T_2$ is densely splitting, then $E_{T_1} \subseteq_{c} E_{T_2}$.

If both are densely splitting then $E_{T_1}$ and $E_{T_2}$ are bi-embeddable.

In particular, $E_1 \subseteq_{c} E_T$ for a densely splitting $T$. 


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If both are densely splitting then $E_{T_1}$ and $E_{T_2}$ are bi-embeddable.

In particular, $E_1 \sqsubseteq_c E_T$ for a densely splitting $T$. 
Lemma

If $T$ is a densely splitting closed treeing with acyclic levels, then $E_1 <_B E_T$. 
Lemma

If $\mathcal{T}$ is a densely splitting closed treeing with acyclic levels, then $E_1 \preceq_B E_\mathcal{T}$.

We can now define the class of treeings which we will use.

Definition

We say that a treeing $\mathcal{T}$ is *sparse* if there is a closed treeing $\mathcal{T}'$ with acyclic levels such that $\mathcal{T} \subseteq \mathcal{T}'$. 
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We need the following technical definition of a particular type of closed treeing with acyclic levels which will allow us to derive potential Wadge class results.
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**Definition**

We say that $E \subseteq (2 \times 2)^{<\mathbb{N}}$ is a frame if:

1. $\forall l \in \mathbb{N} \exists! (s_l, t_l) \in E \cap (2 \times 2)^l$
2. $\forall l, p \in \mathbb{N} \forall t \in 2^{<\mathbb{N}} \exists n ( (s_l0t0^n, t_l1t0^n) \in E \land (|s_l0t0^n| - 1)_0 = p )$
3. $\forall l > 0 \exists q < l \exists t (s_l, t_l) = (s_q0t, t_q1t)$
Fix a pairing function $n = \langle (n)_0, (n)_1 \rangle$.

**Definition**

We say that $E \subseteq (2 \times 2)^{<\mathbb{N}}$ is a frame if:

1. $\forall l \in \mathbb{N} \exists! (s_l, t_l) \in E \cap (2 \times 2)^l$
2. $\forall l, p \in \mathbb{N} \forall t \in 2^{<\mathbb{N}} \exists n \ ( (s_l0t0^n, t_l1t0^n) \in E \land (|s_l0t0^n| - 1)_0 = p ).$
3. $\forall l > 0 \exists q < l \exists t \ (s_l, t_l) = (s_q0t, t_q1t)$

The tree $T$ generated by a frame $E$ is

$$T = \{ (s, t), (t, s) : s = \emptyset \lor \exists l \exists u(s, t) = (s_l0u, t_l1u) \}.$$
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Then $\mathcal{T} = [T]$ generated by a frame is a closed treeing with acyclic levels which is densely splitting.

We let $E_f$ be the equivalence relation induced by $\mathcal{T}$. 

---

**Frames**
Let $\mathcal{I} \supseteq \text{FIN}$ be a free Borel ideal on $\mathbb{N}$. 

Definition

$E_\mathcal{I}$ is the equivalence relation on $2^\mathbb{N}$ given by $x E_\mathcal{I} y$ iff $x \Delta y \in \mathcal{I}$.

Lemma

If $\mathcal{I}$ is a free ideal and $E_f$ is generated by a frame, then $E_\mathcal{I} \cap E_f = \langle E_\mathcal{I} \cap T \rangle$, where $T$ is the tree generated by the frame.

Corollary

For a free Borel ideal $\mathcal{I}$, $E_\mathcal{I} \cap E_f$ is a treeable Borel equivalence relation.
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Vertical invariance

Definition

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is \textit{vertically invariant} if, whenever $h : \mathbb{N} \to \mathbb{N}$ is an injection such that $(h(n))_0 = (n)_0$ for all $n$, then $A \in \mathcal{I}$ iff $h[A] \in \mathcal{I}$. 
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We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is *vertically invariant* if, whenever $h : \mathbb{N} \to \mathbb{N}$ is an injection such that $(h(n))_0 = (n)_0$ for all $n$, then $A \in \mathcal{I}$ iff $h[A] \in \mathcal{I}$.

**Lemma**

*If $T_1$ and $T_2$ are the trees from two frames and $\mathcal{I}$ is a vertically invariant ideal, then $E_\mathcal{I} \cap E_{T_1} \subseteq_c E_\mathcal{I} \cap E_{T_2}$.*
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If $T_1$ and $T_2$ are the trees from two frames and $\mathcal{I}$ is a vertically invariant ideal, then $E_{\mathcal{I}} \cap E_{T_1} \subseteq^c E_{\mathcal{I}} \cap E_{T_2}$.

Hence the following is well-defined (up to bi-embeddibility):

Definition

Let $E^*_{\mathcal{I}} = E_{\mathcal{I}} \cap E_f$.

Then $E^*_{\mathcal{I}}$ is sparsely treed by $\mathcal{T}_{\mathcal{I}} = E_{\mathcal{I}} \cap \mathcal{T} \subseteq \mathcal{T}$. 
Unboundedness of treeable BEQs

The key fact connecting frames to descriptive complexity is:

**Theorem**

If $\mathcal{I}$ is a vertically invariant free ideal which is $\Gamma$-complete for some Wadge class $\Gamma$, then $E^*_\mathcal{I}$ is in $\Gamma$ but is not in $\text{pot}(\bar{\Gamma})$, where $\bar{\Gamma}$ is the dual class of $\Gamma$. 

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There are vertically invariant free Borel ideals of arbitrarily high Wadge class.
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**Lemma**

There are vertically invariant free Borel ideals of arbitrarily high Wadge class.

In fact, we can find an $\omega_1$ length sequence of vertically invariant ideals $\mathcal{I}_\xi$ unbounded in the Borel Wadge classes so that $E^*_{\mathcal{I}_\xi} \preccurlyeq_c E^*_{\mathcal{I}_\eta}$ when $\xi < \eta$. 
This yields our first result about potential complexity:

**Theorem**

*The treeable Borel equivalence relations are unbounded in Wadge class, and hence in $\leq_B$.***
Unboundedness of treeable BEQs

This yields our first result about potential complexity:

**Theorem**

The treeable Borel equivalence relations are unbounded in Wadge class, and hence in $\leq_B$.

In particular, the class of sparsely treed Borel equivalence relations is unbounded in the Borel reducibility hierarchy. But we can ask whether it is cofinal among the treeable BEQs.
Definition

We say that an equivalence relation $E$ is \textit{hyperfinite-on-countable} if $E \upharpoonright B$ is hyperfinite for any Borel set $B$ such that $E \upharpoonright B$ is countable.

Lemma

$E$ is hyperfinite-on-countable iff whenever $F$ is a countable BEQ with $F \leq B$ then $F$ is hyperfinite.

We can extend the definition to analytic equivalence relations in an analogous fashion.

Note

If $E$ is hyperfinite-on-countable, then $E^\infty \not\leq B$. 

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We can extend the definition to analytic equivalence relations in an analogous fashion.

**Note**

If $E$ is hyperfinite-on-countable, then $E_{\infty} T \not\leq_B E$. 


Theorem

Let $T$ be a closed treeing with acyclic levels, and let $T_0 \subseteq T$ be an analytic treeing such that the induced equivalence relation $E_{T_0}$ is countable. Then $E_{T_0}$ is hyperfinite.
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Corollary

If $T$ is a sparse treeing, then $E_T$ is hyperfinite-on-countable.
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Let $T$ be a closed treeing with acyclic levels, and let $T_0 \subseteq T$ be an analytic treeing such that the induced equivalence relation $E_{T_0}$ is countable. Then $E_{T_0}$ is hyperfinite.

Corollary

If $T$ is a sparse treeing, then $E_T$ is hyperfinite-on-countable.

Corollary

$E^*_I$ is hyperfinite-on-countable for any free ideal $I$. In particular, $E^*_\infty T \not\leq_B E^*_I$. 
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The collection of sparsely treed Borel equivalence relations is not cofinal among the treeable Borel equivalence relations under $\leq_B$. 
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### Corollary

The collection of hyperfinite-on-countable equivalence relations is unbounded in the Borel equivalence relations under $\leq_B$. 
A dichotomy for embedding $\mathbb{E}_1$

**Definition**

For $n \in \mathbb{N}$, let $F_n$ denote the equivalence relation on $(2^\mathbb{N})^\mathbb{N}$ given by $x F_n y$ iff $\forall m \geq n \ x(m) = y(m)$. Then $\mathbb{E}_1 = \bigcup_n F_n$. 
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Then $E_1 = \bigcup_n F_n$.

Our main technical dichotomy is the following:

**Theorem**

Suppose that $X$ is a Polish space, $E$ is a treeable Borel equivalence relation on $X$, and $G$ is a Borel treeing of $E$. Then exactly one of the following holds:

1. The equivalence relation $E$ is essentially countable.
2. There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ for which there is a continuous homomorphism $\varphi : (2^\mathbb{N})^\mathbb{N} \rightarrow X$ from $(F_{n+1} \setminus F_n)_{n \in \mathbb{N}}$ to $(G(\leq f(n+1)) \setminus G(\leq f(n)))_{n \in \mathbb{N}}$. 
In many situations case (2) does not tell us anything. But there is an important special case:
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We say that a Borel equivalence relation is *subtreeable-with-$F_\sigma$-iterates* if it has a Borel treeing $T$ contained in an acyclic graphing $G$ so that $G$ has all iterates $F_\sigma$. 
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**Definition**

We say that a Borel equivalence relation is *subtreeable-with-$F_{\sigma}$-iterates* if it has a Borel treeing $T$ contained in an acyclic graphing $G$ so that $G$ has all iterates $F_{\sigma}$.

**Theorem**

Suppose that $X$ is a Polish space, $E$ is a Borel equivalence relation on $X$ which is essentially subtreeable-with-$F_{\sigma}$-iterates. Then exactly one of the following holds:

1. The equivalence relation $E$ is essentially countable.
2. There is a continuous embedding of $\mathbb{E}_1$ into $E$. 
We will not prove the main dichotomy here.
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Note that we can extend these results to analytic equivalence relations in a straightforward manner.
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Note that we can extend these results to analytic equivalence relations in a straightforward manner.

Sparsely treed equivalence relations are subtreeable-with-$F_\sigma$-iterates.
Using that sparsely treed equivalence relations are subtreeable-with-$F_\sigma$-iterates as well as hyperfinite-on-countable, we have:

**Corollary**

Let $\mathcal{T}$ be a sparse treeing, and suppose $E \leq_B E_\mathcal{T}$. Then exactly one of:

1. $E$ is essentially hyperfinite, or 
2. $\mathcal{E}_1 \sqsubseteq_c E$.
Using that sparsely treed equivalence relations are subtreeable-with-$F_\sigma$-iterates as well as hyperfinite-on-countable, we have:

**Corollary**

Let $T$ be a sparse treeing, and suppose $E \leq_B E_T$. Then exactly one of:

1. $E$ is essentially hyperfinite, or
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As this applies to $E_T$ itself, we see that a strong form of Hjorth’s conjecture holds for sparsely treed Borel equivalence relations.
Since $E_1$ is not reducible to any orbit equivalence relations:

**Corollary**

Let $E$ be sparsely treed, and $E^X_G$ a non-essentially hyperfinite orbit equivalence relation. Then $E^X_G \nleq_B E$. In particular, this applies when $E$ is any $E^*_I$. 
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**Corollary**

Let $E$ be sparsely treed, and $E^X_G$ a non-essentially hyperfinite orbit equivalence relation. Then $E^X_G \not\leq_B E$. In particular, this applies when $E$ is any $E^*_I$.

Finally, using that $E_1$ is sparsely treed and that any hypersmooth equivalence relation is reducible to $E_1$, we recover the Kechris-Louveau dichotomy as a special case:

**Corollary**

Let $E$ be hypersmooth. Then exactly one of:

1. $E \leq_B E_0$, or
2. $E_1 \sqsubseteq_c E$. 
No higher level dichotomies

Theorem
Let $\Gamma$ be a Borel Wadge class such that $\Gamma \supseteq \Sigma \sim_0^2$. Then there is no BEQ $\Gamma$ such that for any BEQ $E$ exactly one of:

1. $E \in \text{pot}(\Gamma)$, or
2. $E \Gamma \leq B E$.

Proof.
Suppose there were such an $E \Gamma$. We can find a vertically invariant $I$ with $E^*I/ \in \text{pot}(\Gamma)$, so $E \Gamma \leq B E^*I$. So either $E_1 \leq B E \Gamma$ or $E \Gamma$ is essentially hyperfinite. But the latter implies $E \Gamma \in \text{pot}(\Sigma \sim_0^2) \subseteq \text{pot}(\Gamma)$, which can not happen. So $E_1 \leq B E \Gamma$. 
Theorem

Let $\Gamma$ be a Borel Wadge class such that $\Gamma \supseteq \sum^0_2$. Then there is no BEQ $E_{\Gamma}$ such that for any BEQ $E$ exactly one of:

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A result of Harrington shows that there is an orbit equivalence relation with $E_{G}^{X} \notin \text{pot}(\Gamma)$. 

Hence there is no such $E_{\Gamma}$.
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So $E_\Gamma \leq_B E^X_G$.

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Hence there is no such $E_\Gamma$. \qed
**Unbounded incomparability**

**Theorem**

*Let $E$ be a Borel equivalence relation which is not essentially hyperfinite. Then the class*

\[ \mathcal{F} = \{ F : F \text{ is a BEQ with } F \perp_B E \} \]

*is unbounded in Wadge class (and hence in $\leq_B$).*
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- If $E$ is not reducible to any $E^*_I$, then $\mathcal{F}$ contains unbounded such.
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Proof.

- If $E$ is not reducible to any $E^*_I$, then $\mathcal{F}$ contains unbounded such.
- Otherwise $E \leq_B E^*_I$ for some $I$, so $E_1 \leq_B E$ since $E$ is not essentially hyperfinite.
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- If $E$ is not reducible to any $E^*_I$, then $\mathcal{F}$ contains unbounded such.
- Otherwise $E \leq_B E^*_I$ for some $I$, so $E_1 \leq_B E$ since $E$ is not essentially hyperfinite.
- Then $E$ is not reducible to any $E^X_G$, so $\mathcal{F}$ contains unbounded such. □
Questions

Question (Hjorth) If $E$ is a treeable Borel equivalence relation, is it the case that $E \subseteq^c E$ or $E$ is essentially countable?

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