Weakly pointed trees and partial injections

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Consider the following theorem:

**Theorem (Graf and Mauldin)**

Let $X$ and $Y$ be analytic spaces, $\lambda$ a probability measure on $X$, $\mu$ a probability measure on $Y$, and $R \subseteq X \times Y$ a Borel set such that $R_x$ is uncountable for $\lambda$-a.e. $x \in X$ and $R^y$ is uncountable for $\mu$-a.e. $y \in Y$. Then there exists a Borel set $A \subseteq X$ with $\lambda(A) = 1$, a Borel set $B \subseteq Y$ with $\mu(B) = 1$, and a Borel isomorphism $f$ from $A$ onto $B$ whose graph is contained in $R$.

This says that any sufficiently thick plane set admits an injective partial selector with domain of full measure.

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Topological version

We could ask whether the topological analogue is true, namely:

**Question**

*Suppose $X$ and $Y$ are Polish spaces, and $R \subseteq X \times Y$ is a Borel set such that $R_x$ and $R^y$ are uncountable for comeager sets of $x$ and $y$. Do there exist comeager Borel sets $A \subseteq X$ and $B \subseteq Y$ and a Baire-measurable isomorphism from $A$ onto $B$ whose graph is contained in $R$?*

This turns out to be false (shown independently by C. and Hjorth-Miller). We first consider a recursion-theoretic version of the above question. The techniques for that version can be generalized to settle the topological version.
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Weakly pointed trees

A tree $T$ is *pruned* if every node in $T$ has a proper extension in $T$, and $T$ is *perfect* if every node in $T$ has two incompatible proper extensions in $T$.

We introduce a coding of uniformly branching pruned trees by elements of $3^\omega$.

**Definition**

Let $x \in 3^\omega$. Then $x$ encodes the uniformly branching tree

$$T_x = \{ s \in 2^{<\omega} : \forall n < |s| (x(n) \neq 2 \Rightarrow s(n) = x(n)) \}.$$ 

Note that $T_x \equiv_T x$.

**Definition**

For $x \in 3^\omega$ and $y \in 2^\omega$ we say $y$ is consistent with $x$ if

$$\forall n \in \omega (x(n) \neq 2 \Rightarrow x(n) = y(n)).$$
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We use the same notation for finite strings, imposing requirements only for \( n \) less than the length of the shorter string. Note that \( y \) is consistent with \( x \) if and only if \( y \in [T_x] \).

Recall that a pruned tree \( T \) is pointed if \( T \leq_T y \) for every branch \( y \in [T] \). We introduce the following generalization:

**Definition**

A tree \( T \) is **weakly pointed** if there is some branch \( y \in [T] \) such that \( T \leq_T y \).

For a uniformly branching tree encoded by \( x \in 3^\omega \), this is equivalent to saying that there is a \( y \) consistent with \( x \) such that \( x \leq_T y \). Note that every pointed tree is weakly pointed.
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**n-genericity**

**Definition**

An element $x \in 3^\omega$ is $n$-generic if for every $\Sigma^0_n$ set $A \subseteq 3^{<\omega}$ there is a string $\sigma \sqsubseteq x$ such that either:

(a) $\sigma \in A$, or

(b) $\forall \tau \sqsupseteq \sigma (\tau \notin A)$.

**Definition**

We say that $x$ is weakly $n$-generic if for every dense $\Sigma^0_n$ set $A \subseteq 3^{<\omega}$ there is $\sigma \sqsubseteq x$ such that $\sigma \in A$.

Note that $n$-generic implies weakly $n$-generic, and weakly $(n + 1)$-generic implies $n$-generic.
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We now characterize the amount of genericity necessary to rule out pointedness and weak pointedness. We begin with an easy observation:

**Definition**

Let $B_x = \{ n \in \omega : x(n) = 2 \}$ be the branching levels of $T_x$. Note that $B_x \leq_T x$, and for $y \in [T_x]$ we have $x \leq_T y$ if and only if $B_x \leq_T y$.

**Lemma**

*If $x \in 3^\omega$ is weakly 1-generic then $B_x$ is infinite, i.e. $T_x$ is perfect.*

**Proof.**

For $n \in \omega$ let $A_n = \{ \sigma : \sigma$ contains at least $n$ 2’s\}. Then $A_n$ is a dense r.e. set, and any $x$ meeting all of them must contain infinitely many 2’s.
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Characterizing genericity for pointed trees

For pointed trees, 1-genericity suffices:

Theorem

If $x \in 3^\omega$ is 1-generic, then $T_x$ is not pointed.

Proof.

We see that the leftmost branch of $T_x$ can not compute $T_x$. For $e \in \omega$ define the r.e. set

$$A_e = \{ \sigma \in 3^{<\omega} : \exists n < |\sigma| (\sigma(n) \neq 2 \land \{e\}^{\tilde{\sigma}}(n) \downarrow = 1) \}$$

where $\tilde{\sigma}(n) = \sigma(n)$ if $\sigma(n) \neq 2$ and $\tilde{\sigma}(n) = 0$ if $\sigma(n) = 2$.

Let $y = \tilde{x} \in T_x$ be the leftmost branch.

If there is $\sigma \sqsubseteq x$ with $\sigma \in A_e$ then $\{e\}^y \neq B_x$.

If there is $\sigma \sqsubseteq x$ such that $\forall \tau \sqsupseteq x (\tau \notin A_e)$, then $\{e\}^y$ is either not total or it is finite, so $\{e\}^y \neq B_x$ since $B_x$ must be infinite for $x$ 1-generic. 

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The characterization for weakly pointedness requires more genericity. Weak 2-genericity is necessary:

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There is a 1-generic $x \in 3^\omega$ such that $T_x$ is weakly pointed.

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- Build sequences $\langle \sigma_i \rangle_{i \in \omega}$ with $\sigma_i \in 3^{<\omega}$ and $\langle s_i \rangle_{i \in \omega}$ with $s_i \in 2^{<\omega}$.
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Proof (continued).

- First, check whether there is a $\sigma \sqsupseteq \sigma_0 \bowtie \cdots \bowtie \sigma_n$ with $\sigma \in R_n$, where $R_n = \text{ran}(\varphi_n)$ is the $n$-th r.e. subset of $3^{\omega}$.
- If so, let $\langle i, j \rangle$ be the least pair such that $\varphi_{n,i}(j) \downarrow = \sigma \sqsupseteq \sigma_0 \bowtie \cdots \bowtie \sigma_n$.
- We let $\sigma_{n+1} = \sigma \bowtie \langle 2 \rangle$, and we let $s_{n+1} = \langle 1 \rangle \bowtie \tilde{\sigma}$, where $\tilde{\sigma}(n) = \sigma(n)$ if $\sigma(n) \neq 2$ and $\tilde{\sigma}(n) = 0$ if $\sigma(n) = 2$.
- Otherwise, if there is no such $\sigma$, we let $\sigma_{n+1} = \langle 2 \rangle$ and we let $s_{n+1} = \langle 0 \rangle$.
- Note that $s_0 \bowtie \cdots \bowtie s_n$ is always one digit shorter than $\sigma_0 \bowtie \cdots \bowtie \sigma_n$. 
Characterizing genericity for weakly pointed trees

Proof (continued).

- First, check whether there is a $\sigma \sqsupseteq \sigma_0 \sqsupseteq \cdots \sqsupseteq \sigma_n$ with $\sigma \in R_n$, where $R_n = \text{ran}(\varphi_n)$ is the $n$-th r.e. subset of $3^{<\omega}$.

- If so, let $\langle i, j \rangle$ be the least pair such that $\varphi_{n,i}(j) \downarrow = \sigma \sqsupseteq \sigma_0 \sqsupseteq \cdots \sqsupseteq \sigma_n$.

- We let $\sigma_{n+1} = \sigma \sqcup \langle 2 \rangle$, and we let $s_{n+1} = \langle 1 \rangle \sqcup \tilde{\sigma}$, where $	ilde{\sigma}(n) = \sigma(n)$ if $\sigma(n) \neq 2$ and $\tilde{\sigma}(n) = 0$ if $\sigma(n) = 2$.

- Otherwise, if there is no such $\sigma$, we let $\sigma_{n+1} = \langle 2 \rangle$ and we let $s_{n+1} = \langle 0 \rangle$.

- Note that $s_0 \sqsupseteq \cdots \sqsupseteq s_n$ is always one digit shorter than $\sigma_0 \sqsupseteq \cdots \sqsupseteq \sigma_n$. 
Proof (continued).

- First, check whether there is a $\sigma \sqsupseteq \sigma_0 \bowtie \cdots \bowtie \sigma_n$ with $\sigma \in \mathbb{R}_n$, where $\mathbb{R}_n = \text{ran}(\varphi_n)$ is the $n$-th r.e. subset of $3^{<\omega}$.

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First, check whether there is a $\sigma \sqsupseteq \sigma_0 \triangleleft \cdots \triangleleft \sigma_n$ with $\sigma \in R_n$, where $R_n = \text{ran}(\varphi_n)$ is the $n$-th r.e. subset of $3^{<\omega}$.

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Note that $s_0 \triangleleft \cdots \triangleleft s_n$ is always one digit shorter than $\sigma_0 \triangleleft \cdots \triangleleft \sigma_n$. 
To check that $x \leq_T y$, we reconstruct the sequence $\langle \sigma_i \rangle_{i \in \omega}$ recursively in $y$.

Let $\sigma_0 = \langle 2 \rangle$ and $i_0 = y(0)$. Given $\sigma_n$ and $i_n$, we find $\sigma_{n+1}$ and $i_{n+1}$ as follows.

If $i_n = 1$ then we know that there was a $\sigma \sqsupseteq \sigma_0 \cdot \cdots \cdot \sigma_n$ with $\sigma \in R_n$, so we can calculate the least pair such that $\varphi_{n,i}(j) \downarrow = \sigma \sqsupseteq \sigma_0 \cdot \cdots \cdot \sigma_n$ and let $\sigma_{n+1} = \sigma \cdot \langle 2 \rangle$.

Otherwise, if $i_n = 0$ there was no such $\sigma$ and we let $\sigma_{n+1} = \langle 2 \rangle$. We then let $l = |\sigma_0 \cdot \cdots \cdot \sigma_{n+1}|$ and let $i_{n+1} = y(l - 1)$.

Note that $x <_T 0'$. By adding a second 2 after each stage, we can construct $x$ so that $T_x$ contains a pointed subtree. We do not know whether we can find a branch $y$ such that $x \equiv_T y$. 

Proof (continued).

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Note that $x \prec_T 0'$. By adding a second 2 after each stage, we can construct $x$ so that $T_x$ contains a pointed subtree. We do not know whether we can find a branch $y$ such that $x \equiv_T y$. 
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Characterizing genericity for weakly pointed trees

We see that weak 2-genericity is sufficient:

**Theorem**

If \( x \in 3^\omega \) is weakly 2-generic, then \( T_x \) is not weakly pointed.

**Proof.**

- Let \( x \in 3^\omega \) be weakly 2-generic, and let \( y \in [T_x] \).
- It suffices to show for each \( e \in \omega \) that \( \{ e \}^y \neq B_x \).
- Define the \( \Sigma_2^0 \) set

\[
A_e = \{ \sigma \in 3^{<\omega} : \forall s \in 2^{\vert \sigma \vert} \text{ consistent with } \sigma \\
[(\exists n < \vert \sigma \vert (\sigma(n) \neq 2 \land \{ e \}^s(n) \downarrow = 1)) \lor \\
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\]
Proof (continued).

- Suppose there is $\sigma \subseteq x$ such that $\sigma \in A_e$.
- Let $s = y \upharpoonright |\sigma|$, so $s$ is consistent with $\sigma$.
- If there is $n < |\sigma|$ such that $\sigma(n) = x(n) \neq 2$ and $
\{e\}^s(n) = \{e\}^y(n) \downarrow = 1$, then immediately $\{e\}^y \neq B_x$.
- Otherwise, $\neg \exists t \supseteq s \exists n(|s| \leq n < |t| \land \{e\}^t(n) \downarrow = 1)$, so $\neg \exists n \geq |\sigma|((\{e\}^y(n) \downarrow = 1)$.
- Hence if $\{e\}^y$ is total then it must compute a finite set, and since $B_x$ is infinite we again have $\{e\}^y \neq B_x$.
- It remains to check that $A_e$ is dense.
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- If $A_e$ is not dense, there is $\sigma$ such that $\forall \tau \equiv \sigma(\tau \notin A_e)$. We show this yields a contradiction.

- We have:

$$\forall \tau \equiv \sigma \exists s \in 2^{||\tau||} \text{ consistent with } \tau$$

$$(\neg \exists n < ||\tau|| (\tau(n) \neq 2 \land \{e\}^{s}(n) \downarrow = 1)) \land (\exists t \equiv s \exists n (|s| \leq n < |t| \land \{e\}^{t}(n) \downarrow = 1)).$$

- Specializing this to $\tau = \sigma \triangle t$ where $t \in 2^{<\omega}$ we then have:

$$\forall t \in 2^{<\omega} \exists s \in 2^{||\sigma||} [s \text{ is consistent with } \sigma \land$$

$$\neg \exists n (|\sigma| \leq n < |\sigma \triangle t| \land \{e\}^{s\triangle t}(n) \downarrow = 1) \land$$

$$\exists t' \equiv t \exists n (|\sigma \triangle t| \leq n < |\sigma \triangle t'| \land \{e\}^{s\triangle t'}(n) \downarrow = 1)]$$
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\[
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\[
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\exists t' \sqsupseteq t \exists n(|\sigma \triangleleft t| \leq n < |\sigma \triangleleft t'| \land \{e\}^{s \triangleleft t'}(n) \downarrow = 1)]
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Characterizing genericity for weakly pointed trees

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Proof (continued).

- We now form sequences $\langle t_i \rangle_{i \in \omega}$ and $\langle s_i \rangle_{i \in \omega}$ of elements of $2^{<\omega}$.
- For each $i$ we want $t_i \sqsubseteq t_{i+1}$ and $s_i \in 2^{\sigma}$ consistent with $\sigma$.
- Let $t_0 = \langle \rangle$.
- Given $t_i$, by the previous conclusion we can choose $t_{i+1} \sqsupseteq t_i$ and $s_i \in 2^{\sigma}$ consistent with $\sigma$ such that
  \[
  \neg \exists n(|\sigma| \leq n < |\sigma \cup t_i| \land \{e\}_{s_i}^{t_i}(n) \downarrow = 1)
  \]
  and
  \[
  \exists n(|\sigma \cup t_i| \leq n < |\sigma \cup t_{i+1}| \land \{e\}_{s_i}^{t_{i+1}}(n) \downarrow = 1).
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and

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Proof (continued).

There are only finitely many choices for $s_i$, so there is a fixed $s$ consistent with $\sigma$ and an infinite sequence $n_0 < n_1 < \cdots$ such that $s_{n_i} = s$ for all $i$.

Let $z = s \triangleleft \bigcup_i t_i$. Then for each $i$ we have

$$-\exists n(|\sigma| \leq n < |\sigma \triangleleft t_{n_i}| \land \{e\}^{s \triangleleft t_{n_i}}(n) \downarrow = 1)$$

So $-\exists n \geq |\sigma|((\{e\}^z(n) \downarrow = 1)$, but for each $i$ we have

$$\exists n(|\sigma \triangleleft t_{n_i}| \leq n < |\sigma \triangleleft t_{n_i+1}| \land \{e\}^{s \triangleleft t_{n_i+1}}(n) \downarrow = 1)$$

So $\exists^{\infty} n((\{e\}^z(n) \downarrow = 1)$, a contradiction. \qed
Proof (continued).

- There are only finitely many choices for $s_i$, so there is a fixed $s$ consistent with $\sigma$ and an infinite sequence $n_0 < n_1 < \cdots$ such that $s_{n_i} = s$ for all $i$.

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Proof (continued).

- There are only finitely many choices for $s_i$, so there is a fixed $s$ consistent with $\sigma$ and an infinite sequence $n_0 < n_1 < \cdots$ such that $s_{n_i} = s$ for all $i$.

- Let $z = s \upharpoonright \bigcup_i t_i$. Then for each $i$ we have

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Since the set of weakly 2-generics is comeager, we have:

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The following set is meager:

\[ \{ x \in 3^\omega : \exists y \in 2^\omega (y \geq_T x \land y \text{ is consistent with } x) \} \]

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There is no Baire-measurable partial injection \( F : 3^\omega \rightarrow 2^\omega \) with comeager domain such that

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- The theorem may be proved using a relativized version of the preceding results applied to $\xi$-generics.
- If there were such an $F$, then there would be a countable ordinal $\xi$ and a comeager set of $x$ for which there is a $y$ consistent with $x$ with $x \leq_T y^{(\xi)}$ (namely $y = F(x)$).
- We can also give a purely topological proof.
- Suppose there were such an $F$. We can find a comeager set $C$ such that $F \upharpoonright C$ is continuous (and injective).
- By removing a meager perfect set from $C$, we can in fact assume that we have a comeager set $C$ and a Borel bijection $f : 3^\omega \to 2^\omega$ such that $f(x)$ is consistent with $x$ for all $x \in C$. 

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  \[ A = \{(n, y) : f^{-1}(y)(n) = 2\}. \]

- Then $A$ is Borel, and for each $x \in C$ we have $n \in B_x$ if and only if $(n, f(x)) \in A$.

- We then show that the analytic set
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*If* $x$ *is 1-random, can* $T_x$ *be weakly pointed?*

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*Is there a 1-generic* $x$ *with a branch* $y \in [T_x]$ *with* $x \equiv_T y$?

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*Must a weakly pointed tree contain a pointed subtree?*

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- J.D. Clemens, Weakly pointed trees and partial injections, *preprint*.