# ADVANCED TOPOLOGY SEMINAR: TOPOLOGICAL METHODS IN GROUP THEORY

JOHANNES EBERT, ULRICH PENNIG

- Time/location: Wed, 14 s.t. until 15.30, seminar room SR 1D.

**Talk 0.** (On demand: A crash course on group homology, Johannes Ebert)
$K(\pi;1)$-complexes. Existence, uniqueness, functoriality, representing property. Group homology as homology of $K(\pi;1)$-complex. Algebraic version: Homological algebra, resolutions of modules. Connection between both approaches.

## Bass-Serre Theory

**Talk 1.** (Grushko's Theorem, Georg Frenck)
Grushko's Theorem: $f : F \to G_0 * G_1$ an epimorphism from a finitely generated free group. Then $F = F_0 * F_1$ such that $f(F_i) = G_i$. Literature: [5], §2. Kurosh subgroup theorem; [5], Thm 3.1. Uniqueness of free product decompositions [5], Thm 3.5.

**Talk 2.** (Bass-Serre Theory I, Nils Leder)
Graphs of groups and their realization. [7], §1.B. Amalgamated products and HNN-extensions as special cases. The situation of [5], p.138f is a geometric motivation. A special case of Thm 1.B.11 loc. cit. proves Thm II.7.3 in [6], the *Whitehead asphericity theorem* and the Lemma 7.4. The same material is covered in [5], p. 154–157. Proof of the "subgroup theorem", [5], Thm 3.15.

**Talk 3.** (Bass-Serre theory II, Arne Peeters)
Groups acting on graphs; [5], §4. See also [12]. A classical example is provided by the group $\mathrm{SL}_2(\mathbb{Z})$, see [12] or [6], II.7.

## Ends of a group and Stallings' Theorem

**Talk 4.** (Ends of a group, Robert Schielek)
Ends of a simplicial complex. Ends of a group versus homology. The number of ends of a finitely generated group is $0, 1, 2$ or $\infty$. A torsionfree group with 2 ends is infinite cyclic. [5], §5. An alternative source for the same material is [4].

**Talk 5.** (Stallings' Theorem I, Lukas Wolzendorff)
The Theorem is [5], Thm 6.1. Explain how the formulation in [10], Thm 0.1, is equivalent. First application: Theorem 6.2 [5]. Then begin the proof, starting with lemma 6.3 (which is the easy direction). The hard part of the proof is to find a tree on which the group $G$ acts, and finding the splitting by Bass-Serre theory; most of it will be covered in the next talk.

**Talk 6.** (Proof of Stallings' Theorem, Sonja Hannibal)
Continuation of the proof, as in [5]. It goes without saying that you should communicate with the speaker of the previous talk.

**Talk 7.** (Background in 3-dimensional Topology, Raphael Reinauer)
Proof of the results in [11], §5.C. For more details see the relevant parts of [8] and [9].

**Talk 8.** (The sphere theorem, Svenja Knopf)
The sphere theorem says that if $M$ is an oriented compact 3–manifold with $\pi_2(M) \neq 0$, then there is an embedding $S^2 \to M$ that is nontrivial in homotopy. A proof of it can be given using Stallings' Theorem, see [11], §5.C and §2. See also [5], p. 180f.

## Finiteness conditions and dimension

**Talk 9.** (A survey on finiteness conditions, Sarah Humberg)
Finiteness conditions for group, homological dimension and perhaps Euler characteristic of groups. See [1], [4] for what is needed. Proofs can be found in [6].

**Talk 10.** (Homological dimension versus geometric dimension, Christoph Winges)
More or less by definition, $\mathrm{coh}-\dim(G) \leq \mathrm{geo}-\dim(G)$. Here we see that $\mathrm{geo}-\dim(G) \leq \mathrm{coh}-\dim(G)$, with the exception that there could exist a group with $\mathrm{geo}-\dim(G) = 3$ and $\mathrm{coh}-\dim(G) = 2$. The case $d = 0$ is easy ([6], p. 188, exercise 1). The case $d = 1$ is due to Stallings (for finitely generated groups and this uses his Theorem), see [10], §6. Swan extended the result to the nonfinitely generated case, but there won't be time to discuss this. The case $d \geq 3$ is due to Eilenberg and Ganea [3], see [6], VIII.7 for proofs.

**Talk 11.** (Euler characteristics of groups, Adam Mole)
For groups $G$ that are virtually geometrically finite, there is a notion of an Euler characteristic. Reference. [6]. Some examples. An easy and intructive example is again given by the group $\mathrm{SL}_2(\mathbb{Z})$. Solve the following exercises: $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$. The abelianization is $\mathbb{Z}/12$; the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/12$ is a homology isomorphism and its kernel is $\mathbb{Z} * \mathbb{Z}$.

## The Bestvina-Brady theorem

**Talk 12.** (Discrete Morse Theory and RAAGs, Urs Fuchs)
This talk has two independent parts, that will be combined later. 1. Morse Theory on cell complexes; [4], p. 187–190. Also consult the relevant sections of [1]. Then introduce flag complexes and the associated right angled Artin groups, [1], p. 454. Discuss some easy examples of the groups $G_L$, $H_L$ ($L = S^0$ is already interesting).

**Talk 13.** ($CAT(0)$-geometry, Lukas Buggisch)
The relevant parts of $CAT(0)$-geometry are described in [4], p. 191–192. For proofs, see [2]. Try to give as much credibility to Theorems 8.3.7 and 8.3.8 of [4].

**Talk 14.** (The Bestvina-Brady Theorem, Ulrich Pennig)
State and prove the Bestvina-Brady Theorem (Thm 8.3.12 of [4]). A curious consequence is Thm 8.7 of [1]; stating that either the Eilenberg-Ganea conjecture ($\mathrm{coh}-\dim(G) = 2 \Rightarrow \mathrm{geo}-\dim(G) = 2$) or the Whitehead conjecture (a subcomplex of an aspherical 2-complex is aspherical) is false.

## References

[1] Bestvina, Brady: *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), 445–470.
[2] Bridson, Haefliger: *Metric space of nonpositive curvature*, Springer-Verlag, Berlin, 1999
[3] S. Eilenberg, T. Ganea: *On the Lusternik-Schnirelmann category of abstract groups.* Ann. of Math. 65 (1957), 517–518.
[4] R. Geoghegan: *Topological methods in group theory*, Graduate Texts in Mathematics, 243. Springer, New York, 2008
[5] C. T. C. Wall: *Topological methods in group theory*, Homological group theory (Proc. Sympos., Durham, 1977), pp. 137–203, London Math. Soc. Lecture Note Ser., 36, Cambridge Univ. Press, Cambridge-New York, 1979.
[6] K. Brown: *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994
[7] A. Hatcher: *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
[8] J. Hempel: 3-*Manifolds* (AMS Chelsea Publishing).
[9] J. Lurie: *Topics in geometric topology.* Lecture notes, http://www.math.harvard.edu/ lurie/937.html
[10] J. Stallings: *On torsion-free groups with infinitely many ends*, Ann. Math. 88 (1968), p. 312–334.
[11] J. Stallings: *Group theory and Three-dimensional manifolds*, Yale mathematical monographs 4.
[12] J.-P. Serre: *Trees*, Springer-Verlag, Berlin-New York, 1980.

Mathematisches Institut, Einsteinstrasse 62, 48149 Münster
*E-mail address*: johannes.ebert@uni-muenster.de, u.pennig@uni-muenster.de