# ADVANCED TOPOLOGY SEMINAR: HOMOLOGICAL STABILITY THEOREMS

JOHANNES EBERT, MICHAEL WEISS

- Time/location: Wed, 16-18, room 503 (5th floor)

Suppose that $G_1 \to G_2 \to \ldots$ is a sequence of groups and homomorphisms. We say that the sequence satisfies *homological stability* if the induced maps $H_q(BG_n) \to H_q(BG_{n+1})$ are isomorphisms for $q \leq f(n)$, where $f(n)$ tends to $\infty$ with increasing $n$. In practise, the condition will be $q \leq n/a + b$ with $a, b \in \mathbb{N}$. Slightly more generally, we can ask the same for a sequence $X_n$ of spaces.

Such theorems have been established for an astonishing variety of groups or spaces, including symmetric groups, linear groups, orthogonal groups, diffeomorphism groups of highly-connected high-dimensional manifolds and mapping class groups of oriented surfaces, to name only these results that we will discuss. Besides these, there are results for automorphism groups of free groups ([16], [17]), mapping class groups of 3-manifolds ([15]), mapping class groups of nonorientable surfaces [35] and configuration spaces of manifolds [27]. For many of these groups, there are inumerably many different extended version to nonconstant coefficients and various decorations.

In many of the above cases, the stable homology is known ([23], [26], [3], [8], [11], [22], [9]), but we leave these questions aside as they require entirely different methods.

All the proofs follow a blueprint that was introduced by Quillen in the context of linear groups [24]. The crucial point is to find a highly connected simplicial complex $C_n$ with an action of $G_n$, such that the action of $G_n$ on the $p$-simplices is transitive and the stabilizers are smaller groups $G_{n-p-1}$. A careful study of the spectral sequence of the action proves stability. We discuss four sequences of groups (or spaces) in this seminar, in increasing degree of complexity: symmetric groups, general linear groups, orthogonal groups, diffeomorphism groups of highly connected high-dimensional manifolds, mapping class groups of surfaces.

As a warm-up, we discuss symmetric groups, thereby introducing the spectral sequence argument (talk 1). The next talk 2 introduces the class of simplicial complexes we will mainly talk about.

The following talks, 3, 4, 5 establish the connectivity results for the complexes used for general linear and orthogonal groups (among these complexes is the *Tits building* of a vector space). The methods are quite different from each other and the third one depends on both of the previous ones. The following talk 6 harvests the results and proved stability for linear and orthogonal groups.

So far, we saw mainly complexes and groups of an algebraic nature, but from talk 7 on, we move to diffeomorphism groups. The first case is the recent result by Galatius and Randal-Williams on diffeomorphisms of highly-connected high-dimensional ($dim \geq 6$) manifolds. This makes crucial use of the connectivity result of talk 5.

The last part of the seminar is devoted to the more classical case of mapping class groups. This is originally due to Harer [13] and was later improved by Ivanov [18] [19], Boldsen [2] and Randal-Williams [28] and others. Here we follow the truely excellent exposition [34].

## 1. Techniques

**Talk 1.** (Warm-up: homological stability for symmetric groups; Christoph Winges)

Homological stability for symmetric groups was established by Nakaoka [23] as a corollary of his actual computation of the homology. A different proof was given by Segal [30]. There are other proofs that follow the blueprint and this talk should discuss these proofs. There is an algebraic version [20] and a geometric version [27], §5 . The arguments are really isomorphic. Nevertheless, present both versions to a certain extent, as this sheds light on the machinery.

The key is the high connectivity of the semisimplicial set denoted $C_*(m)$ in [20] or $F(C)$ in [27]. There are dozens proofs in the literature and later in the seminar, we will see one of of them. The focus of this talk should lie on the way the spectral sequence is built and used, so if there is a time problem, sacrifice the proof of the connectivity theorem.

**Talk 2.** (Cohen-Maccaulay-complexes, Svenja Knopf)

Most of the simplicial complexes we will see are Cohen-Maccaulay complex and this talk should introduce them. Begin with a general discussion of simplicial complexes versus Δ-sets versus partially ordered sets, as all three notions and their relationship is used later on. Then introduce the Cohen-Maccaulay-condition. Source: [25] §1,8,9. Then present the "poset fibre theorem". The version needed in talk 5 is Proposition 1.2 of [4], a slight strengthening of Theorem 9.1 [25]. The talk should conclude by introducing the main characters of the following three talks: the posets (complexes) of [31], Thm. 2.6, the Tits building of a vector space [25], page 118, the complex of Thm 1.6 of [32], [4], Theorems 2.9 and 3.2. State the connectivity result in each case.

## 2. Linear and orthogonal groups

**Talk 3.** (Van der Kallen-Maazen's connectivity theorem; Fabian Hebestreit)

This talk should discuss the proof of Theorem 2.6 of [31] in full detail. This is used to establish stability for general linear groups and also to deduce Charneys theorem in the next two talks. Please discuss the ring- and module theoretic basics as well, but keep in mind that we do loose much if the ring is restricted to be a (commutative) field or principal ideal domain. Some of the technical results of loc. cit. have been improved by [5], §1 (quoted as Lemmata 1.4, 1.5, 1.6 in [4]). As they are used later in talk 5 and as Lemma 1.5 implies Theorem 1 of [20] that has been used in talk 1, you should try to combine both sources [31] and [5]. The original source is Maazen's unpublished thesis [21].

**Talk 4.** (Charney's connectivity theorem: preparations; Felix Springer)

Charney's connectivity theorem ([4], Theorem 3.2) will be used for stability of orthogonal groups and also for diffeomorphism groups of high-dimensional manifolds. This talk should present: The Solomon-Tits theorem, see [25], bottom of page 118 and built on that Theorem 1.6 of [32]. Question: can you prove Theorem 1 of [20] using the method offered by Quillen? Then cover [4], §2 prove Theorem 2.9 and do not forget to say something about quadratic forms on modules over general rings (again, fields and principal ideal domains would suffice). The speaker should communicate with the speaker of the next talk to balance the material.

**Talk 5.** (Charney's connectivity theorem: conclusion; Daniel Kasprowski)

Theorem 3.2 and Corollary 3.3 of [4]. This is obtained by a combination of Theorem 2.9 loc. cit (talk 4) and the main result of talk 3, the poset fibre theorem (talk 2) and a new technique. (Maybe too naive) question: can you prove Theorem 3.2 loc. cit. using the technique of [15], Theorem 3.8?

**Talk 6.** (Spectral sequence argument for linear and orthogonal groups).

The complexes studied in the previous talks are used to prove stability for linear and orthogonal groups, [4], §4 and [31], §3,4,5. Remark: you might want to skip the parts concerned with nonconstant coefficients, at least partially. Moreover, the spectral sequence arguments in both cases are rather similar. You find an axiomatization of them in [15], §5.

## 3. Diffeomorphism groups of highly connected high-dimensional manifolds

**Talk 7.** (Statement of the result and spectral sequence argument; Ulrich Pennig)

State the result [10], Theorem 1.2. Then define the complex that is used (Definition 4.1) and state the connectivity result (Theorem 4.6) which will be proven in the next talk. Then present the spectral sequence argument (which is rather straightforward in this case), §5. You also should give some background on the classification of highly connected high-dimensional manifolds.

**Talk 8.** (Connectivity result of Galatius, Randal-Williams; Urs Fuchs)
Proof of the connectivity result Theorem 4.6 of [10]. The proof is in sections §1,2,3,4 (most of §3 has been covered in talk 5). The proof works from Charneys algebraic result to a complex of embedded spheres, via the Whitney trick and the simplicial complex techniques of §2.1, to prove connectivity of a discretized version. To get at the version of interest (Theorem 4.6) one uses microfibrations and a new technique. Even if you like to avoid the technicalities of Prop. 2.5 and 2.7 of [10], describe the method precisely. The same method can be used in talk 13.

## 4. Mapping class groups

**Talk 9.** (Theorems of Earle-Eells and Gramain; Michael Weiss)
Give the definition of mapping class groups of surfaces, as well as a short survey that includes Dehn twists and the classification of smooth surfaces; this is an essential ingredient for the stability theorem. The rest of the proof is devoted to another fundamental ingredient: the theorem by Earle-Eells [6] that asserts that the components of the diffeomorphism group of a surface are contractible. Give a short description of how this fits into Teichmüller theory; a full proof belongs to the realm of geometric analysis and is well beyond the scope of this seminar. There is a topological proof of this fact by Gramain [12]. Show how the Earle-Eells theorem is equivalent to Théorème 5 of [12]. If time permits, say something on the topological proof of that fact.

**Talk 10.** (Statement of the Harer stability theorem)
Statement of the stability theorem, see [34], Theorems 1.1 and 1.2. Introduce the simplicial complexes that are used for the proof (Definition 2.1) and state the connectivity result Proposition 2.8, whose proof covers the next two talks. Then go through the proofs of the other ingredients in §2 in detail.

**Talk 11.** (Connectivity argument I)
Explain the different steps of the connectivity argument (first page of §4, [34]). Then prove Theorem 4.1 of [34]. You might want to consult the original source [14].

**Talk 12.** (Connectivity argument II)
Theorems 4.3, 4.8 and 4.9 of [34] (the last one is what is needed for the stability theorem). Question: Can you unify some parts of these proofs using Theorem 6.2 of [1] and find a proof of that using the arguments in [34]? Consult also [7].

**Talk 13.** (Spectral sequence argument and closing the last hole)
[34], §3 and §5. The spectral sequence argument for Theorem 1.1 is a bit more sophisticated than that of talks 1, 6 and 7 but should pose no problems by now. Finally, give the proof of Theorem 1.2 (closing the last hole). This proof is due to [28], §11 and it can be simplified using the technique of [10], Theorem 4.6, see also [7]. You might want to read the original proof, e.g. [13], [19]: it is much more involved.

## References


[1] K. Baclawski: *Cohen-Macaulay ordered sets.* J. Algebra 63 (1980), no. 1, 226–258.
[2] S. Boldsen: *Improved homological stability for the mapping class group with integral or twisted coefficients*, Math. Z. 270 (2012), no. 1—, 297–329.
[3] A. Borel: *Stable real cohomology of arithmetic groups.* Ann. Sci. École Norm. Sup. (4) 7 (1974), 235–272 (1975).
[4] R. Charney: *A generalization of a theorem of Vogtmann.* J. Pure Appl. Algebra 44 (1987), no. 1–3, 107–125.
[5] R. Charney: *On the problem of homology stability for congruence subgroups.* Communications in Algebra 12(17) 2081–2123 (1984).
[6] Earle, Eells: *A fibre bundle description of Teichmüller theory.* J. Differential Geometry 3 1969 19–43.
[7] J. Ebert: *Private communication.*
[8] S. Galatius: *Stable homology of automorphism groups of free groups.* Ann. of Math. (2) 173 (2011), no. 2, 705–768.
[9] S. Galatius, I. Madsen, U. Tillmann, M. Weiss: *The homotopy type of the cobordism category.* Acta Math. 202 (2009), no. 2, 195–239.
[10] S. Galatius, O. Randal-Williams: *Homological stability for moduli spaces of high dimensional manifolds* arXiv:1203.6830.

[11] S. Galatius, O. Randal-Williams: *Stable moduli space of high dimensional manifolds.* arXiv:1201.3527

[12] A. Gramain: *Le type d'homotopie du groupe des difféomorphismes d'une surface compacte*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 53–66.

[13] J. Harer: *Stability of the homology of the mapping class groups of orientable surfaces.* Ann. of Math. (2) 121 (1985), no. 2, 215–249

[14] A. Hatcher: *On triangulations of surfaces*, Topology Appl. 40 (1991), no. 2, 189–194. Updated version: http://www.math.cornell.edu/ hatcher/Papers/TriangSurf.pdf.

[15] A. Hatcher, N. Wahl: *Stabilization for mapping class groups of 3-manifolds* Duke Math. J. 155 (2010), 205-269 and Allen Hatchers homepage

[16] A. Hatcher, N. Wahl: *Stabilization for the automorphisms of free groups with boundaries* Geometry and Topology 9 (2005), 1295–1336 and erratum Allen Hatchers homepage.

[17] A. Hatcher, K. Vogtmann: *Homology stability for outer automorphism groups of free groups* Alg. & Geom. Topology 4 (2004), 1253–1272 and erratum Allen Hatchers homepage.

[18] N. Ivanov: *On stabilization of the homology of Teichmüller modular groups*, Leningrad Math. J. Vol 1 (1990), 675–691.

[19] N. Ivanov: *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients.* Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 149–194, Contemp. Math., 150, 1993.

[20] M. Kerz: *The complex of words and Nakaoka stability.* Homology Homotopy Appl. 7 (2005), no. 1, 77–85.

[21] H. Maazen: *Homology stability for the general linear group.* Unpublished PhD thesis, Utrecht, 1979, available at http://www.staff.science.uu.nl/ kalle101/maazen1979.pdf

[22] I. Madsen, M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture.* Ann. of Math. (2) 165 (2007), no. 3, 843–941.

[23] Nakaoka: *Homology of the infinite symmetric group.* Ann. of Math. (2) 73 1961 229–257.

[24] D. Quillen: *Finite generation of the groups $K_i$ of rings of algebraic integers.* Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 179–198. Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973.

[25] D. Quillen: *Homotopy properties of the poset of nontrivial p-subgroups of a group.* Adv. in Math. 28 (1978), no. 2, 101–128.

[26] D. Quillen: *On the cohomology and K-theory of the general linear groups over a finite field.* Ann. of Math. (2) 96 (1972), 552–586.

[27] O. Randal-Williams: *Homological stability for unordered configuration spaces*, arXiv:1105.5257.

[28] O. Randal-Williams: *Resolution of moduli spaces and homological stability*, arXiv:0909.4278

[29] G. Segal: *Classifying spaces and spectral sequences.* Publ. Math I.H.E.S 34 (1968), 105–112.

[30] G. Segal: *The topology of spaces of rational functions.* Acta Math. 143 (1979), no. 1-2, 39–72.

[31] W. van der Kallen: *Homology stability for linear groups.* Invent. Math. 60 (1980), no. 3, 269–295.

[32] K. Vogtmann: *Spherical posets and homology stability for $O_{n,n}$.* Topology 20 (1981), no. 2, 119–132.

[33] K. Vogtmann: *Homology stability for $O_{n,n}$.* Comm. Algebra 7 (1979), no. 1, 9–38.

[34] N. Wahl: *Homological stability for mapping class groups of surfaces*, arXiv:1006.4476.

[35] N. Wahl: *Homological stability for the mapping class groups of non-orientable surfaces.* Invent. Math. 171 (2008), no. 2, 389–424.

[36] M. Weiss: *What does the classifying space of a category classify?.* Homology Homotopy Appl. 7 (2005), no. 1, 185–195.

Mathematisches Institut, Einsteinstrasse 62, 48149 Münster
*E-mail address*: johannes.ebert@uni-muenster.de, m.weiss@uni-muenster.de