A LECTURE COURSE ON THE ATIYAH-SINGER INDEX THEOREM

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Contents

Date: April 8, 2014.

[1]I thank Paul Breutmann and Matthias Wink for many helpful comments on this set of lecture notes

1. Index theory in abstract functional analysis

The meaning of the word "abstract" is that we consider operators on abstract Hilbert spaces, not differential operators. This chapter is intended as a warm-up to index theory. Besides reviewing some of the basic principles from linear functional analysis and learning the definition of a Fredholm operator, we will prove the first theorem of this course: the Toeplitz index theorem. To each map $f : S^1 \to \mathbb{C}^\times$, we will define a Fredholm operator T_f whose index is $-\deg(f)$, the classical winding number. Thus we see that one of the most basic topological invariants have a nice interpretation as an index. This will be an important ingredient of the proof of the Bott periodicity theorem, which in turn is fundamental for the Atiyah-Singer index theorem.

1.1. Generalities on Fredholm operators and the statement of the Toeplitz index theorem.

Definition 1.1.1. Let V and W be two vector spaces (usually over $\mathbb{C}$). A linear map $F : V \to W$ is called a *Fredholm operator* if $\ker(F)$ and $\operatorname{coker}(F) := W/\operatorname{Im}(F)$ are both finite-dimensional. The *index* of F is by definition $\operatorname{ind}(F) := \dim\ker(F) - \dim\operatorname{coker}(F)$.

Lemma 1.1.2. *If V and W are finite dimensional vector spaces, then any linear map $F : V \to W$ is Fredholm and its index is* $\operatorname{ind}(F) := \dim(V) - \dim(W)$.

Proof. Recall the rank-nullity theorem from Linear Algebra I; it says that $\dim\operatorname{Im}(F) = \dim(V) - \dim\ker(F)$. Thus $\operatorname{ind}(F) = (\dim(V) - \dim\operatorname{Im}(F)) - (\dim(W) - \dim\operatorname{Im}(F)) = \dim(V) - \dim(W)$. □

Lemma 1.1.3. *If $U \xrightarrow{G} V \xrightarrow{F} W$ be two linear maps. If two of the three operators G, F, $F \circ G$ are Fredholm, then so is the third, and*

$$\operatorname{ind}(F \circ G) = \operatorname{ind}(F) + \operatorname{ind}(G).$$

Proof. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "{(\mathrm{id},G)}"] \arrow[d, "G"] & U \oplus V \arrow[r, "-G+\mathrm{id}"] \arrow[d, "{(F\circ G,\mathrm{id})}"] & V \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & V \arrow[r, "{(F,\mathrm{id})}"] & W \oplus V \arrow[r, "-\mathrm{id}+F"] & W \arrow[r] & 0 \end{tikzcd} \tag{1.1.4}$$

and both rows are exact sequences. Now we view the columns as chain complexes and get a six-term exact sequence

$$0 \to \ker(G) \to \ker(FG) \to \ker(F) \to \operatorname{coker}(G) \to \operatorname{coker}(FG) \to \operatorname{coker}(F) \to 0,$$

using that $\ker(FG) \cong \ker(FG \oplus \mathrm{id})$, and the analogous relation for the cokernels. This is the (not very) long exact homology sequence of the short exact sequence 1.1.4 of chain complexes. Now an exercise in linear algebra shows:

$$\dim\ker(G) - \dim\ker(FG) + \dim\ker(F) - \dim\operatorname{coker}(G) + \dim\operatorname{coker}(FG) - \dim\operatorname{coker}(F) = 0,$$

which is what we wanted to show. □

Exercise 1.1.5. *If you do not understand how the exact sequence arose in the above proof, take a homological algebra book and read the section on the long exact homology sequence. Do the linear algebra exercise.*

This bourbakian approach cannot be pursued much longer: by means of pure linear algebra, we cannot say more on indices of operators. In the sequel, we will only study *continuous linear maps* of Banach spaces, in fact, only of *Hilbert spaces*. We have to recall some notions and results from basic functional analysis. Consider a vector space V over $\mathbb{C}$, together with a scalar product $V \times V \to \mathbb{C}$, $(x,y) \mapsto (x,y)$. The scalar product is $\mathbb{C}$-sesquilinear and positive definite.

We define the *norm* induced by the scalar product by $\|x\| := \sqrt{(x,x)}$. V is called a Hilbert space if the norm is complete, i.e. if each Cauchy sequence converges.

Lemma 1.1.6. *A linear map $f : V \to W$ of normed vector spaces is continuous if and only if there is a $C \geq 0$ with $\|f(x)\| \leq C\|x\|$ for all x. The smallest such C is called the* operator norm $\|f\|$. *An alternative word for continuous linear map is "bounded operator", and* $\mathrm{Lin}(V;W)$ *is the set of bounded linear maps.*

This is Lemma 5.6 in [13]. An important class of operators on a Hilbert space are the *projection operators*. Let V be a Hilbert space and $U \subset V$ be a closed subspace. By $U^{\perp}$, we denote the orthogonal complement of U in V. Any element $v \in V$ can be written uniquely as $v = Pv + (v - PV)$, $Pv \in U$, $(v - Pv) \in U^{\perp}$. The map $v \mapsto Pv$ is the projection operator. It has the properties $P^2 = P$ and $\mathrm{Im}(P) = U$. Moreover, $\|P\| = 1$ if $0 \neq U$.

Exercise 1.1.7. *Show that* $\mathrm{Lin}(V, W)$*, together with the operator norm, is a normed vector space. Prove that $\|fg\| \leq \|f\|\|g\|$. Prove that* $\mathrm{Lin}(V;W)$ *is complete if W is complete. Is it a Hilbert space?*

The archetypical Hilbert space is $L^2(X;\mu)$ for a measure space (X,μ). Special cases: $X = S$ a discrete set and μ the counting measure. In that case, we call it $\ell^2(S)$. The shift operator $T_- : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by setting $T_- e_i := e_{i+1}$, where e_i is the canonical ith basis vector. It is bounded with norm 1 and is a Fredholm operator with index -1. There is another shift operator $T_+ : \ell^2 \to \ell^2$, $T_+ e_i := e_{i-1}$ and $T_+ e_1 = 0$: It has index 1.

This is not a linear algebra class; we want to geometrize these examples. Let us look at the space $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. This is a Riemann manifold with volume form $\frac{1}{2\pi i}\frac{dz}{z}$. We look at the space $L^2(S^1)$ of complex valued square integrable functions $S^1 \to \mathbb{C}$; the scalar product is given by

$$(f,g) := \frac{1}{2\pi i}\int_{S^1} \bar{f} g \frac{dz}{z}.$$

An orthonormal basis is given by the functions $f_k(z) = z^k$, $k \in \mathbb{Z}$. By means of this basis, we identify $L^2(S^1)$ with $\ell^2(\mathbb{Z})$ (Fourier series!). You might it find more convenient to identify $L^2(S^1)$ with the space of all 1-periodic functions on $\mathbb{R}$; the scalar product has the alternative form $\int_0^1 \bar{f} g dx$, the above orthonormal basis corresponds to $e^{2\pi i k x}$. You are mathematically mature and should not try to separate real and imaginary part of a function.

Inside $L^2(S^1)$, we find an important subspace $H(S^1)$; it is the closure of the linear span of all the functions f_k with $k \geq 0$. The space $H(S^1)$ is also called the *Hardy space*. There is a linear orthogonal projection operator $P : L^2(S^1) \to H(S^1)$.

Note that by a standard abuse of terminology, a projection P is called "orthogonal" if it is selfadjoint (see below). Under the above isometry $L^2(S^1) \cong \ell^2(\mathbb{Z})$, the subspace $H(S^1)$ corresponds to $\ell^2(\mathbb{N})$.

Another important operator is given when $f : S^1 \to \mathbb{C}$ is a continuous function; it sends $u \in L^2(S^1)$ to $M_f u := fu$. This is a bounded operator with $\|M_f\| \le \|f\|_{C^0}$.

Definition 1.1.8. Let $f : S^1 \to \mathbb{C}$ be a continuous function. The *Toeplitz operator* $T_f : H(S^1) \to H(S^1)$ is given by $T_f u := PM_f u$.

Example: if $f(z) = z^{\pm 1}$, then T_f is the shift $T_\mp$. More generally, one can consider powers of these operators; for example, $T_{z^k} = (T_-)^k$ if $k \ge 0$, but *not* if $k < 0$.

What can we say about continuous maps $f : S^1 \to \mathbb{C}^\times$? There is an important topological invariant, the *winding number* or *mapping degree*. We have the following (equivalent) definitions, see "Manifolds and differential forms" and "Topology I".

- The fundamental group $\pi_1(\mathbb{C}^\times)$ is isomorphic to $\mathbb{Z}$ via the isomorphism $\psi : \mathbb{Z} \to \pi_1(\mathbb{C}^\times)$, which sends the number n to the (homotopy class of the) closed loop $t \mapsto e^{2\pi i n t}$. If $f : S^1 \to \mathbb{C}^\times$ is any map, the closed loop $t \mapsto f(e^{2\pi i t})/f(1)$ represents an element $[[f]] \in \pi_1(\mathbb{C}^\times)$, and we put $\deg(f) := \psi^{-1}([[f]])$.
- Any map $S^1 \to S^1$ induces a self-map of the first homology group $H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$; it is given by mutiplication with an integer n.
- Assume that f is smooth, and consider a regular value z of the function $g = \frac{f}{|f|} : S^1 \to S^1$ and count preimages $g^{-1}(z)$ with sign (the sign is the sign of the derivative of g). If f is not smooth, take a smooth approximation.
- $\deg(f) := \int_{S^1} f^*(\frac{dz}{2\pi i z})$ if f is smooth.

Remark 1.1.9. The maps $\pi_1(\mathbb{C}^\times) \to [S^1;\mathbb{C}^\times] \xrightarrow{\deg} \mathbb{Z}$ (the first one is the most obvious one) are both isomorphisms. This is notable since the group structures in the first two groups have two sources: in the fundamental group, you take composition of loops, in the second one, the pointwise product of functions. The second set has a group structure since $\mathbb{C}^\times$ is a topological (even Lie) group.

From these considerations, we see that for $f_k(z) = z^k$, $\operatorname{ind}(T_{f_k}) = -\deg(f_k) = -k$. The first real theorem of this lecture course is

Theorem 1.1.10. *(The Toeplitz index theorem) If $f : S^1 \to \mathbb{C}^\times$ is continuous, then T_f is a Fredholm operator and* $\operatorname{ind}(T_f) = -\deg(f)$.

Note that we just proved the Toeplitz index theorem for the special functions f_k. A concrete description of T_f as a infinite matrix is not available and not practical, we need more clever tools. The first thing we need is a general principle to prove that an operator is Fredholm. This means, we have to absorb a crash course on some parts of functional analysis.

1.2. **Some functional analysis 1: the open mapping theorem and its consequences.** A basic reference that contains (almost) all the abstract functional analysis we need is Hirzebruch-Scharlau, "Einführung in die Funktionalanalysis" [13]. You should have a copy on your desk.

The first thing we recall is the *open mapping theorem.*

Theorem 1.2.1. *(The open mapping theorem) Let V and W be two Banach spaces and let $F : V \to W$ be a continuous linear map. If F is surjective, then F is an open map (i.e, images of open sets in V are open in W).*

This is Satz 9.1 in [13].

Exercise 1.2.2. *Read the proof of Theorem 1.2.1.*

Exercise 1.2.3. *Give counterexamples that show that the completeness of both spaces is essential in the theorem.*

The converse of the open mapping theorem ("an open linear map is surjective") is easy (why?). The most important consequence of the open mapping theorem is:

Corollary 1.2.4. *A bijective continuous linear map F of Banach spaces is a homeomorphism. In particular, the inverse F^{-1} is bounded as well.*

Lemma 1.2.5. *If $F : V \to W$ is Fredholm, the image $F(V) \subset W$ is a closed subspace.*

Proof. Let $U \subset W$ be a complement of $f(V)$. Since U is finite-dimensional, there is, up to equivalence, exactly one norm on U (Analysis II). The operator $F_1 : V \oplus U \to W$, $(v,u) \mapsto F(v) + u$, is surjective and bounded, hence an open map by 1.2.1. The subset $V \oplus U \setminus V \oplus 0$ is open and so is $F_1(V \oplus U \setminus V \oplus 0) = W \setminus F(V)$. □

We denote by $\mathrm{Lin}(V,W)^\times \subset \mathrm{Lin}(V,W)$ the subset of all invertible operators.

Proposition 1.2.6. *The subset $\mathrm{Lin}(V,W)^\times \subset \mathrm{Lin}(V,W)$ is open and the inversion map $F \mapsto F^{-1}$ is continuous. More precisely, if $F \in \mathrm{Lin}^\times(V,W)$ and $R \in \mathrm{Lin}(V,W)$ with $\|R\| < \|F^{-1}\|^{-1}$, then $F - R \in \mathrm{Lin}^\times(V,W)$.*

Proof. (compare [13], 23.2.) The geometric series $F^{-1}\sum_{k=0}^{\infty}(RF^{-1})^k$ converges to $(F-R)^{-1}$. □

Theorem 1.2.7. *(Homotopy invariance of the index) Let V, W be Hilbert spaces and let $\mathrm{Fred}(V,W)$ be the set of all Fredholm operators. Then $\mathrm{Fred}(V,W) \subset \mathrm{Lin}(V,W)$ is an open subset and the index function $\mathrm{ind} : \mathrm{Fred}(V,W) \to \mathbb{Z}$, $F \mapsto \mathrm{ind}(F)$ is locally constant.*

Proof. Let $F \in \mathrm{Fred}(V,W)$. Let $G : \ker(F)^\perp \to V$ be the inclusion and $H : W \to \mathrm{Im}(F)$ be the orthogonal projection which exists by Lemma 1.2.5 and because W is a Hilbert space. These two are Fredholm operators with $\mathrm{ind}(G) = -\dim(\ker(F))$ and $\mathrm{ind}(H) = \dim\mathrm{coker}(F)$. The composition HFG is invertible. By Proposition 1.2.6, we get that for all F_1 sufficiently close to F, the composition HF_1G is invertible. Thus HF_1G and H are Fredholm, and so is F_1G by Lemma 1.1.3; and $\mathrm{ind}(F_1G) + \dim\mathrm{coker}(F) = \mathrm{ind}(F_1G) + \mathrm{ind}(H) = 0$.

Again by Lemma 1.1.3, F_1 is Fredholm and $\mathrm{ind}(F_1) = \mathrm{ind}(F_1G) - \mathrm{ind}(G) = -\dim\mathrm{coker}(F) + \dim\ker(F) = \mathrm{ind}(F)$. □

1.3. **Some functional analysis 2: the adjoint operator.** Let V be a normed vector space and V' the dual space, i.e., the vector space of all continuous linear functions $V \to \mathbb{C}$. This is a normed vector space, and moreover complete, since $\mathbb{C}$ is complete.

Definition 1.3.1. Let $F : V \to W$ be a linear continuous map. The *transpose* operator is $F' : W' \to V'$, $F'(\phi)(v) := \phi(F(v))$.

This has some obvious properties (linearity, $(FG)' = G'F'$, etc) which we will not recall. It is not absolutely clear that $\|F'\| = \|F\|$. That $\|F'\| \le \|F\|$ follows from the definitions, and $\|F\| \le \|F'\|$ follows from the Hahn-Banach theorem.

A special property of Hilbert spaces is that they are self-dual:

Proposition 1.3.2. *Let V be a Hilbert space. Then the $\mathbb{C}$-antilinear map $V \to V'$, $v \mapsto \langle v, \_\rangle$ is an isometry.*

This is Satz 20.9 in [13]. The following lemma will be used only much later, in the discussion of Sobolev spaces, but fits thematically.

Lemma 1.3.3. *Let W, V be Hilbert spaces and $F : W \to V'$ a $\mathbb{C}$-linear or antilinear bounded operator. Suppose there exists C, C' such that*

$$\|v\| \le C \sup_{w \in W, |w| \le 1} |F(w)(v)|$$

and

$$\|w\| \le C' \sup_{v \in V, |v| \le 1} |F(w)(v)|$$

hold for all $v \in V$, $w \in W$. Then F is bijective (and hence a homeomorphism)

Proof. By the definition of the various norms and the assumption, we have, by the second estimate,

$$\|Fw\| = \sup_{w \in W, |w| \le 1} |F(w)(v)| \ge \frac{1}{C}\|w\|.$$

Therefore F is injective (clear), and the image of F is closed, because if $Fw_n \to v'$, then $\|w_n - w_m\| \le C\|Fw_n - Fw_m\| \to 0$ and w_n is a Cauchy sequence.

For the surjectivity, assume that $v'' \in V''$ is a linear form such that $v'' \circ F = 0$. We have to prove that $v'' = 0$. By duality, v'' is given by scalar product with an $v \in V$: $v''(v') = \langle v, v'\rangle$. We knwow that $F(w)(v) = 0$ for all $w \in W$. Therefore, by the first estimate in the assumption of the Lemma, $\|v\| = 0$, as claimed. □

Using Proposition 1.3.2, we can define the adjoint operator F^* of a linear operator $F : V \to W$.

Definition 1.3.4. Let $F : V \to W$ be a bounded operator of Hilbert spaces. The *adjoint* F^* of F is the composition $W \cong W' \overset{F'}{\to} V' \cong V$.

Proposition 1.3.5.

(1) *$F \mapsto F^*$ is antilinear.*
(2) *$F^{**} = F$.*
(3) *$(FG)^* = G^* F^*$.*
(4) *$\langle Fv, w\rangle = \langle v, F^* w\rangle$.*
(5) *$\|F\| = \|F^*\|$.*
(6) *$\|F^* F\| = \|F\|^2$.*

Proof. Everything is clear except perhaps the last equation. For each $v, w \in V$ of norm ≤ 1, we have

$$|\langle v, F^* F w\rangle| = |\langle Fv, Fw\rangle| \le \|F\|^2$$

and

$$\|Fv\|^2 = |\langle Fv, Fv\rangle| = |\langle v, F^* F v\rangle| \le \|F^* F\|.$$

□

Moreover, we often need the following relations.

Proposition 1.3.6. *Let V, W be Hilbert spaces and $F : V \to W$ be a bounded operator. Then*

(1) $\ker(F^*) = \overline{\operatorname{Im} F}^{\perp}$
(2) $\overline{\operatorname{Im} F} = \ker(F^*)^{\perp}$.

Note that taking the closure is necessary for the second relation.

1.4. **Some functional analysis 3: Compact operators.**

Definition 1.4.1. Let V, W be Banach spaces. A bounded operator $F : V \to W$ is called *compact* if one of the following equivalent conditions hold.

(1) The image of each bounded $X \subset V$ is relatively compact in W.
(2) The image of the open unit ball $B_1(V)$ is relatively compact in W.
(3) If v_n is a bounded sequence, then Fv_n does have a convergent subsequence.

We denote by $\operatorname{Kom}(V;W) \subset \operatorname{Lin}(V,W)$ the subset of all compact operators

Let us recall some well-known equivalent formulations of compactness for a metric space X. We say that X is *totally bounded* if for each $\epsilon > 0$, there exist finitely many $x_1, \dots, x_n \in X$ such that the ϵ-balls around the x_i's cover X, i.e.

$$\bigcup_{i=1}^{n} D_\epsilon(x_i) = X.$$

A metric space X is compact if and only if it is complete and totally bounded.

Examples 1.4.2.
Each operator with a finite-dimensional image is compact.
The identity on V is compact iff V is finite dimensional.

Proof. The first follows from the Heine-Borel theorem, as well as one half of the second statement. For the converse, see Lemma 24.2 [13]. □

Theorem 1.4.3. *Let V, W, U, X be Banach spaces. Then:*

(1) $\operatorname{Kom}(V,W) \subset \operatorname{Lin}(V,W)$ *is a closed subspace.*
(2) *If $F \in \operatorname{Kom}(V,W)$ and $G \in \operatorname{Lin}(W,X)$; $H \in \operatorname{Lin}(U;V)$, then $GFH \in \operatorname{Kom}(U,X)$.*
(3) *If $F \in \operatorname{Kom}(V,W)$, then $F' \in \operatorname{Kom}(W',V')$.*

Proof. (Compare [13], Lemma 24.3)

Part (2): Let $A \subset U$ be bounded. Then $H(A) \subset V$ is bounded and $FH(A) \subset W$ relatively compact. Thus $\overline{FH(A)} \subset W$ and hence $G\overline{FH(A)} \subset X$ is compact. But $\overline{GFH(A)} \subset \overline{G\overline{FH(A)}} = G\overline{FH(A)}$ is compact.

Part (1): If F is compact, then clearly so is aF, $a \in \mathbb{C}$. Moreover, if F and G are compact, then $F \oplus G$ is compact as an operator $V \oplus V \to W \oplus W$. The diagonal $\Delta : V \to V \oplus V$ and the sum $\mu : W \oplus W \to W$ are bounded, and thus $\mu \circ (F \oplus G) \circ \Delta = F + G$ is compact by part (2). So $\operatorname{Kom}(V,W) \subset \operatorname{Lin}(V,W)$ is a subspace. To prove that it is closed, assume that $F \in \overline{\operatorname{Kom}(V,W)}$ and let $\epsilon > 0$. Pick $G \in \operatorname{Kom}(V,W)$ with $\|F - G\| < \epsilon/3$.

Then $G(B_1(V))$ can be covered by finitely many $\epsilon/3$-balls around $Gv_1, \dots, Gv_n$ since it is relatively compact. Then for each $v \in B_1(V)$, there exists an i such that $\|Gv - Gv_i\| < \epsilon/3$ and therefore

$$\|Fv - Fv_i\| \leq \|Fv - Gv\| + \|Gv - Gv_i\| + \|Gv_i - Fv_i\| < \epsilon;$$

therefore $F(B_1(V))$ can be covered by finitely many ϵ-balls. Since ϵ was arbitrary, the set $F(B_1(V))$ is relatively compact.

The third statement uses the Arzela-Ascoli theorem, which we first state. $\square$

Theorem 1.4.4. *(Arzela-Ascoli theorem) Let K be a topological space and X be a complete metric space. A set $A \subset C(K,X)$ of continuous functions $K \to X$ is called* equicontinuous *if for all $\epsilon > 0$ and each $y \in K$, there exists a neighborhood $U \subset K$ of y such that for all $f \in A$ and $z \in U$, one has $d(f(y), f(z)) < \epsilon$. Let $A \subset C(K,X)$ be equicontinuous.*

1. *If K is compact and if for all $y \in K$, the set $A_y = \{f(y) | f \in A\} \subset X$ is relatively compact, then $A \subset C(K,X)$ is relatively compact, where $C(K,X)$ carries the metric $d(f,g) = \sup_{y \in K} d(f(y), g(y))$.*
2. *If K has a countable dense subset S such that for each $y \in S$, the set A_y is relatively compact, then any sequence f_n has a subsequence f_{n_k} which converges uniformly on each compact subset of K.*

The first part is proven in [13], Korollar 3.1. The second part (which is very similar), is Theorem 11.28 in [24].

Example 1.4.5. Let $K \subset V$ be any subset of a Banach space and $A \subset \mathrm{Lin}(V,W)$ be bounded. Then $A \subset C(K,W)$ is equicontinuous.

Proof. Let $x, z \in K$ and $F \in A$. Then $\|Fz - Fx\| \le \|F\| \|z - x\| \le C\|z - x\|$ with C a global bound on A. Then $U = D_{\epsilon/2C}$ is the desired neighborhood. $\square$

Proof of Theorem 1.4.3 (3). Let $X \subset W'$ be bounded. Then $X|_{\overline{F(B_1(V))}}$ is relatively compact, by Arzela-Ascoli and since F is compact. Thus if $(\ell_n)_n \subset W'$ is a bounded sequence, then there is a subsequence ℓ_{n_k} such that ℓ_{n_k} converges uniformly on $\overline{F(B_1(V))}$. Hence $\ell_{n_k} \circ F = F'(\ell_{n_k})$ converges uniformly on $B_1(V)$. $\square$

In the Hilbert space setting, we can phrase Theorem 1.4.3 by saying that $\mathrm{Kom}(V) \subset \mathrm{Lin}(V)$ is a 2-sided $*$-ideal in the C^*-algebra $\mathrm{Lin}(V)$.

Proposition 1.4.6. *Let V, W be separable Hilbert spaces. Then* $\mathrm{Kom}(V,W)$ *is the closure of the space of finite rank operators.*

Before giving the proof, we introduce terminology. We say that a sequence x_n in a metric space X is *subconvergent* if x_n has a convergent subsequence. In arguments that involve picking a subsequence, we will also often denote the subsequence by x_n as well, instead of using stacked indices such as x_{n_k} or worse.

Proof. Let F be a compact operator. Pick an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of W and let $P_n : W \to W$ be the orthogonal projection operator onto $\mathrm{span}\{e_i | i \le n\}$. Clearly $P_n F$ has finite rank, and we claim that $P_n F$ subconverges to F.

Let $K = \overline{F(B_1(V))}$, which by assumption is compact. Consider the sequence $P_n|_K$ of functions $K \to W$. The family $\{P_n|_K\}$ is equicontinuous because it is bounded (Example 1.4.5). Moreover, for each $x \in K$, the sequence $P_n x$ converges to x, which is why $\{P_n x\}$ is relatively compact in W. Thus the Arzela-Ascoli theorem applies and proves that the family $\{P_n|_K\}$ is relatively compact; thus a subsequence of $P_n|_K$ is uniformly convergent. Therefore, $P_n F$ is uniformly subconvergent on $B_1(V)$, and the pointwise limit is F, and so $P_n F \to F$, which is why F is in the closure of the finite-rank operators. $\square$

1.5. **Atkinsons Lemma and its consequences.** The next result is our main tool to prove that a given operator is Fredholm. We now restrict to separable Hilbert spaces. The exposition follows [12] and [2].

Theorem 1.5.1. *(Atkinson's Lemma) Let V, W be separable Hilbert spaces and $F \in \mathrm{Lin}(V, W)$. Then F is Fredholm if and only if there exists $G \in \mathrm{Lin}(W, V)$ such that $GF - 1$and $FG - 1$ are compact. Such an operator G is called* parametrix.

Proof. Assume first that F is Fredholm. The operator

$$F_0 : \ker(F)^{\perp} \to \mathrm{Im}(F)$$

is a bijective operator of Hilbert spaces (by 1.2.5, the target is complete) and thus its inverse is bounded by the open mapping theorem 1.2.1. Let G be the composition $W \to \mathrm{Im}(F) \overset{F_0^{-1}}{\to} \ker(F)^{\perp} \subset V$. It is easy to see that $FG - 1$ and $GF - 1$ have finite rank and are thus compact.

For the converse direction, let $GF = 1 + K$ and $FG = 1 + L$ with compact K, L. Chose finite rank operators R, S with $\|R - K\|, \|S - L\| < 1$, by Proposition 1.4.6. Then $1 - R + K$ and $1 - S + L$ are invertible by Proposition 1.2.6. Now compute that

$$(1 - R + K)^{-1} GF = (1 - R + K)^{-1}(1 - R + K + R) = 1 + (1 - R + K)^{-1} R =: 1 + P$$

with P an operator of finite rank. Thus if $Fv = 0$, then $v + Pv = 0$, i.e. $\ker(F) \subset \mathrm{Im}(P)$; in particular, the kernel of F is finite-dimensional. On the other hand,

$$FG(1 - S + L)^{-1} = (1 - S + L + S)(1 - S + L)^{-1} = 1 + S(1 - S + L)^{-1} =: 1 + Q$$

with Q of finite rank. Thus $\mathrm{Im}(1 + Q) \subset \mathrm{Im}(F)$. But $\ker(Q) \subset \mathrm{Im}(1 + Q)$, and since Q has finite rank, $\ker(Q)$ has finite codimension, and therefore $\mathrm{Im}(F)$ has finite codimension as well. □

Corollary 1.5.2. *Let $F \in \mathrm{Fred}(V, W)$ and $K \in \mathrm{Kom}(V, W)$. Then*

1. *$F + K \in \mathrm{Fred}(V, W)$ and $\mathrm{ind}(F + K) = \mathrm{ind}(F)$.*
2. *A self-adjoint Fredholm operator has index 0.*
3. *$F^* \in \mathrm{Fred}(W, V)$ and $\mathrm{ind}(F^*) = -\mathrm{ind}(F)$.*

Proof. Part (1): Let G be a parametrix for F. Then $G(F + K) - 1 = GF - 1 + GK$ is compact; similarly, $(F + K)G - 1$ is compact and one can apply Atkinson's lemma to see that $F + K$ is Fredholm. For each $t \in [0, 1]$, the operator tK is also compact, and thus $\mathrm{ind}(F + tK)$ does not not depend on t, by Theorem 1.2.7.
Part (2): Let F be self-adjoint and Fredholm. Then $\mathrm{Im}(F)$ is closed, and $\mathrm{Im}(F)^{\perp} = \ker(F^*) = \ker(F)$. Thus the index is zero.
Part (3): If G is a parametrix for F, then G^* is a parametrix for F^*, showing that F^* is Fredholm. To compute the index, we consider the operator F^*F which is self-adjoint and thus has index 0. Therefore $\mathrm{ind}(F) + \mathrm{ind}(F^*) = 0$. □

1.6. **Proof of the Toeplitz index theorem.** We now have amassed enough knowledge to prove the Toeplitz index theorem quite easily. Recall that P is the projection onto $H(S^1) \subset L^2(S^1; \mathbb{C})$. It turns out that it is formally simpler to consider the operator is $PM_fP + (1 - P)$, as an operator $L^2(S^1)$ to itself. This is the direct sum of the Toeplitz operator and the identity and thus has the same index. In particular, we redefine $T_f := PM_fP + (1 - P)$.

Lemma 1.6.1. *For each $f \in C^0(S^1)$, the operator $[P, M_f] := PM_f - M_f P$ is compact.*

Proof. Let $\mathcal{A} \subset C^0(S^1;\mathbb{C})$ the subset of all f such that $PM_f - M_f P$ is compact. We verify the hypotheses of the Stone-Weierstraß theorem 1.6.2 in order to show that $\mathcal{A} = C^0(S^1;\mathbb{C})$. Let

$$\Phi : C^0(S^1;\mathbb{C}) \to \mathrm{Lin}(L^2(S^1));\ f \mapsto [P, M_f].$$

This is a continuous linear map because $\|[P,M_f]\| = \|PM_f - M_f P\| \le 2\|f\|_{C^0}$ and hence $\mathcal{A} := \Phi^{-1}(\mathrm{Kom}(L^2(S^1)))$ is a closed subspace by Theorem 1.4.3 (1).

It is clear that the constant function $f = 1$ is in $\mathcal{A}$. If $f, g \in \mathcal{A}$, then

$$PM_{fg} - M_{fg}P = PM_f M_g - M_f M_g P = PM_f M_g - M_f PM_g + M_f PM_g - M_f M_g P = [P; M_f]M_g + M_f[P, M_g];$$

and this is compact, so $fg \in \mathcal{A}$. If $f \in \mathcal{A}$, then

$$PM_{\bar{f}} - M_{\bar{f}}P = [M_f, P]^*$$

because $P = P^*$ and $M_{\bar{f}} = M_f^*$. For the function $f(z) = z$, direct computation shows that

$$(PM_z - M_z P)(z^k) = \begin{cases} 0 & k < -1 \\ 1 & k = -1 \\ 0 & k \ge 0 \end{cases}$$

and therefore $PM_z - M_z P$ has finite rank. Apply the Stone-Weierstraß theorem. □

Theorem 1.6.2. *(The Stone-Weierstrass theorem) Let X be a compact Hausdorff space and $\mathcal{A} \subset C^0(X, \mathbb{C})$. Assume*

(1) *$\mathcal{A}$ is a closed subalgebra,*
(2) *$1 \in \mathcal{A}$,*
(3) *$f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$,*
(4) *for all $x \ne y \in X$, there is $f \in \mathcal{A}$ with $f(x) \ne f(y)$.*

Then $\mathcal{A} = C^0(X, \mathbb{C})$.

The proof can be found in [17], Theorem III.1.4.

Lemma 1.6.3. *$T_{fg} - T_f T_g$ is compact.*

Proof.
$$T_{fg} - T_f T_g = P[M_f, P]M_g P.$$
□

Corollary 1.6.4. *If $f : S^1 \to \mathbb{C}^\times$ is continuous, then T_f is Fredholm.*

Proof. $T_f T_{f^{-1}} - 1$ and $T_{f^{-1}} T_f - 1$ are compact by Lemma 1.6.3. Apply Atkinson's theorem. □

Corollary 1.6.5. *If $f, g : S^1 \to \mathbb{C}^\times$, then $\mathrm{ind}(T_{fg}) = \mathrm{ind}(T_f) + \mathrm{ind}(T_g)$.*

Proof. This follows from Lemma 1.6.3, 1.5.2 (1) and 1.1.3. □

Proof of the Toeplitz index formula. We know, from Topology I, that f is homotopic to z^k, for $k = \deg(f)$. Since $\|T_f\| \leq \|f\|$, the map $C^0(S^1;\mathbb{C}) \to \mathrm{Lin}(L^2(S^1), L^2(S^1))$, $f \mapsto T_f$, is continuous. Therefore, by Theorem 1.2.7, the index of T_f is equal to $\mathrm{ind}(T_{z^k})$.

Since $T_{fg} = T_f T_g + r$ for a compact r, we see that $\mathrm{ind}(T_{fg}) = \mathrm{ind}(T_f) + \mathrm{ind}(T_g)$. Thus $\mathrm{ind}(T_{z^k}) = k\,\mathrm{ind}(T_z)$, but this index was computed directly. □

Exercise 1.6.6. *We did not use that P is the projection onto H that often. Prove: if Q is another orthogonal projection such that $P - Q$ is compact, then the operator $QM_fQ + (1-Q)$ is Fredholm. What is its index?*

1.7. **A generalization.** We now take a slightly more abstract viewpoint on the proof of the Toeplitz index theorem. This will be the germ of the Bott periodicity theorem. We have shown that $[S^1, \mathbb{C}^\times] \to \mathbb{Z}$, $f \mapsto \mathrm{ind}(T_f)$ is a well-defined homomorphism (the source has a group structure by pointwise multiplication of functions), by Corollary 1.6.5.

Moreover, recall that the natural map $\pi_1(\mathbb{C}^\times) \to [S^1, \mathbb{C}^\times]$ is a group isomorphism. This is easy, but has some content; composition in the fundamental group is defined by concatenation of paths, and in the group of free homotopy classes, it is defined by multiplication.

Thus a different version of the Toeplitz index theorem is that

$$\pi_1(S^1) \to \mathbb{Z}; [f] \mapsto \mathrm{ind}(T_f)$$

is an isomorphism that is equal to minus the degree.

Now we define Toeplitz operators to matrix-valued functions $S^1 \to \mathrm{Mat}_{n,n}(\mathbb{C}) =: \mathbb{C}(n)$. Let $H(S^1)^n \subset L^2(S^1;\mathbb{C}^n)$ be the space spanned of all functions whose Fourier coefficients with negative indices are zero and let P_n be the orthogonal projection onto $H(S^1)^n$. Of course, we could write P_n as an $n \times n$-matrix of linear operators

$$P_n = \begin{pmatrix} P & \ldots & \ldots \\ \ldots & P & \ldots \\ \ldots & \ldots & P \end{pmatrix}.$$

For any matrix valued function $f : S^1 \to \mathbb{C}(n)$, we can form

$$T_f := P_n M_f P_n + (1 - P_n).$$

For two such functions f, g, we compute that $T_f T_g - T_{fg}$ is the matrix of operators whose (i,k)-entry is

$$\sum_{j=1}^{n} (P f_{ij} P g_{jk} P - P f_{ij} g_{jk} P) \equiv P \sum_{j=1}^{n} [f_{ij}; P] g_{jk} \pmod{\mathrm{Kom}},$$

which is compact by Lemma 1.6.1. This proves:

Lemma 1.7.1. *If $f : S^1 \to \mathrm{GL}_n(\mathbb{C})$ is continuous, then T_f is Fredholm. If f, g are two such functions, then* $\mathrm{ind}(T_{fg}) = \mathrm{ind}(T_f) + \mathrm{ind}(T_g)$.

Now if G is any Lie group (here $\mathrm{GL}_n(\mathbb{C})$), then $\pi_1(G)$ is abelian and $\pi_1(G) \to [S^1; G]$ is an isomorphism.

Lemma 1.7.2. *If $n \geq 1$, then $\pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{Z}$. More precisely, the inclusion $\mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$, $z \mapsto \mathrm{diag}(z, 1, \ldots, 1)$ and the determinant $\mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$ induce mutually inverse isomorphisms.*

Proof. Observe that the maps

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C});\ A \mapsto (\mathrm{diag}(\det A^{-1}, 1, \dots, 1)A; \det(A))$$

and

$$\mathrm{SL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C});\ (B, z) \mapsto \mathrm{diag}(z, 1, \dots, 1)B$$

are mutually inverse homeomorphisms (but not group homomorphisms). Thus the claim follows from the fact that $\pi_1(\mathrm{SL}_n(\mathbb{C})) = 1$. There are various insightful proofs of this fact. All proofs begin with the observation that $SU(n) \to \mathrm{SL}_n(\mathbb{C})$ is a homotopy equivalence by polar decomposition.

The standard proof of $\pi_1(SU(n)) = 1$ uses the long exact homotopy sequence, see [4] §VII.8. In the lecture "Topology I", I gave a proof that uses only π_1 and the Seifert-Van Kampen theorem. Here is a sketch: we argue by induction on n, the case $n = 2$ serving as induction beginning ($SU(2) \cong S^3$, and this is simply connected). Consider the map $p : SU(n) \to S^{2n-1}$ that takes a matrix A to Ae_1. The map p is a fibre bundle with fibre $SU(n-1)$, as we will see later. Cover S^{2n-1} by the complements U_i, $i = 0, 1$ of two different points. Then U_i is homeomorphic to $\mathbb{R}^{2n-1}$ and one can show that p is trivial over both subsets, in other words, there are homeomorphisms $p^{-1}(U_i) \cong U_i \times SU(n-1)$ over U_i. Now let $V_i := p^{-1}(U_i)$. The sets $V_i \cong \mathbb{R}^{2n-1} \times SU(n-1)$ are simply connected by induction hypothesis, and the indersection $V_0 \cap V_1 \cong (\mathbb{R}^{2n-1} \setminus 0) \times SU(n-1)$ is connected. So, by Seifert-van Kampen, the union $SU(n) = V_0 \cup V_1$ is simply connected.

Another interesting proof that uses a bit of the structure of the Lie group $SU(n)$ is due to Hermann Weyl and can be found in Rossmann's book [22]. □

There are stabilization maps $\mathrm{st} : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$ which induce, by the previous lemma, isomorphisms $\mathrm{st}_* : [S^1, \mathrm{GL}_n(\mathbb{C})] \to [S^1; \mathrm{GL}_{n+1}(\mathbb{C})]$. If $J_n : [S^1; \mathrm{GL}_n(\mathbb{C})] \to \mathbb{Z}$ denotes the map $[f] \mapsto \mathrm{ind}(T_f)$, we get that

$$J_{n+1} \circ \mathrm{st}_* = J_n.$$

This is nothing else that the observation that $T_{\mathrm{st} \circ f} = T_f \oplus \mathrm{id}$.

Corollary 1.7.3. *$J_n : \pi_1(\mathrm{GL}_n(\mathbb{C})) \to \mathbb{Z}$; $[f] \mapsto \mathrm{ind}(T_f)$ is an isomorphism.*

Let us switch the perspective a bit further. Suppose that $f, g : S^1 \to \mathrm{GL}_n(\mathbb{C})$ are two maps. We might now consider the *direct sum* $f \oplus g : S^1 \to \mathrm{GL}_{2n}(\mathbb{C})$.

Lemma 1.7.4. *There are homotopies $f \oplus g \sim fg \oplus 1$, $f \oplus g \sim g \oplus f$.*

Proof. We only give the first one, as the second is similar in spirit and equally easy to find. Look at

$$\begin{pmatrix} f & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} \begin{pmatrix} 1 & \\ & g \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$$

For $t = 0$, we get $f \oplus g$, for $t = \pi/2$, we get $fg \oplus 1$. □

We now get another, almost trivial, proof of the fact that $\mathrm{ind}(T_{fg}) = \mathrm{ind}(T_f) + \mathrm{ind}(T_g)$ (this took some work above!). Namely: $\mathrm{ind}(T_{f \oplus g}) = \mathrm{ind}(T_f) + \mathrm{ind}(T_g)$ *is* obvious, and the above homotopy shows that $\mathrm{ind}(T_{f \oplus g}) = \mathrm{ind}(T_{fg} \oplus 1) = \mathrm{ind}(T_{fg})$.

Passing to the colimit, we get a new description of the group structure on $\pi_1(\mathrm{GL}_\infty)$, namely the one given by direct sum, and it agrees with the old one.

We draw a lesson from these observations. By "stabilizing", we often have the possibility to exchange the operation of composition by the operation of direct sum, which is often much easier to handle.

1.8. **An example from ordinary differential equations.** Let us discuss the one single case when the index of a differential operator can be computed by hand, namely the case of an ordinary differential operator on S^1. What we do in effect is to prove the Atiyah-Singer index theorem for the manifold S^1 by bare hands: each elliptic differential operator on S^1 of order 1 has index zero (the final result is unfortunately quite boring). Via the usual map $\mathbb{R}/\mathbb{Z} \to S^1$, $t \mapsto e^{2\pi i t}$, we can identify (vector-valued) functions on S^1 with 1-periodic functions $C^\infty(\mathbb{R};\mathbb{C}^n)_1$. Now let $A : \mathbb{R} \to \mathrm{Mat}_{n,n}(\mathbb{C})$ be a smooth, 1-periodic, matrix valued function. We consider the linear differential operator

$$D : C^\infty(\mathbb{R};\mathbb{C}^n) \to C^\infty(\mathbb{R};\mathbb{C}^n);\ f \mapsto f' + Af. \tag{1.8.1}$$

This is in fact an *elliptic* differential operator on S^1, as we will learn soon. Because A is 1-periodic, D maps $C^\infty(\mathbb{R},\mathbb{C})_1$ to itself, and we denote the restriction by

$$D^{per} = D : C^\infty(\mathbb{R};\mathbb{C}^n)_1 \to C^\infty(\mathbb{R};\mathbb{C}^n)_1. \tag{1.8.2}$$

Recall from Analysis II the solution theory of linear ODEs of order 1, forgetting for the moment that A is assumed to be periodic. There exists a (unique) function $W : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ such that $W(0) = 1$ and $W' = -AW$, the *fundamental solution*. If $v \in \mathbb{C}^n$, then $f(t) = W(t)v$ is the unique solution to the initial value problem

$$Df = 0;\ f(0) = v.$$

We also need to talk about *inhomogeneous* solutions, namely solutions f of the ODE

$$Df = u. \tag{1.8.3}$$

Let us try to solve the equation 1.8.3, first with the intial value $f(0) = 0$. To find the solution, we make the ansatz $f(t) = W(t)c(t)$ for a yet to be determined function $c : \mathbb{R} \to \mathbb{C}^n$ (with $c(0) = 0$). Applying the equation 1.8.3, we find that

$$c' = W^{-1}u \text{ or } c(t) = \int_0^t W(s)^{-1}u(s)ds.$$

The general solution to the initial value problem $Df = u$, $f(0) = v$ is then given by

$$f(t) = W(t)v + W(t)\int_0^t W(s)^{-1}u(s)ds. \tag{1.8.4}$$

We have proven so far that $D : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is surjective and has n-dimensional kernel. But we want to talk about *periodic solutions*. Assume that A

is 1-periodic and let $W(t)$ be the fundamental solution. For all $t \in \mathbb{R}$, the identity

$$W(t+1) = W(t)W(1) \tag{1.8.5}$$

holds, as one proves by differentiating both sides of the equation and comparing the values for $t = 0$. We consider the linear map $\psi : \mathbb{C}^n \to C^\infty(\mathbb{R}; \mathbb{C}^n)$, given by

$$v \mapsto W(t)v$$

If v is in the eigenspace $\ker(W(1) - 1)$, then $W(t+1)v = W(t)W(1)v = W(t)v$, and so $W(t)v$ is a periodic function. But $W(t)v$ is also a solution of the ODE $Df = 0$, and so ψ maps $\ker(W(1) - 1)$ to $\ker(D^{per})$, and ψ is injective. But we know that any periodic solution of $Df = 0$ can be written in the form $W(t)v$, and this is periodic if and only if v is an eigenvector. Thus

$$\psi : \ker(W(1) - 1) \to \ker D^{per} \text{ is an isomorphism.}$$

Now turn to the determination of the cokernel of the operator D^{per}. Let u be a periodic function and suppose that there is a periodic solution of $Df = u$. Then

$$f(0) = f(1) = W(1)f(0) + W(1)\int_0^1 W(s)^{-1}u(s)ds$$

or

$$W(1)^{-1}(1 - W(1))f(0) = \int_0^1 W(s)^{-1}u(s)ds.$$

In other words, $\int_0^1 W(s)^{-1}u(s)ds$ lies in $\mathrm{Im}(W(1)^{-1}(1-W(1))) = \mathrm{Im}(1-W(1))W(1)^{-1}) = \mathrm{Im}(1 - W(1))$. The previous manipulations can be read in the opposite direction, which proves that a periodic solution $Df = u$ exists iff $Ju := \int_0^1 W(s)^{-1}u(s)ds \in \int_0^1 W(s)^{-1}u(s)ds \in \mathrm{Im}(1 - W(1))$. The linear map

$$J : C^\infty(\mathbb{R}, \mathbb{C}^n)_1 \to \mathbb{C}^n;\ Ju = \int_0^1 W(s)^{-1}u(s)ds$$

is surjective. To see this, take a function $a \in C^\infty(\mathbb{R})$ with compact support in $(0,1)$ with $\int_0^1 a(s)ds = 1$. For $v \in \mathbb{C}^n$, form $u(s) := a(s)W(s)v$ which has compact support and extend it to all of $\mathbb{R}$ by 1-periodicity. But

$$Ju = \int_0^1 W(s)^{-1}W(s)a(s)vds = v.$$

These arguments show that the image of D^{per} is the preimage $J^{-1}(\mathrm{Im}(1 - W(1))) \subset C^\infty(\mathbb{R}, \mathbb{C})_1$; and this preimage has the same codimension as $\mathrm{Im}(1 - W(1)) \subset \mathbb{C}^n$. But this codimension is, by the rank-nullity theorem, the same as $\dim(\ker(W(1) - 1))$, and hence

$$\mathrm{ind}(D^{per}) = 0.$$

We go one step further. The vector space $C^\infty(\mathbb{R}, \mathbb{C}^n)_1$ has an inner product $\langle f; g\rangle := \int_0^1 (f(t); g(t))\,dt$, using the integral and the inner product on $\mathbb{C}^n$. Now we consider the *adjoint operator* to D:

$$D^* f(t) := -f'(t) + A(t)^* f(t).$$

Let $V : \mathbb{R} \to \mathrm{Mat}_{n,n}(\mathbb{C})$ be the fundamental solution for D^*, i.e. $V(0) = 1$ and $V' = A^* V$.

Exercise 1.8.6. *Prove:*

(1) D^* *is indeed the adjoint of* D *in the sense that* $\langle D^* f; g\rangle = \langle f; Dg\rangle$ *holds for all functions* f, g *(partial integration).*
(2) $V^* W = 1$ *(differentiate!).*
(3) $\mathrm{Im}(W(1) - 1) = (\ker(V(1) - 1))^{\perp}$.
(4) *Conclude that* $u \in \mathrm{Im}(D)$ *if and only for all* $w \in \ker(V(1) - 1)$*, the equation* $\int_0^1 (V(s)w, u(s))\, ds = 0$ *holds.*
(5) *Prove that there is an orthogonal sum decomposition* $C^\infty(\mathbb{R}; \mathbb{C}^n)_1 = \mathrm{Im}(D) \oplus \ker(D^*)$.

In two cases, there are explicit formulae for the solution operator. If $n = 1$, then $W(t) = \exp(-\int_0^t A(s)ds)$. The other easy case is when $A(s) \equiv A$ is constant, in which case the fundamental solution is $\exp(At)$.

Exercise 1.8.7. *Assume that* $n = 1$. *Prove that* $\dim \ker(D) = 1$ *if and only if* $\int_0^1 a(s)ds \in 2\pi i$ *(in the other case, the kernel is trivial). Assume that* $n \geq 1$ *and* A *is constant. Show that* $\dim(\ker(D)) = \sum_{k \in \mathbb{Z}} \mathrm{Eig}(A, 2\pi k)$.

1.9. **Literature and remarks.** To aquire the neccessary background in functional analysis for the index theorem, you do not have to delve into the formidable treatise [23]; the nice book [13] contains all material. The proof of the basic properties of compact operators is taken from that source. The treatment of Fredholm operators is a "best of" the (relevant sections) of the texts [13], [2], [12].

2. Differential operators on manifolds and the de Rham complex revisited

The index theorem will give a formula for the index of a *general* elliptic differerntial operator, but the main interest lies in *special operators* that are associated with any manifold or any manifold with some extra structure. The father of most of these natural operators is the *exterior derivative*, which we briefly recall. Then we move on to the general definition of a differential operator on a manifold, introduce the *symbol* and the notion of ellipticity.

2.1. **The de Rham operator.** We are very brief in this section. The material is standard and can be found in many textbooks, of which I most recommend [15] for a first orientation and [31] for a more detailed exposition. Let V be an n-dimensional real vector space and let $\Lambda^p V^*$ be the space of alternating p-multilinear forms on V. There is the *wedge product*

$$\Lambda^p V^* \otimes \Lambda^q V^* \to \Lambda^{p+q} V^*; \omega \otimes \eta \mapsto \omega \wedge \eta,$$

which turns $\Lambda^* V^*$ into a *graded commutative algebra*, i.e.

$$\omega \wedge (\eta \wedge \zeta) = (\omega \wedge \eta) \wedge \zeta;\ \omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega$$

where $|\omega|$ denotes the degree of ω. Moreover, for each $v \in V$, we have the *insertion operator* $\iota_v : \Lambda^p V^* \to \Lambda^{p-1} V^*$, defined by $(\iota_{v_1}\omega)(v_2, \ldots, v_p) := \omega(v_1, \ldots, v_p)$. It is an *antiderivation*, i.e.

$$\iota_v(\omega \wedge \eta) = (\iota_v \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (\iota_v \eta).$$

For $\xi \in V^* = \Lambda^1 V^*$, let $\epsilon_\xi(\omega) := \xi \wedge \omega$. Both structures, the exterior product and the insertion operator, are intertwined by the easily verified identity

$$\epsilon_\xi \iota_v + \iota_v \epsilon_\xi = \xi(v) \tag{2.1.1}$$

(left hand-side is the operator that multiplies by $\xi(v)$). Let M^n be a smooth manifold. A *vector field* on M is a section of $TM \to M$, and $\mathcal{V}(M) := \Gamma(M; TM)$ denotes the space of all vector fields on M. Tangent vectors to a manifold have a schizophrenic nature (as derivatives of curves and directional derivatives). This means that $\mathcal{V}(M)$ has the alternative expression as the set of all linear map $X : C^\infty(M) \to C^\infty(M)$ such that $X(fg) = (Xf)g + f(Xg)$ holds for all functions f, g. The commutator

$$[X, Y] := XY - YX$$

of two vector fields is again a vector field. The commutator is also called *Lie bracket* and it turns $\mathcal{V}(M)$ into a *Lie algebra*. Let $\mathcal{A}^p(M)$ be the space of all smooth p-forms on M. One can interprete $\mathcal{A}^p(M)$ as the space of smooth sections of the bundle $\Lambda^p T^* M \to M$ of exterior forms on the tangent bundle and therefore the linear algebraic structure on the exterior algebra (wedge product and insertion operator) carries over to forms on manifolds.

The most important structure is the exterior derivative, a sequence of linear maps $d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$, uniquely characterized by the properties

- On $\mathcal{A}^0(M) = C^\infty(M)$, d is the total differential.
- $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$ (Leibniz rule)
- $d^2 = 0$.

A smooth map $f : M \to N$ induces a map $f^* : \mathcal{A}(N) \to \mathcal{A}(M)$, compatible with d and $\wedge$. The quotient space

$$H^p_{dR}(M) := \frac{\ker(d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M))}{\mathrm{Im}(d : \mathcal{A}^{p-1}(M) \to \mathcal{A}^p(M))}$$

is called *de Rham cohomology* of M. The de Rham cohomology has a deep topological meaning, which shall not bother us right now. Instead, we look at the operator d in more detail. Let $x : M \supset U \to \mathbb{R}^n$ be a local coordinate system on M. For a subset $I = \{i_1, \ldots, i_p\} \subset \underline{n}$, $|I| = p$, let $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. In these coordinates, we can write each p-form (locally) as

$$\omega = \sum_{I \subset \underline{n}; |I| = p} a_I dx_I$$

for smooth functions a_I. The exterior derivative is then given by

$$d\omega = \sum_{I \subset \underline{n}; |I| = p} \sum_{j=1}^n \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I.$$

This formula shows that d is a *linear partial differential operator of order* 1, a notion with which we will have to familiarize us next. We can combine the exterior derivative with the insertion operator to get the *Lie derivative*

$$L_X(\omega) := d(\iota_X \omega) + \iota_X d\omega.$$

The Lie derivative takes p-forms to p-forms, and has the following properties

Proposition 2.1.2.

(1) *L_X commutes with ι_X and d.*
(2) *$L_X(\omega \wedge \eta) = (L_X \omega) \wedge \eta + \omega \wedge L_X \eta$ (no sign).*
(3) *if $f \in \mathcal{A}^0(M)$, then $L_X f = Xf$.*

2.2. **Differential operators in general.**

Notation 2.2.1. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\underline{n}}$, we let $|\alpha| := \sum_i \alpha_i$. For $x \in \mathbb{R}^n$, we let $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ and $\frac{\partial^{|\alpha|}}{\partial x_\alpha} := \frac{\partial^{\alpha_1}}{\partial x_1} \ldots \frac{\partial^{\alpha_n}}{\partial x_n}$. Moreover, $D^\alpha := (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_\alpha}$. Furthermore, $\alpha! := \prod_{i=1}^n \alpha_i!$.

If $E \to M$ is a vector bundle over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\Gamma(M, E)$ the vector space of smooth sections of E.

Definition 2.2.2. Let M be a smooth manifold and $E_i \to M$ be two smooth vector bundles. A *differerential operator* $P : \Gamma(M, E_0) \to \Gamma(M; E_1)$ of order k is a linear map which satisfies the following properties:

(1) P is local in the sense that if $s \in \Gamma(M, E_0)$ vanishes on the open subset $U \subset M$, then so does Ps.
(2) If $x : U \to \mathbb{R}^n$ is a chart and $\phi_i : E_i|_U \to U \times \mathbb{K}^{p_i}$ a trivialization, then the localized operator $\phi_1 \circ P \circ (\phi_0)^{-1}$ can be written as

$$(\phi_1 \circ P \circ (\phi_0)^{-1})(f)(y) = \sum_{|\alpha| \leq k} A^\alpha(y) \frac{\partial^\alpha}{\partial x_\alpha} f(y)$$

for each $f \in C^\infty(U, \mathbb{K}^{p_0})$, where $A^\alpha : U \to \mathrm{Mat}_{p_1, p_0}(\mathbb{K})$ is a smooth function.

Examples 2.2.3. *Composition with a vector bundle homomorphism induces an operator of order* 0. *The exterior derivative is an operator of order* 1.

There is a coordinate-free description of differential operators, which is sometimes useful.

Theorem 2.2.4. *Let* $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ *be linear. Then*

(1) P *is a differential operator of order* 0 *if and only if the commutator with the multiplication by any function* $f \in C^\infty(M)$ *is zero;* $[P, f] = 0$.
(2) P *is a differential operator of order* k *if and only if the commutator* $[P, f]$ *is an operator of order* $k - 1$, *for each* $f \in C^\infty(M)$.

Proof. In both parts, the "only if" direction is easy and we turn to the "if" direction. Part (1). First we prove that P is local. If $s \equiv 0$ near x, there is a function f with $f \equiv 1$ near x and $fs = 0$. Then

$$Ps(x) = fPs(x) = P(fs)(x) = 0$$

and hence P is local. Thus we can compute in local coordinates. Let $s_1, \dots, s_{p_0}$ be a local basis of E_0 and $a_i \in C^\infty$. We find

$$P(\sum_i a_i s_i)(x) = \sum_i a_i(x) P s_i(x).$$

Ps_i is a section of E_1 and we can write it as a linear combination of a given local basis of E_1 with smooth coefficients. This gives the desired presentation of P.

Part (2). Assume that $[P, f]$ is an operator of order $k-1$, for each $f \in C^\infty(M)$. We first prove that P is local. As above, assume that $f \equiv 1$ and $s \equiv 0$ near x. Then

$$Ps(x) = fPs(x) = [f, P]s(x) + P(fs)(x) = [f, P]s(x).$$

By induction hypothesis, $[f, P]$ is local and so $[f, P]s(x) = 0$. Let $x_0 \in M$ and x a local chart such that $x(x_0) = 0$. Next, we recall a lemma that was crucial in proving that the tangent space of a manifold, defined using derivations, was an n-dimensional vector space. Let $0 \in U \subset \mathbb{R}^n$ be convex and $f : U \to \mathbb{R}$ be smooth. Then

$$f(x) = f(0) + \int_0^1 \frac{\partial}{\partial t} f(tx)dt = f(0) + \sum_{i=1}^n \int_0^1 x_i \frac{\partial f}{\partial x_i}(tx)dt =: f(0) + \sum_{i=1}^n x_i g_i(x)$$

with g_i smooth and $g_i(0) = \frac{\partial}{\partial x_i} f(0)$. Iteratively, we find that there is a unique polynomial $p(x)$ of degree k and smooth functions g_α, $|\alpha| = k+1$ with $g_\alpha(0) = \frac{\partial^{|\alpha|}}{\partial x_\alpha} f(0)$ such that

$$f(x) = p(x) + \sum_{|\alpha|=k+1} x^\alpha g_\alpha(x). \tag{2.2.5}$$

Moreover, if $[P, f]$ has order $k-1$ for each f, we find that for all functions $f_0, \dots, f_k$ with $f_i(0) = 0$, we have

$$P(f_0 \dots f_k s)(0) = 0$$

for each section s, because

$$P(f_0 \dots f_k s)(0) = [P, f_0](f_1 \dots f_k s)(0) + f_0(0) P(f_1 \dots f_k s)(0) = 0$$

by induction hypothesis. If $s_1, \dots, s_{p_0}$ is a local basis and a_i smooth and $x(x_0) = 0$, then

$$P(\sum_i a_i s_i)(x_0) = \sum_i P(a_i s_i)(x_0) = \sum_i P((p_i + \sum_{|\alpha|=k+1} x^\alpha g_{i,\alpha}) s_i)(x_0)$$

But as $[P, f]$ has order $k-1$, we find $P((p_i + \sum_{|\alpha|=k+1} x^\alpha g_{i,\alpha}) s_i)(x_0) = 0$ and hence $P(\sum_i a_i s_i)(x_0) = \sum_i P(p_i s_i)(x_0)$. But $p_i(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x_\alpha} a_i(0) x^\alpha$ and thus $P(a_i s_i)(x_0) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x_\alpha} a_i(0) P(x^\alpha s)(x_0)$. Rearranging all terms gives the desired presentation.

□

Example 2.2.6. Let $E_0 = E_1 = \underline{\mathbb{R}} := M \times \mathbb{R}$ be the trivial bundle. Let X be a vector field, i.e. a derivation, in other words a map $X : C^\infty(M) \to C^\infty(M)$ such that $X(fg) = (Xf)g + f(Xg)$ for all $f, g \in C^\infty(M)$. The commutator $[X, f]$ is the operator

$$[X, f]g = (Xf)g$$

and so it is an operator of order 0. Thus, X is a differential operator of order 1.

Example 2.2.7. According to the list of axioms for the exterior derivative, we can compute the commutator $[d, f]$ as

$$[d, f]\omega = d(f\omega) - f d\omega = df \wedge \omega$$

and so d is a differential operator of order 1.

Example 2.2.8. If $E \to M$ is an arbitrary vector bundle, a *connection* on E is a linear map $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ such that for all $s \in \Gamma(M, E)$ and $f \in C^\infty(M)$, we have $\nabla(fs) = df \otimes s + f \nabla s$, which means that

$$[\nabla, f]s = df \otimes s$$

which is why a connection is a differential operator of order 1, characterized by the condition $[\nabla, f] = df \otimes \_$.

Definition 2.2.9. Let M be a manifold and E_0, E_1 be two vector bundles. We denote by $\mathrm{Diff}^k(E_0, E_1)$ the set of all differential operators of order k.

Let us note some obvious properties.

Lemma 2.2.10.

(1) $\mathrm{Diff}^k(E_0, E_1)$ *is a vector space.*
(2) $\mathrm{Diff}^k(E_0, E_1) \subset \mathrm{Diff}^{k+1}(E_0, E_1)$.
(3) *If* $P \in \mathrm{Diff}^k(E_0, E_1)$ *and* $Q \in \mathrm{Diff}^m(E_1, E_2)$, *then* $Q \circ P \in \mathrm{Diff}^{k+m}(E_0, E_2)$.

We remark that the order of P is not really a well-defined number. Some of the most important information about a differential operator can be encoded in the terms of highest order, the *symbol*. From now on, we assume that the vector bundles E_i are over $\mathbb{C}$.

Definition 2.2.11. Let $P \in \mathrm{Diff}^k(E_0, E_1)$. Let $y \in M$, $\xi \in T_y^* M$ and $e \in (E_0)_y$. Pick $f \in C^\infty(M;\mathbb{R})$ with $f(y) = 0$ and $d_y f = \xi$, and pick $s \in \Gamma(M, E_0)$ with $s(y) = e$. We define the *symbol* of P to be

$$\mathrm{smb}_k(P)(y,\xi)(e) := \frac{i^k}{k!} P(f^k s)(y) \in (E_1)_y.$$

Lemma 2.2.12. *The expression* $\mathrm{smb}_k(P)(y,\xi)(e)$ *only depends on* y, ξ *and* e *(assuming that* P *and* k *are fixed).*

Proof. For the purpose of this proof, we compute in local coordinates. Pick a local chart with $x(y) = 0$. We can assume that the bundles E_0 and E_1 are trivial over the domain of the chart x. We write $\xi = \sum_i \xi_i dx_i$, $\xi_i \in \mathbb{R}$ and compute

$$\frac{i^k}{k!} P(f^k s)(x) = \frac{i^k}{k!} \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_\alpha}(f^k s)(x) = \frac{i^k}{k!} \sum_{|\alpha| \le k} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} A^\alpha(x) \frac{\partial^{|\beta|}}{\partial x_\beta}(f^k)(x) \frac{\partial^{|\gamma|}}{\partial x_\gamma}(s)(x).$$

Since the derivative $\frac{\partial^{|\beta|}}{\partial x_\beta} f^k(x)$ is zero for $|\beta| < k$ (the argument from the proof of Theorem 2.2.4), the sum equals

$$\frac{i^k}{k!} \sum_{|\alpha|=k} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_\alpha}(f^k)(x) s(x).$$

But $\frac{\partial^{|\alpha|}}{\partial x_\alpha}(f^k)(x) = k! \xi^\alpha$, and we conclude that

$$\mathrm{smb}_k(P)(y,\xi)(e) = i^k \sum_{|\alpha|=k} A^\alpha(x) \xi^\alpha e. \tag{2.2.13}$$

This equation can be read in two directions: it shows that the left-hand-side does not depend on the concrete choice of f and s, and that the right-hand-side has a coordinate-invariant meaning. ☐

Remark 2.2.14. There are in principle two approaches to calculus on manifolds (the coordinate-free one and the one using coordinates). As a general rule, the coordinate-free approach is more modern and preferred by most pure mathematicians (for good reasons). The above proof shows that the combination of both can be a useful argument, and that one should not stick ideologically to the coordinate-free approach.

Exercise 2.2.15. *If you think that using local coordinates is a stupid thing to do, try to give a proof of Lemma 2.2.12 and Proposition 2.2.18 avoiding choices of coordinates (this is possible).*

Now we give a more invariant interpretation of the symbol. Let V be a finite-dimensional real vector space (think about $T_x M$). By the symbol $\mathrm{Sym}^k V$, we denote the vector space of degree k homogeneous polynomial functions $V^* \to \mathbb{R}$. Given a manifold M, we can form the vector bundle $\mathrm{Sym}^k TM \to M$, whose fibre over x is precisely $\mathrm{Sym}^k T_x M$. What does the symbol do? It assigns to any given $\xi \in T_x^* M$ a linear map $\mathrm{smb}_k(P)(x,\xi) : (E_0)_x \to (E_1)_x$, and $\xi \mapsto \mathrm{smb}_k(P)(x,\xi)$ is a polynomial function $T_x^* M \to \mathrm{Hom}((E_0)_x, (E_1)_x)$. Moreover, this polynomial function is homogeneous of degree k. To see this, simply look at the right-hand side of 2.2.13. We might now form the vector bundle $\mathrm{Sym}^k TM \otimes \mathrm{Hom}(E_0, E_1) \to M$, and the symbol $\mathrm{smb}_k(P)$ defines a section of this vector bundle: at a point $x \in M$, the value of this section is the polynomial function $\xi \mapsto \mathrm{smb}_k(P)(x,\xi)$. Another look at 2.2.13 proves that this is a smooth section.

Definition 2.2.16. Let $\mathrm{Smbl}_k(E_0,E_1)$ be the space of smooth sections of $\mathrm{Sym}^k TM \otimes \mathrm{Hom}(E_0,E_1) \to M$. The symbol of an order k operator is a well-defined element of $\mathrm{Smbl}_k(E_0,E_1)$ and we have produced a map $\mathrm{smb}_k : \mathrm{Diff}^k(E_0,E_1) \to \mathrm{Smbl}_k(E_0,E_1)$.

Example 2.2.17. It is worth to work out the meaning of all this in the case when $M = U \subset \mathbb{R}^n$ is an open subset, both vector bundles are trivialized and P is a differerential operator of order k. We write

$$Pu(x) = \sum_{|\alpha|\leq k} A^{\alpha}(x) D^{\alpha} u(x)$$

(note that we used D^{α} here, instead of $\frac{\partial^{|\alpha|}}{\partial x_{\alpha}}$ as above). The symbol is now given (using 2.2.13) by the formula

$$\mathrm{smb}_k(P)(x,\xi) = \sum_{|\alpha|=k} A^{\alpha}(x)\xi^{\alpha}.$$

We can represent a differerential operator P on $U \subset \mathbb{R}^n$ by a function $p(x,\xi)$ which is smooth in x and polynomial (of degree k) in ξ. The operator P is given by $p(x,D)$ (replace ξ_i by D^i). Some caution is necessary because D^i does not commute with multiplication by smooth functions. One sometimes calles the polynomial $p(x,\xi)$ the *complete symbol*, and the leading part the *principal symbol*. On a manifold, the complete symbol does not have an easily identified meaning, only the principal symbol (which we call "symbol") does.

It is obvious that smb_k is a linear map (if you are unsure, look at the formula 2.2.13). If E_i, $i = 0,1,2$, are three vector bundles and k,l two natural numbers, there is a composition map

$$\mathrm{Smbl}_k(E_0,E_1) \times \mathrm{Smbl}_l(E_1,E_2) \to \mathrm{Smbl}_{k+l}(E_0,E_2).$$

This is defined by means of linear algebra: If V is a finite-dimensional real vector space and E_i, $i = 0,1,2$, complex vector spaces, then a bilinear map

$$(\mathrm{Sym}^k V \otimes \mathrm{Hom}(E_0,E_1)) \times (\mathrm{Sym}^l V \otimes \mathrm{Hom}(E_1,E_2)) \to (\mathrm{Sym}^{k+l} V \otimes \mathrm{Hom}(E_0,E_2))$$

is given by

$$(p \otimes a, q \otimes b) \mapsto pq \otimes (b \circ a).$$

More concretely, we compose an order l homogeneous polynomial p on $T_x^* M$ with values in $\mathrm{Hom}((E_1)_x,(E_2)_x)$ with an order k homogeneous polynomial q with values in $\mathrm{Hom}((E_0)_x,(E_1)_x)$. This is done by $\xi \mapsto (p \circ q)(\xi) := p(\xi) \circ q(\xi)$.

Proposition 2.2.18. *Let* $P \in \mathrm{Diff}^k(E_0,E_1)$ *and* $Q \in \mathrm{Diff}^l(E_1,E_2)$. *Then* $\mathrm{smb}_{k+l}(Q \circ P) = \mathrm{smb}_l(Q)\mathrm{smb}_k(P)$.

Proof. In concrete terms, this means that for each cotangent vector $\xi \in T_x^* M$, we have $\mathrm{smb}_{k+l}(Q \circ P)(\xi) = \mathrm{smb}_l(Q)(\xi) \circ \mathrm{smb}_k(P)(\xi)$, which is easy to see using formula 2.2.13. ∎

Proposition 2.2.19. *The sequence*

$$0 \to \mathrm{Diff}^{k-1}(E_0, E_1) \to \mathrm{Diff}^k(E_0, E_1) \to \mathrm{Smbl}_k(E_0, E_1) \to 0$$

is exact.

Proof. Everything is easy (for example in local coordinates) except exactness at the right. We have to show: if $p \in \mathrm{Smbl}_k(E_0, E_1)$, then there is a differential operator P with symbol p. Locally (in local coordinates) the problem is easy to solve, because if p is given by

$$\sum_{|\alpha|=k} A^\alpha(x)\xi^\alpha$$

with some matrix-valued functions A^α, the operator

$$\sum_{|\alpha|=k} A^\alpha(x)D^\alpha$$

has the required symbol. To glue the local solutions together, one uses a partition of unity.

□

All this becomes more transparent if we consider operators of order 1. Assume that P has order 1, $x \in M$, $f \in C^\infty(M)$ and $s \in \Gamma(M, E_0)$. Then compute

$$[P, f]s(x) = [P, f - f(x)]s(x) + [P, f(x)]s(x) = [P, f - f(x)]s(x).$$

(note that $f(x)$ denotes the constant function, and note that P therefore commutes with $f(x)$). Moreover,

$$[P, f-f(x)]s(x) = P((f-f(x)(s))(x)-(f-f(x))(x)Ps(x) = P((f-f(x))(s))(x) = -i\mathrm{smb}_1(P)(df)s(x).$$

Thus:

Lemma 2.2.20. *If P is an operator of order 1, the symbol can be computed as* $\mathrm{smb}_1(P)(df)s := i[P, f]s$.

Now, by definition, $\mathrm{Smbl}_1(E_0, E_1)$ is the space of sections in the vector bundle

$$\mathrm{Sym}^1 TM \otimes \mathrm{Hom}(E_0, E_1) = TM \otimes \mathrm{Hom}(E_0, E_1) = \mathrm{Hom}(T^*M \otimes E_0; E_1)$$

and in this description, the symbol of P becomes $\mathrm{smb}_1(P)(df)s := i[P, f]s$. Here we used a section and the derivative of a function as a variable, but Lemma 2.2.12 proves that this is really a well-defined section of vector bundles.

We now come to one of the central definitions of this lecture course.

Definition 2.2.21. Let M be a smooth manifold, $E_0, E_1 \to M$ two complex vector bundles and $P \in \mathrm{Diff}^k(E_0, E_1)$, $x \in M$. Then we say that P is *elliptic at x* if for each $\xi \in T_x^*M$, $\xi \neq 0$, the homomorphism $\mathrm{smb}_k(P)(\xi) : (E_0)_x \to (E_1)_x$ is invertible. We say that P is *elliptic* if it is elliptic at each point of M.

Example 2.2.22. Let $M = \mathbb{R}$ and $A, B : \mathbb{R} \to \mathrm{Mat}_{p,p}(\mathbb{C})$. Consider the operator $P : C^\infty(\mathbb{R}; \mathbb{C}^p) \to C^\infty(\mathbb{R}; \mathbb{C}^p)$ given by

$$Pf := Bf' + Af = B\frac{\partial}{\partial x}f + Af.$$

Let us compute the symbol. A typical cotangent vector vector is $(x, \xi dx)$, $x \in \mathbb{R}(= M)$, $\xi \in \mathbb{R}$. In this local coordinate, we find that the symbol is given by $\mathrm{smb}_1(P)(\xi dx) = i\xi B(x) \in \mathrm{Mat}_{p,p}(\mathbb{C})$. Thus we find that P is elliptic at $x \in \mathbb{R}$ if and only if $B(x)$ is invertible.

Example 2.2.23. From basic complex analysis, one knows the *Wirtinger* or *Cauchy-Riemann* operators on $C^\infty(U;\mathbb{C})$, where $U \subset \mathbb{C}$ is open. They are defined by

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}); \ \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

(x and y are the real coordinates $z = x + iy$). Let $adx + bdy$ be a (real) cotangent vector to U. We find that the symbol of $\frac{\partial}{\partial \bar{z}}$ is given by

$$\mathrm{smb}_1(\frac{\partial}{\partial \bar{z}})(adx + bdy) = i\frac{1}{2}(a + ib).$$

(substitute the jth cotangent coordinate for the partial derivative D^i). We can identify $T^*\mathbb{C}$ with $\mathbb{C}$ in the canonical way, this corresponds to $adx + bdy \mapsto a + ib$. Call the resulting coordinate ζ (this is a complex-valued linear form on T^*U). Thus the symbol of $\frac{\partial}{\partial \bar{z}}$ is multiplication by $\frac{i}{2}\zeta$.

Similarly, one finds that

$$\mathrm{smb}_1(\frac{\partial}{\partial z})(adx + bdy) = i\frac{1}{2}(a - ib) = \frac{i}{2}\bar{\zeta}.$$

Both operators are elliptic. The holomorphic functions on U are precisely the solutions of the PDE $\frac{\partial}{\partial \bar{z}} f = 0$, and this remark shows that complex analysis in one variable is a special case of elliptic operator theory (in higher dimensions, the situation is much more subtle).

In the theory of Riemann surfaces, there is an important operator combining the two Wirtinger operators. Namely, let $\mu \in C^\infty(U)$ be a smooth function and consider $P = \frac{\partial}{\partial \bar{z}} + \mu\frac{\partial}{\partial z}$. The symbol is

$$\mathrm{smb}_1(P)(\zeta) = \frac{i}{2}(\zeta + \mu\bar{\zeta}).$$

For which μ is this operator elliptic? To get to the important point, write $V = T_x^*U$ and note that $\zeta : V \to \mathbb{C}$ is a (real-linear) isomorphism. In this reformulation, the problem becomes to find under which conditions on a complex number $\mu = \mu(x)$, the equation $\zeta + \mu\bar{\zeta}$ has no nontrivial solutions for $\zeta \in \mathbb{C}$. Observe that $|\zeta| = |\bar{\zeta}|$ and therefore, if $\zeta + \mu\bar{\zeta} = 0$ for $\zeta \neq 0$, we must have $|\mu| = 1$. Thus if P is not elliptic at x, then $|\mu| = 1$. Vice versa, if $|\mu(x)| = 1$, then P is not elliptic at x.

Thus: the operator P is elliptic on U if $|\mu| \neq 1$ on U. The relevant case is when $|\mu| < 1$. The operator P is relevant for the problem of integrability of almost-complex structures.

Example 2.2.24. Let us compute the symbol of the exterior derivative $d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$. Let $f \in C^\infty(M)$. Then

$$\mathrm{smb}_1(d)(df)\omega = i(d(f\omega) - fd\omega) = i(df \wedge \omega).$$

Thus the symbol of d, viewed as a bundle map $T^*M \otimes \Lambda^p T^*M \to \Lambda^{p+1}T^*M$, is given by $(\xi, \omega) \mapsto i\xi \wedge \omega$.

Thus, the exterior derivative is *not* elliptic unless $\dim M = 1$. However, it is relatively close to being elliptic.

Lemma 2.2.25. *Let V be a finite dimensional real vector space of dimension n. For $\xi \in V^*$, denote by $\epsilon_\xi : \Lambda^p V^* \to \Lambda^{p+1} V^*$ the map $\epsilon_\xi \omega := \xi \wedge \omega$. Then if $\xi \neq 0$, the sequence*

$$0 \to \Lambda^0 V^* \xrightarrow{\epsilon_\xi} \Lambda^1 V^* \xrightarrow{\epsilon_\xi} \dots \xrightarrow{\epsilon_\xi} \Lambda^n V^* \to 0$$

is exact.

Proof. For $v \in V$, we get the map $\iota_v : \Lambda^p V^* \to \Lambda^{p-1} V^*$ which inserts v as the first argument. It satisfies $\iota_v(\omega \wedge \eta) = (\iota_v \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge \iota_v \eta$. By 2.1.1

$$(\iota_v \epsilon_\xi + \epsilon_\xi \iota_v)\omega = (\xi(v))\omega.$$

Now pick v such that $\xi(v) = 1$. If $\xi \wedge \omega = 0$, we find that

$$\omega = \xi(v)\omega = (\iota_v \epsilon_\xi + \epsilon_\xi \iota_v)\omega = \epsilon_\xi(\iota_v \omega)$$

and this proves the exactness. □

This lemma proves that the de Rham complex fits into the following definition.

Definition 2.2.26. Let M be a smooth manifold. An *elliptic complex* of length n is a sequence

$$0 \to \Gamma(M, E_0) \xrightarrow{P_1} \Gamma(M, E_1) \xrightarrow{P_2} \dots \xrightarrow{P_n} \Gamma(M, E_n) \to 0$$

of differential operators of order 1 between complex (or real) vector bundles such that

(1) $P_i \circ P_{i-1} = 0$ and
(2) for each nonzero cotangent vector $\xi \in T_x^* M$, the sequence

$$0 \to (E_0)_x \xrightarrow{\mathrm{smb}_1(P_1)(\xi)} (E_1)_x \xrightarrow{\mathrm{smb}_1(P_2)(\xi)} \dots \xrightarrow{\mathrm{smb}_1(P_n)(\xi)} (E_n)_x \to 0$$

is exact.

We write the elliptic complex as (E_*, P).

Remark 2.2.27. An elliptic complex of length one is the same as an elliptic operator on M. The length and the dimension of M are completely unrelated in general. One can formulate a more general definition where the operators P_i have order > 1, but for applications this is irrelevant. It is important that all operators have the same order.

Out of an elliptic complex, we can extract an elliptic operator, but that requires Riemann metrics on M and hermitian bundle metrics on the bundles E_i.

2.3. **The formal adjoint.**

Assumptions 2.3.1. The manifold M comes equipped with a Riemann metric, and the complex vector bundles are equipped with hermitian bundle metrics.

We first recall how functions can be integrated on a Riemannian manifold. First assume that M is an *oriented* Riemann manifold of dimension n. There is a unique n-form $\mathrm{vol} \in \mathcal{A}^n(M)$, characterized by the property that if $(e_1, \dots, e_n)$ is an oriented orthonormal basis of $T_x^* M$, for some $x \in M$, then $\mathrm{vol}(e_1, \dots, e_n) = 1$. The form vol is called the *volume form*.

Definition 2.3.2. Let $f \in C_c^\infty(M)$ be a function with compact support. We define

$$\int_M f(x)dx := \int_M f\mathrm{vol}.$$

This notion of integral uses the orientation on M, but it does use the orientation *twice*: to define the volume form and to define the integral. Both dependences cancel out: assume that $T : M \to N$ is an orientation-reversing isometry. Then

$$\int_M f(Tx)dx = \int_M (T^* f)\mathrm{vol}_M \overset{1}{=} -\int_M T^*(f\mathrm{vol}_N) \overset{2}{=} \int_N f\mathrm{vol}_N = \int_N f(x)dx.$$

The equation 1 holds since $T^*\mathrm{vol}_N = -\mathrm{vol}_M$ because T reverses orientation; the equation 2 holds because the integral of forms depends on the orientation. Another way to express this is by saying that $f \geq 0$, then $\int_M f \geq 0$. We might extend the definition of the integral to nonoriented Riemann manifolds as follows. Let $\pi : \tilde{M} \to M$ be the orientation cover. The manifold $\tilde{M}$ has a canonical orientation. The Riemann metric on M gets pulled back to $\tilde{M}$, and the unique nontrivial Deck transformation T of the covering $\tilde{M} \to M$ becomes an orientation-reversing isometry of $\tilde{M}$. Now we define

$$\int_M f(x)dx := \frac{1}{2}\int_{\tilde{M}} (f \circ \pi)(x)dx.$$

The factor 1/2 guarantees that the new integral coincides with the old one on oriented manifolds. It is easy to see $\int_M f(x)dx \geq 0$ for $f \geq 0$, and also that if $f \geq 0$ has integral 0, then $f = 0$.

Note that this procedure does *not* give a sensible procedure to integrate *n-forms* on a nonoriented manifold. If $\omega \in \mathcal{A}^n(M)$, then $\int_{\tilde{M}} \pi^*\omega = -\int_{\tilde{M}} T^*\pi^*\omega = -\int_{\tilde{M}} \pi^*\omega$, and so the integral is zero.

Definition 2.3.3. Let $E \to M$ be a hermitian vector bundle on a Riemannian manifold. Let $s, t \in \Gamma_c(M; E)$. By

$$\langle s, t\rangle := \int_M (s(x), t(x))dx,$$

we define an inner product on the space of compactly supported sections of E, which therefore becomes a pre-Hilbert space. We let $L^2(M; E)$ be the Hilbert space obtained by completing $\Gamma_c(M; E)$ with respect to the norm given by this scalar product.

Remark 2.3.4. We can make contact to measure theory as follows. The integral is a functional $C_c^0(M) \to \mathbb{C}$ and has the property that $\int_M f(x)dx \geq 0$ if $f \geq 0$. Thus, by the Riesz representation theorem, [24], Thm 2.14, there is a unique measure on the σ-algebra of Borel sets that gives this functional by integration. The usual theorems from Analysis III show that we can view the elements of the Hilbert space $L^2(M, E)$ as measurable sections of the vector bundle E.

Definition 2.3.5. Let M be riemannian and $E_0, E_1 \to M$ be hermitian, and $P \in \mathrm{Diff}^k(E_0, E_1)$. A *formal adjoint* P^* is a differential operator $P^* : \Gamma(M, E_1) \to \Gamma(M; E_0)$ such that for all compactly supported sections s, t, we have $\langle s, P^*t\rangle = \langle Ps, t\rangle$. A differerential operator P is *formally selfadjoint* if $E_0 = E_1$ and if $P^* = P$.

Theorem 2.3.6.

(1) *The adjoint, if it exists, satisfies* $(PQ)^* = Q^*P^*$, $(Q+P)^* = Q^* + P^*$, $(aP)^* = \bar{a}P^*$.
(2) *The adjoint is uniquely determined.*
(3) *Each differential operator* $P \in \mathrm{Diff}^k(E_0, E_1)$ *has an adjoint in* $\mathrm{Diff}^k(E_1, E_0)$.
(4) *The symbol of the adjoint can be computed pointwise:* $\mathrm{smb}_k(P^*)(\xi) = (\mathrm{smb}_k(P)(\xi))^*$, *where we used the adjoint of a vector bundle homomorphism.*

Before we give the proof, let us remark that $\langle s, t\rangle$ is defined if only one of the sections s and t has compact support. Moreover, if P^* is an adjoint of P, then $\langle Ps, t\rangle = \langle s, P^*t\rangle$ holds if only one section has compact support.

Proof. The first part is trivial. If P^*, P' are two adjoints, one has

$$\langle P^*s, t\rangle = \langle s, Pt\rangle = \langle P's, t\rangle$$

or

$$\langle (P^* - P')s, t\rangle = 0$$

for all sections s and t, which proves that $P^*s = P's$ for all s.

For the existence of adjoints, we first prove that forming adjoints is a *local* procedure, despite its appearance. So let $U, V \subset M$ be open, and let $(P|_U)$ and $(P|_V)$ be the restrictions of P to U, V. Assume that there exist adjoints $(P|_U)^*$ and $(P|_V)^*$. We now assert that $(P|_U)^*|_{U\cap V} = (P|_V)^*|_{U\cap V}$, in other words, the restrictions of the adjoints to the intersection agree. Let s, t be sections supported in $U \cap V$. Then

$$\langle (P|_U)^*s, t\rangle = \langle s, Pt\rangle = \langle (P|_V)^*s, t\rangle$$

which proves the assertion. Assume that (U_i) is an open covering of M and P_i^* an adjoint of $P|_{U_i}$. These operators fit together to a (differential) operator Q. We claim that Q is an adjoint of P. Let s, t be compactly supported sections and let (μ_i) be a partition of unity subordinate to (U_i). Then

$$\langle Ps, t\rangle = \sum_{i,j} \langle P\mu_i s, \mu_j t\rangle = \sum_{i,j} \langle \mu_i s, Q\mu_j t\rangle = \langle s, Qt\rangle$$

(note that the sums are finite).

To find the adjoints of the localized operators, we can assume that $M = \mathbb{R}^n$, but we cannot assume that the metric on M is the standard metric. Moreover, each complex vector bundle has local trivializations which are *isometric*, i.e. preserve the inner product. To see this, begin with any local trivialization (which yields a local basis) and apply the Gram-Schmidt process to it. The Gram-Schmidt process produces a smooth orthonormal local basis. So, we assume that P is a differential operator on trivial vector bundles, with the standard bundle metric, over $\mathbb{R}^n$, with some metric.

We can write $P = \sum_{|\alpha|\leq k} A^\alpha(x) D^\alpha$, with some smooth matrix-valued functions A^α. One way to argue from here is that D^i has an adjoint. Another way is that P is a sum of operators, each of which is given by matrix-multiplication (order 0 operator) and a composition of vector fields. Thus it is enough to prove that order 0 operators have adjoints (which is clear: take the pointwise adjoint) and that vector fields X have adjoints.

Let f, g be two compactly supported functions. By Stokes theorem, we have

$$0 = \int_M d(\iota_X(\bar{f} g \mathrm{vol})) = \int_M L_X(\bar{f} g \mathrm{vol}) = \int_M (X\bar{f})(g\mathrm{vol}) + \int_M \bar{f}(Xg)\mathrm{vol} + \int_M \bar{f} g L_X \mathrm{vol}.$$

We define the *divergence* $\mathrm{div}(X)$ of the vector field X as the unique function such that $\mathrm{div}(X)\mathrm{vol}_M = L_X \mathrm{vol}_M$. This, together with $X\bar{f} = \overline{Xf}$, shows that

$$\langle Xf, g\rangle = \int_M \overline{Xf} g \mathrm{vol} = -\int_M \bar{f}(Xg)\mathrm{vol} - \int_M \bar{f} g L_X \mathrm{vol} = -\langle f, Xg\rangle - \langle f, \mathrm{div}(X) g\rangle;$$

in other words, that $-X - \mathrm{div}(X)$ is an adjoint of X.

The formula for the symbols follows because it is true for order 0 operators and for vector fields:

$$\mathrm{smb}_1(X^*) = -\mathrm{smb}_1(X).$$

Since the symbol $\mathrm{smb}_1(X)$ is skew-adjoint (since it is purely imaginary), the formula for the symbol of an adjoint follows. □

Let (E_*, P) be an elliptic complex over the Riemannian manifold M. We assume that each bundle $E_i \to M$ has a hermitian bundle metric. We get a bundle metric on the direct sum $\bigoplus_i E_i$, by requiring that the vector bundles E_i and E_j are orthogonal if $i \neq j$. By taking the direct sum of the operators P_i, we get an operator $P : \bigoplus_i \Gamma(M; E) \to \bigoplus_i \Gamma(M; E_i)$ and the adjoint P^*. We write $E^{ev} := \bigoplus_i E_{2i}$ and $E^{odd} := \bigoplus_i E_{2i+1}$. Note that P maps $\Gamma(M, E^{ev})$ into $\Gamma(M, E^{odd})$ and vice versa. The same applies to the adjoint.

Proposition 2.3.7. *Let (E_*, P) be an elliptic complex and $E = \oplus_{i\geq 0} E_i$. Then the operator $P + P^*$ on $\Gamma(M; E)$ is elliptic (and formally self-adjoint). Hence the restricted operators $P + P^* : \Gamma(M, E^{ev}) \to \Gamma(M, E^{odd})$ and $P + P^* : \Gamma(M, E^{odd}) \to \Gamma(M, E^{ev})$ are elliptic.*

This follows immediately, using Theorem 2.3.6 (4), from the following linear algebraic lemma.

Lemma 2.3.8. *Let $0 \to V_0 \overset{f_1}{\to} V_1 \overset{f_2}{\to} \ldots V_n \to 0$ be a cochain complex of finite dimensional hermitian vector spaces. Then the following are equivalent:*

1. *The complex is exact.*
2. *The linear map $f + f^* : V_* \to V_*$ is an isomorphism.*

Proof. $2 \Rightarrow 1$: The maps f^* define a chain homotopy, from 0 to $f^* f + f f^* = (f + f^*)^2$. If $f + f^*$ is an isomorphism, then so is $(f + f^*)^2$ and this isomorphism is chain homotopic to 0. Thus the zero map induces an isomorphism on cohomology and the complex is exact.

$1 \Rightarrow 2$: Let $(f + f^*)x = 0$. Then $(f f^* + f^* f)x = 0$ and therefore

$$0 = \langle f f^* x; x\rangle + \langle f^* f x; x\rangle = \langle f^* x; f^* x\rangle + \langle f x; f x\rangle,$$

which is why $fx = f^* x = 0$. Since the complex is exact, there is y with $fy = x$, and y satisfies $f^* f y = 0$. Thus

$$0 = \langle f^* f y; y\rangle = \langle f y; f y\rangle$$

which implies $x = fy = 0$. Therefore $f + f^*$ is injective, and surjective by the finiteness assumption.

$\square$

3. Analysis of elliptic operators I

We now have to delve into some nontrivial analysis. The goal is to state precisely and prove the following two theorems.

Theorem 3.0.9. *(Local regularity) If P is an elliptic differential operator, f a smooth section and let $Pu = f$. Then u is smooth.*

At the moment, we only know what P should do to a smooth section, and so as stated, the Theorem is quite tautological. One way to phrase the theorem in a nontrivial way is to assume that u is only C^k (in which case Pu still makes sense, if k is the order of P). One example for this result is the well-known result from complex analysis that a holomorphic function (which is assume to be C^1) has to be C^∞.

In fact, this is not the intended precise formulation of the local regularity theorem. What we will do is to introduce Hilbert spaces of "weakly differentiable functions", the *Sobolev spaces*, on which we can give a meaning to the equation $Pu = f$, when u and f are Sobolev functions. The local regularity theorem will then say that any solution in the Hilbert space sense is actually a smooth section.

The second main theorem of this section is

Theorem 3.0.10. *If P is an elliptic differential operator on a closed manifold, then $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ is a Fredholm operator.*

Even if the theorem as stated is perfectly true, what we really prove is that P induces a map of certain Sobolev spaces, and that this map is a Fredholm operator (in the Hilbert space sense).

3.1. Preliminaries: Convolution and Fourier transformation.

Convention 3.1.1. Let dx be the normalized Lebegue measure on $\mathbb{R}^n$, such that the unit cube $[0,1]^n$ has measure $(2\pi)^{-n/2}$. The effect is that the Gaussian integral is normalized

$$\int_{\mathbb{R}^n} e^{-x^2/2} dx = 1$$

and that a lot of factors of the form $(2\pi)^{\pm n/2}$ disappear.

For a complex valued function f on $\mathbb{R}^n$, we have the Lebesgue norms

$$\|f\|_{L^p} := (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$$

whenever this makes sense. If f is a C^k function, we let

$$\|f\|_{C^k} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)|$$

which is only meaningful if all derivatives up to order k are bounded.

Definition 3.1.2. Let $f, g \in L^1(\mathbb{R}^n)$. The *convolution* $f * g$ is the function

$$f * g(x) := \int_{\mathbb{R}^n} f(y) g(x-y) dy = \int_{\mathbb{R}^n} f(x-z) g(z) dz = g * f(x).$$

The following important properties will be used many times. For the proof, see [17], p. 223 ff.

Proposition 3.1.3.

(1) *$f * g \in L^1$.*
(2) *The convolution is associative, commutative and bilinear.*
(3) *If $f \in L^1$, $g \in L^p$, then $f * g \in L^p$ and $\|f * g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$ $(1 \le p \le \infty)$.*
(4) *If f is smooth with compact support and $g \in L^1$, then $f * g$ is smooth and $D^\alpha(f * g) = (D^\alpha f) * g$.*

Proposition 3.1.4. *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a function with $\phi \ge 0$ and $\int \phi(x)dx = 1$. Let, for $t > 0$, $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$. Let $f \in L^p$. Then $\phi_t * f$ converges, for $t \to 0$, to f, in the L^p-norm. $(p < \infty)$.*

If f is a bounded continuous function, then $\phi_t * f \to f$ uniformly on all compact subsets.

Corollary 3.1.5. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p < \infty$.*

Proof. Let a_n be sequence of bump functions with increasing support. Then, for each $f \in L^p$, $a_n f$ is in L^p, and $a_n f \to f$ in L^p. Thus it is enough to approximate a function with compact support by smooth functions. Let f be such a function. The function $\phi_t * f$ has compact support, and for $t \to 0$, it converges to f. $\square$

Definition 3.1.6. The space $\mathcal{S}(\mathbb{R}^n)$ of *Schwartz functions* is the space of all smooth functions f such that for all multiindices α, β, the function $x^\alpha D^\beta f$ is bounded.

Examples: functions with compact support, e^{-x^2}.

Definition 3.1.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$. The *Fourier transform* of f is defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx.$$

We remark that one should really view ξ as a point in the dual space of $\mathbb{R}^n$. Also, the functions $e^{-ix\xi}$ are the *characters* of the group $\mathbb{R}^n$, i.e. the continuous homomorphisms $\mathbb{R}^n \to U(1)$. From that viewpoint, the Fourier transformation is a special case of harmonic analysis on locally compact abelian groups.

For $j \in \underline{n}$, we compute

$$\widehat{D^j f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} D^j f(x)dx = -\int_{\mathbb{R}^n} \frac{-i}{i}\xi_j e^{-ix\xi} f(x)dx$$

by partial integration, from which one sees that

$$\widehat{D^j f}(\xi) = \xi_j \hat{f}(\xi)$$

and inductively

$$\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi) \tag{3.1.8}$$

For each Schwartz function f, one clearly has $|\hat{f}(\xi)| \le \|f\|_{L^1}$. Because $D^j f$ is again a Schwartz function, we find that $\xi_j \hat{f}(\xi)$ is a bounded function, and iteratively, we find that $\xi^\alpha \hat{f}$ is a bounded function, for all α. Thus the Fourier transform of a Schwartz function is rapidly decreasing. By the Lebesgue dominated convergence theorem, the function $\hat{f}$ is differentiable (since the ξ-derivative of the integrand is a Schwartz function and hence L^1), and by differentiating under the integral:

$$D^j \hat{f}(\xi) = -\widehat{x_i f}(\xi).$$

By induction, one finds that $\hat{f}$ is smooth and

$$D^\alpha \hat{f} = (-1)^{|\alpha|} \widehat{x^\alpha f}(\xi). \tag{3.1.9}$$

Therefore, the Fourier transform of a Schwartz function is a Schwartz function.

Example 3.1.10. The function $h(x) = e^{-x^2/2}$ is its own Fourier transform. One checks this by differentiating the function $e^{\xi^2/2}\hat{h}(\xi)$ using the rules just found and the fact that $\hat{h}(0) = 1$ (normalization of the integral!).

We introduce two scaling operators. Let f be a function on $\mathbb{R}^n$. We set, for $t > 0$,

$$f_t(x) := \frac{1}{t^n} f(x/t);\; f^t(x) := f(tx).$$

It is easy to check that

$$\widehat{f_t} = (\hat{f})^t \text{ and } \widehat{f^t} = (\hat{f})_t. \tag{3.1.11}$$

Proposition 3.1.12.

(1) *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$.*
(2) $\widehat{f * g} = \hat{f}\hat{g}$.

Proof. The first part can be found in [17], p. 239. For the other part, compute

$$\widehat{f * g}(\xi) = \int \int f(y) g(x-y) e^{-ix\xi} dy dx.$$

Since f and g are Schwartz functions, the integrand is in $L^1(\mathbb{R}^{2n})$, and we can apply Fubini's theorem and the change of variables $y \to y$, $x \to z + y$ to obtain

$$\int \int f(y) g(z) e^{-i(z+y)\xi} dz dy = \hat{f}(\xi)\hat{g}(\xi).$$

□

For a Schwartz function f, we define $f^-(x) := f(-x)$. It is straightforward to show

$$\hat{f}^- = \widehat{f^-};\; (f * g)^- = f^- * g^-;\; (fg)^- = f^- g^-.$$

Theorem 3.1.13. *(The Fourier inversion formula) For all Schwartz functions f, we have $\hat{\hat{f}} = f^-$. More explicitly, $f(x) = \int_{\mathbb{R}^n} e^{ix\xi}\hat{f}(\xi) d\xi$.*

Proof. Let $g \in \mathcal{S}(\mathbb{R}^n)$. We compute

$$\int \hat{f}(\xi) e^{-ix\xi} g(\xi) d\xi = \int \int f(y) e^{-i(y+x)\xi} g(\xi) dy d\xi = \int f(y) \hat{g}(x+y) dy$$

since the integrand is in $L^1(\mathbb{R}^{2n})$ (this was the purpose of introducing the integrating factor g). Inserting g^t for g, we obtain, using 3.1.11

$$\int \hat{f}(\xi) e^{-ix\xi} g^t(\xi) d\xi = \int f(y) \frac{1}{t^n} \hat{g}(\frac{x+y}{t}) dy \overset{y=tu-x}{=} \int f(tu - x) \hat{g}(u) du. \tag{3.1.14}$$

Now specialize to the case $g(\xi) = e^{-|\xi|^2/2}$ and consider the limit $t \to 0$. The integrand on the right-hand side of 3.1.14 converges pointwise to $f(-x)\hat{g}$, and is bounded by $\|f\|_{L^\infty} g$, and so by the dominated convergence theorem, the limit becomes

$$\lim_{t\to 0} \int f(tu - x)\hat{g}(u)du = f(-x) \int \hat{g}(u)du = f(-x)$$

by the normalization of the Lebesgue measure. The integrand of the left-hand side of 3.1.14 is bounded by the L^1-function $\hat{f}$, and $g^t(\xi) \to 1$ as $t \to 0$. So by the dominated convergence theorem we obtain

$$\lim_{t\to 0} \int \hat{f}(\xi)e^{-ix\xi} g^t(\xi)d\xi = \int \hat{f}(\xi)e^{-ix\xi}d\xi = \hat{\hat{f}}(x),$$

which was to be shown. □

Corollary 3.1.15.

(1) $\widehat{fg} = \hat{f} * \hat{g}$.
(2) *The map $f \mapsto \hat{f}$ is a bijective map $\mathcal{S} \to \mathcal{S}$.*

Proof. The second part is clear; since twice the Fourier transform is the reflection map $f \mapsto f^-$ and so bijective. By the bijectivity, we find using 3.1.12

$$\widehat{fg} = \hat{f} * \hat{g} \Leftrightarrow \widehat{\widehat{fg}} = (\hat{f} * \hat{g})\hat{} = \hat{\hat{f}}\hat{\hat{g}} = f^- g^-$$

which is true. □

Recall that the L^2-inner product is given by

$$\langle f, g\rangle := \int f(\bar{x})g(x)dx.$$

Theorem 3.1.16. *(The Plancherel theorem) For all Schwartz functions f, g, we have*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle.$$

Proof. Let us momentarily denote $(f, g) := \langle \bar{f}, g\rangle$. The first step is

(3.1.17)
$$(\hat{f}, g) = \int \hat{f}(x)g(x)dx = \int\int f(\xi)e^{-ix\xi}g(x)d\xi dx = \int f(\xi)\hat{g}(\xi)d\xi = (f, \hat{g})$$

where we used Fubini. Note that there are no conjugation signs. It is easy to see that

$$\overline{\hat{f}(\xi)} = \hat{\bar{f}}^-(\xi).$$

Compute, using 3.1.17,

$$(\bar{f}, g) = (\bar{f}, \hat{\hat{g}}^-) = (\hat{\bar{f}}, \hat{g}^-) = ((\overline{\hat{f}})^-, \hat{g}^-) = (\overline{\hat{f}}, \hat{g})$$

and this proves the theorem. □

Corollary 3.1.18. *The Fourier transform extends to an isometry $L^2 \to L^2$*

This is clear because the Schwartz space lies dense in L^2 (since it contains $C_c^\infty(\mathbb{R}^n)$). Warning: the defining formula for the Fourier transform only holds for functions in $L^1 \cap L^2$.

3.2. Sobolev spaces in $\mathbb{R}^n$.

Definition 3.2.1. On the Schwartz space, we introduce the *Sobolev norm*, for each $s \in \mathbb{R}$, by

$$\|f\|^2_{W^s} := \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

If it is clear that we mean the Sobolev norm (and not an L^p or C^k-norm), we write $\|f\|_s$. The *Sobolev space* $W^s(\mathbb{R}^n)$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the s-norm. If $U \subset \mathbb{R}^n$ is open, we let $W^s(U)$ be the closure of $C^\infty_c(U) \subset \mathcal{S}(\mathbb{R}^n)$. We can define Sobolev spaces of vector valued functions, using a scalar product on the value space.

We often also omit the domain of definition of the functions and write simply W^s.

Lemma 3.2.2.
(1) $W^0 = L^2$.
(2) *For $s \geq t$, we have $\|f\|_t \leq \|f\|_s$ and hence get a continuous map $W^s \to W^t$.*
(3) *This inclusion map is injective.*

Proof. The first and second part are obvious (by the Plancherel theorem and because the smooth functions with compact support lie dense in L^2, so does the Schwartz space). For the third part, let f_n be a $\| \|_s$-Cauchy sequence of Schwartz functions and assume that $\|f_n\|_t \to 0$. Let $g_n := |\hat{f}_n|^2(1+|\xi|^2)^t$ and let $\mu = (1+|\xi|^2)^{s-t}$, which is a nonzero function.Taking the Fourier transforms and the definition of the Sobolev norm, we obtain from our assumptions that

$$\|g_n\|_{L^1} \to 0; \; \|\mu g_n - \mu g_m\|_{L^1} \to 0.$$

Since μg_n is an L^1-Cauchy sequence, it converges almost everywhere (by [17], Theorem VI.5.2), and also g_n converges almost everywhere, namely to 0. Since μg_n is a Cauchy sequence, and its pointwise limit is 0, it follows that $\|\mu g_n\|_{L^1} \to 0$, by [17], Corollary VI.5.4. □

Remark 3.2.3. We will use the third part only once, but in a crucial way: it is this statement that allows to go from arguments in the completion to actual functions. The use of the nontrivial relationship of L^1-convergence with pointwise convergence is important. Viewing L^1 in a purely abstract fashion as a completion is possible, but insufficient for the applications. If $s > 0$, we have the inclusion $W^s \to L^2$, which allows us to consider elements of W^s as actual functions. If $s < 0$, no such interpretation is possible. In fact, we will see soon that the famous Dirac δ-"function" is an element of W^s for $s << 0$. The only way to reinterprete elements in W^s for negative s is as distributions. However, the theory of distributions requires more background and we rather avoid it. This does not mean that the Sobolev spaces W^s, $s < 0$, are irrelevant for us, but need to be treated with care.

Lemma 3.2.4. *If $s > 0$ is an integer, the Sobolev norm is equivalent to the norm $\|f\|_s = \sum_{\alpha \leq s} \|D^\alpha f\|_{L^2}$. Hence any differential operator P with constant coefficients of order k induces a continuous map $W^s \to W^{s-k}$.*

Proof. There are constants $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^n$

$$c_1(1+|x|)^s \leq \sum_{|\alpha| \leq s} x^\alpha \leq c_2(1+|x|)^s$$

holds (this is the standard growth estimate for polynomials). Together with the definition of the Sobolev norm, the rules for the Fourier transform and the Plancherel theorem, we get that the Soboloev norm is equivalent to the norm $\|f\|^2 = \sum_{\alpha\leq s} \|D^\alpha f\|_{L^2}^2$. Then one uses the inequalities (true for all $a_1,\dots,a_k \in \mathbb{R}$) $\sum_{i=1}^k a_i^2 \leq (\sum_{i=1}^k |a_i|)^2 \leq k \sum_{i=1}^k a_i^2$ which follow from the Cauchy-Schwarz inequality. □

Example 3.2.5. Let L be the differential operator

$$Lf := f - \Delta f = f + \sum_{i=1}^n (D^i)^2 f.$$

By the basic rules for the Fourier transform, we have

$$\widehat{Lf}(\xi) = (1+|\xi|^2)\hat{f}(\xi).$$

Therefore, $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is invertible with inverse M given by

$$\widehat{Mf}(\xi) := (1+|\xi|^2)^{-1}\hat{f};$$

moreover, the identity

$$\|Lf\|_s = \|f\|_{s+2} \tag{3.2.6}$$

holds. This will be used later.

Theorem 3.2.7. *(The Sobolev embedding theorem) Let $s > k + \frac{n}{2}$. Then there is a constant C such that $\|u\|_{C^k} \leq C\|u\|_s$ for all $u \in \mathcal{S}$. Hence (since C^k is complete), we get a continuous inclusion $W^s \to C^k$.*

Proof. Let $|\alpha| = l \leq k$ and $x \in \mathbb{R}^n$ and $u \in \mathcal{S}$. Then

$$|D^\alpha u|(x) = |\int \xi^\alpha \hat{u} d\xi| \leq \int |\xi|^l |\hat{u}(\xi)|(1+|\xi|^2)^{s/2}(1+|\xi|^2)^{s/2} d\xi \leq$$

$$(\int |\hat{u}(\xi)|^2(1+|\xi|^2)^s)^{1/2}(\int |\xi|^{2l}(1+|\xi|^2)^{-s})^{1/2} \leq (\int |\xi|^{2l}(1+|\xi|^2)^{-s})^{1/2}\|u\|_s$$

by the Cauchy-Schwarz inequality, provided that $|\xi|^l(1+|\xi|^2)^{s/2}$ is an L^2-function. But

$$\int |\xi|^{2l}(1+|\xi|^2)^{-s} \leq \int_0^\infty \int_{S^{n-1}} r^{2l}(1+r^2)^{-s} r^{n-1},$$

and this is finite if (and only if) $2l - 2s + n - 1 < -1$, i.e. $s > l + \frac{n}{2}$. □

Theorem 3.2.8. *(The Rellich compactness theorem) Let $U \subset \mathbb{R}^n$ be relatively compact and $s > t$. Then $W^s(U) \to W^t$ is a compact operator.*

Proof. We have to prove: if u_n is a sequence of smooth functions supported in U and if $\|u_n\|_s \leq 1$, then a subsequence of u_n converges in the t-norm (think a moment on the definition of a compact operator to see why this is enough). Pick a compactly supported function a such that $a|_U \equiv 1$, so that $au_k = u_k$ for all k. It follows that $\hat{u_k} = \hat{a} * \hat{u_k}$ and

$$D^j \hat{u_k} = (D^j \hat{a}) * \hat{u_k}.$$

Thus we can estimate $D^j \hat{u_k}(\xi)$ by

$$|D^j \hat{a} * \hat{u_k}(\xi)| \le \int |D^j \hat{a}(\xi - \eta)\hat{u_k}(\eta) d\eta| \le$$

$$(\int |D^j \hat{a}(\xi-\eta)|^2 (1+|\eta|^2)^{-s}))^{1/2} (\int |\hat{u_k}(\eta)|^2 (1+|\eta|^2)^s)^{1/2} \le C\|u_k\|_s$$

by Cauchy-Schwarz. The first integral exists since a is a Schwartz function. So $D^j \hat{u_k}(\xi)$ is uniformly (in k!) bounded, and by a similar argument shows $\hat{u_k}(\xi)$ is bounded. It follows that the family $\hat{u_k}$ is equicontinuous, and Arzela-Ascoli provides a subsequence, also called $\hat{u_k}$, that converges uniformly on compact subsets of $\mathbb{R}^n$. We claim that u_k is a W^t-Cauchy sequence. Let $\epsilon > 0$. Let us compute

$$\|u_k - u_l\|_t^2 = \int_{|\xi| \le R} |\hat{u_k} - \hat{u_l}|^2 (1+|\xi|^2)^t d\xi + \int_{|\xi|\ge R} |\hat{u_k} - \hat{u_l}|^2 (1+|\xi|^2)^t d\xi.$$

On the part $|\xi| \ge R$, we estimate $(1+|\xi|^2)^t \le (1+|\xi|^2)^s (1+R^2)^{t-s}$. Thus

$$\int_{|\xi|\ge R} |\hat{u_k} - \hat{u_l}|^2 (1+|\xi|^2)^t d\xi \le (1+R^2)^{t-s} (\|u_k\|_s^2 + \|u_l\|_s^2) \le 2(1+R^2)^{t-s}$$

and since $t-s<0$, we can make this term smaller than $\epsilon/2$, by choosing R sufficiently large. Because $\hat{u}_k$ is uniformly convergent on $\{|\xi| \le R\}$ and $(1+|\xi|^2)^t$ is bounded on this set, the integral

$$\int_{|\xi|\le R} |\hat{u_k} - \hat{u_l}|^2 (1+|\xi|^2)^t d\xi$$

converges to zero as $k, l \to \infty$. □

We already said several times that Hilbert spaces are self-dual via the scalar product, and the Sobolev spaces are Hilbert spaces. However (at the moment this is not yet clear), the actual norm on the Sobolev space is negotiable (it is only a "Hilbertian space" in Bourbakian rigor). Only when we consider operators on Riemannian manifolds, the scalar product on L^2 will have an intrinsic meaning. Nevertheless, the self-duality is important for Sobolev spaces, and it takes the form of a perfect pairing $W^s \times W^{-s} \to \mathbb{C}$. Often, a statement is easier to prove on one side of the Sobolev chain, and duality allows us to pass to the other side.

Proposition 3.2.9. *(Duality) The sesquilinear form*[2] $\mathcal{S} \times \mathcal{S} \to \mathbb{C}$, $(f,g) \mapsto \int \overline{f(x)} g(x) dx$ *extends to a pairing* $W^s \times W^{-s} \to \mathbb{C}$ *and satisfies*

(1) $|(f,g)| \le \|f\|_s \|g\|_{-s}$.
(2) $\|f\|_s = \sup_{g \ne 0} \frac{|(f,g)|}{\|g\|_{-s}} = \sup_{\|g\|_{-s} \le 1} |(f,g)|$.
(3) *The induced map* $W^{-s} \to (W^s)'$, $g \mapsto (f \mapsto (f,g))$ *is an isometric isomorphism.*

Proof. By Plancherel and Cauchy-Schwarz:

$$(f,g) = (\hat{f}, \hat{g}) = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} (1+|\xi|^2)^{s/2} \hat{g}(\xi) (1+|\xi|^2)^{-s/2} d\xi \le \|f\|_s \|g\|_{-s}.$$

The inequality $\sup_{\|g\|_{-s}\le 1} |(f,g)| \le \|f\|_s$ follows immediately. If f is a Schwartz function, define g by $\hat{g} = \hat{f}(1+|\xi|^2)^s$. Then $(f,g) = (\hat{f}, \hat{g}) = \|f\|_s^2$ by the definition of the Sobolev norm and the Plancherel theorem. Moreover $\|g\|_{-s}^2 = \|f\|_s^2$, in other words

[2]Before, we have denoted this by $\langle f, g \rangle$. The reason for the notation switch is that here the pairing plays a different role.

$$\frac{|(f,g)|}{\|g\|_{-s}} = \|f\|_s$$

and this proves the other inequality.

For the third part, it follows easily from what we already proved that $\phi : W^{-s} \to (W^s)^*$ is norm-preserving. Thus it has closed image. What we have to show is the following claim

- If V is a Hilbert space, and $H \subset V^*$ a closed subspace, such that for all $v \in V$, we have $\|v\| = \sup_{x \in H} \frac{|x(v)|}{\|x\|}$. Then $H = V^*$.

To prove the claim, translate it using the self-duality of Hilbert spaces to the following statement:

- H Hilbert space, $W \subset H$ closed, such that $\|x\|_H = \sup_{w \in W, \|w\|=1} |(x,w)|$. Then $W = H$.

This is easy. Assume that $W^\perp \neq 0$ and pick an element in $W^\perp$ of norm 1 to get a contradiction. □

3.3. **The fundamental elliptic estimate.** In this subsection, we will prove the following two results.

Proposition 3.3.1. *Let P a differential operator of order k, with compact support. Then for each $s \in \mathbb{Z}$, P induces a bounded operator $P : W^s \to W^{s-k}$. Moreover, if the coefficients of P depend smoothly on a parameter $t \in \mathbb{R}$, then $t \mapsto P_t$ is a continuous map $\mathbb{R} \to \mathrm{Lin}(W^s, W^{s-k})$.*

The other result is of fundamental importance for the proof of the regularity theorem.

Theorem 3.3.2. *(Garding inequality) Let P be a differential operator of order k on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be relatively compact and assume that P is elliptic over $\bar{U}$. Then there exists a constant C, depending on P, U and $s \in \mathbb{Z}$, such that for each $u \in C_c^\infty(U)$ with support in U, the* elliptic estimate

$$\|u\|_s \leq C(\|u\|_{s-k} + \|Pu\|_{s-k})$$

holds.

The proofs of both results rely on an estimate for the multiplication operator $f \mapsto af$ on $\mathcal{S}$, when a has compact support. We will show below that for each $a \in C_c^\infty(\mathbb{R}^n)$, multiplication by a induces a continuous map $W^s \to W^s$, for each $s \in \mathbb{Z}$. Together with Lemma 3.2.4, this proves the result. However, we need two more precise estimates on the operator norm of D. The first estimate is used to show that if the coefficients of D depend smoothly on a parameter, then the induced operators depend continuously on the parameter. This will eventually prove that the indices of two operators whose symbols are homotopic will agree. The second estimate will be used in the proof of Gardings inequality.

Proposition 3.3.3. *Let $a \in C_c^\infty$. Then $f \mapsto af$ extends to a bounded operator $M_a : W^s \to W^s$, for each $s \in \mathbb{Z}$. More precisely, we have the following estimates:*

1. *$\|au\|_s \leq C\|a\|_{C^{|s|}}\|u\|_s$, for all $s \in \mathbb{Z}$ and a constant C independent of a.*
2. *$\|au\|_s \leq \|a\|_{C^0}\|u\|_s + C(a)\|u\|_{s-1}$. In other words, the "leading term" can be estimated by the C^0-norm of a, with a "lower perturbation", whose norm depends on higher derivatives of a.*

Proof. For nonnegative k, we compute

$$\|au\|_k^2 = \sum_{|\alpha|\le k} \|D^\alpha(au)\|_0^2 \le \sum_{|\alpha|\le k}\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\gamma!\beta!}\|(D^\beta a)(D^\gamma u)\|_0^2.$$

We can estimate

$$\|(D^\beta a)(D^\gamma u)\|_0 \le \|D^\beta a\|_{C^0}\|D^\gamma u\|_0 \le \|a\|_{C^{|\beta|}}\|u\|_{W^{|\gamma|}}.$$

Thus we find

$$\|au\|_k \le \|a\|_{C^0}\|u\|_k + C\|a\|_{C^k}\|u\|_{k-1}.$$

This is the second estimate for $k \ge 0$, the constants C depends on k and n alone. To get the first estimate, we estimate further

$$\|a\|_{C^0}\|u\|_k + C\|a\|_{C^k}\|u\|_{k-1} \le C'\|a\|_{C^k}\|u\|_k,$$

obtaining the first estimate for positive k. For $-k$, the estimate follows by duality:

$$\|af\|_{-k} = \sup_{\|g\|_k\le 1} |\langle af, g\rangle| = \sup_{\|g\|_k\le 1} |\langle f, \bar{a}g\rangle| \le \sup_{\|g\|_k\le 1} \|f\|_{-k}\|\bar{a}g\|_k \le \sup_{\|g\|_k\le 1} \|f\|_{-k}\|a\|_{C^k}\|g\|_k.$$

Before we prove the second estimate for negative values of k, we note a corollary of the first estimate. $\square$

Corollary 3.3.4. *Let P be a differential operator with compact support, of order k. Then, for each $s\in\mathbb{Z}$, there is a constant $C = C(P,s)$ such that $\|Pu\|_{s-k} \le C\|u\|_s$ holds. The constant C can be bounded by the sum of the C^l-norms of the coefficients of P, with $l = |s-k|$.*

If $I = (t_0, t_1) \subset \mathbb{R}$, and if P_t is a family of order k differential operators that depend smoothly on t, then $I \to \mathrm{Lin}(W^s, W^{s-k})$, $t \mapsto P_t$ is continuous.

Proof. By Proposition 3.3.3, one estimates

$$\|Pu\|_{s-k} \le \sum_{|\alpha|\le k} \|A^\alpha D^\alpha u\|_{s-k} \le \sum_{|\alpha|\le k} \|A^\alpha\|_{C^{|s-k|}}\|u\|_{s-|\alpha|}.$$

This proves the first assertion. The second is an easy consequence, as the differentiability assumption on P_t shows that the derivatives of the coefficients of P_t depend continuously on t. $\square$

End of the proof of Proposition 3.3.3. We have to prove the estimate $\|au\|_s \le \|a\|_{C^0}\|u\|_s + C(a)\|u\|_{s-1}$ for negative integers s and do so by downwards induction on s; the case $s \ge 0$ has been settled before. We make use of the operator L discussed in 3.2.5 and the fact that L is an isometry $W^{s+2} \to W^s$, for all s. Namely, we estimate (under the assumption that the proof has been given for all $t > s$)

$$\|aLu\|_s \le \|[a,L]u\|_s + \|L(au)\|_s \le C(a)\|u\|_{s+1} + \|L(au)\|_s =$$

(by Corollary 3.3.4)

$$= C(a)\|Lu\|_{s-1} + \|au\|_{s+2} \le$$

(by 3.2.6)

$$\leq C(a)\|Lu\|_{s-1} + \|a\|_{C^0}\|u\|_{s+2} + C'(a)\|u\|_{s+1} \leq C''(a)\|Lu\|_{s-1} + \|a\|_{C^0}\|Lu\|_s.$$

Since L is an isomorphism, this finishes the proof. □

Corollary 3.3.5. *Let P be a differential operator with compact support of order k and $s \in \mathbb{Z}$. Then, for all $u \in W^s$,*

$$\|Pu\|_{s-k} \leq C_1\|u\|_s + C_2\|u\|_{s-1},$$

where the constant C_1 can be bounded by the sum of the C^0-norms of the coefficients of P, and C_2 does depend on the higher derivatives of the coefficients of P (but not on u).

Proof. We can write $P = \sum_{|\alpha|=k} D^\alpha a_\alpha + Q$ with an operator of order $k-1$. By Corollary 3.3.4, $\|Qu\|_{s-k} \leq C\|u\|_{s-1}$ for some constant $C = C(Q)$ depending on the coefficients of Q. On the other hand, by the second estimate of Proposition 3.3.3,

$$\|D^\alpha a_\alpha u\|_{s-k} \leq \|a_\alpha u\|_s \leq \|u\|_s\|a_\alpha\|_{C^0} + C\|u\|_{s-1}$$

with $C = C(a)$ depending on a and its derivatives. Putting both estimates together, we obtain the claimed estimate. □

For the proof of the Garding inequality, we need another preliminary fact.

Lemma 3.3.6. *(Peter and Paul estimate) Let $r < s < t \in \mathbb{R}$. Then for each $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that for all $u \in \mathcal{S}$, the estimate*

$$\|u\|_s \leq \epsilon\|u\|_t + C(\epsilon)\|u\|_r$$

holds.

Proof. For $y \geq 1$, the inequality

$$1 \leq y^{t-s} + (1/y)^{s-r}$$

holds because either y or $1/y$ is ≥ 1 and both exponents are positive. For $y = (1+|\xi|^2)\epsilon^{\frac{1}{t-s}}$, we obtain

$$1 \leq (1+|\xi|^2)^{t-s}\epsilon + (1+|\xi|^2)^{r-s}\epsilon^{\frac{r-s}{t-s}}$$

or

$$(1+|\xi|^2)^s \leq (1+|\xi|^2)^t\epsilon + (1+|\xi|^2)^r\epsilon^{\frac{r-s}{t-s}}$$

which implies the claim by integration. □

Proof of Theorem 3.3.2. The proof is a prototypical example of a "local-to-global" argument in analysis. We proceed in three steps:

(1) P has *constant coeffients.* This is much easier (that this is so is one of the two reasons why the whole proof of the local regularity theorem is much simpler for the classical operators on $\mathbb{R}^n$, as the Cauchy-Riemann operator and the Laplace operator).
(2) P has variable coefficients, but the functions are required to have small support.
(3) The general case.

First step: the coefficients are constant

Recall that P can be represented by a function $p(x,\xi)$ which is smooth in the x-variable and a degree k polynomial in the ξ-variable (and has values in matrices).

In the first step, we assume that $p(x,\xi) = p(\xi)$ does not depend on x, in other words, P has constant coefficients. Ellipticity states that there exist c, $R > 0$ such that $|p(\xi)| \geq c(1+|\xi|^2)^{k/2}$ for all $|\xi| \geq R$. Recall that in the Fourier picture, the operator is written as

$$\widehat{Pu}(\xi) = p(\xi)\hat{u}(\xi).$$

Now

$$\|u\|_s^2 = \int_{\mathbb{R}^n - B_R(0)} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s + \int_{B_R(0)} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi.$$

The second summand estimates as

$$\int_{B_R(0)} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi \leq \sup_{|\xi|\leq R} (1+|\xi|^2)^k \int_{B_R(0)} |\hat{u}(\xi)|^2 (1+|\xi|^2)^{s-k} d\xi \leq C_0 \|u\|_{s-k}^2$$

with $C_0 = (1+R^2)^k$. The first summand is

$$\leq \int_{\mathbb{R}^n - B_R(0)} |p(\xi)\hat{u}(\xi)|^2 \frac{1}{c}(1+|\xi|^2)^{s-k} d\xi \leq \frac{1}{c}\|Pu\|_{s-k}^2$$

and this settles the case of constant coefficients (the restriction on the supports of u was not necessary, and it works for each $s \in \mathbb{R}$).

The general case, local version

Let $x_0 \in U$. We claim that there exists a neighborhood $V \subset U$ of x_0 and a constant $C = C(x_0, s)$ such that for all $u \in C_c^\infty(V)$, the estimate $\|u\|_s \leq C(\|u\|_{s-k} + \|Pu\|_{s-k})$ holds.

Let $\delta > 0$. Let P_0 be the differential operator with constant coefficients associated with $p_0(\xi) \coloneqq p(x_0, \xi)$. As we assumed that P is elliptic over U, the operator P_0 is elliptic. Now pick a neighborhood W of x_0 such that all coefficients of $P - P_0$ are bounded by δ on W. Moreover, we pick $x_0 \in V \subset \bar{V} \subset W$ and a bump function λ that is 1 on V and has support in W. If $\operatorname{supp}(u) \subset V$, then by the first part of the proof (and the triangle inequality), we obtain

$$\|u\|_s \leq C_1(\|u\|_{s-k} + \|P_0 u\|_{s-k}) \leq C_1(\|u\|_{s-k} + \|(P_0 - P)u\|_{s-k} + \|Pu\|_{s-k})$$

with C_1 depending on x_0 alone. We now care about the summand $\|(P_0 - P)u\|_{s-k}$. Observe that

$$\|(P_0 - P)u\|_{s-k} = \|(P_0 - P)\lambda u\|_{s-k}$$

since $\lambda u = u$; the operator $(P_0 - P)\lambda$ has compact support and the C^0 norm of all coefficients is bounded by δ. By Corollary 3.3.5, we find

$$\|(P_0 - P)\lambda u\|_{s-k} \leq \delta C_2 \|u\|_s + C_3 \|u\|_{s-1},$$

with a universal constant C_2 and C_3 depending on P *and* δ, because the higher derivatives of the bump function λ become large when δ is small. Let $\epsilon > 0$ and assume $k > 1$ for the moment. By the Peter-Paul estimate, we find C_4 such that

$$C_1C_3\|u\|_{s-1} \le \epsilon\|u\|_s + C_4\|u\|_{s-k}.$$

Putting everything together, we obtain

$$\|u\|_s \le (C_1 + C_4)\|u\|_{s-k} + (\delta C_1C_2 + \epsilon)\|u\|_s + C_1\|Pu\|_{s-k}),$$

and note that all constants except C_4 are independent of δ and ϵ (C_4 depends on both; C_2 is universal and C_1 only depends on x_0). If $k = 1$, the above estimate is still true even with $\epsilon = 0$, without appealing to the Peter-Paul inequality. Pick δ and ϵ small enough so that $\delta C_1C_2 + \epsilon < 1$. Thus

$$(1 - \delta C_1C_2 + \epsilon)\|u\|_s \le (C_1 + C_4)\|u\|_{s-k} + C_1\|Pu\|_{s-k}).$$

Dividing by $(1 - \delta C_1C_2 + \epsilon)$ finishes the second step.

General case, global version

The third step deals with the general case. There exists a larger open $W \supset \bar{U}$ so that P is elliptic over W. Cover $\bar{U}$ by finitely many $V_1, \dots V_m \subset W$ as found in the second part, such that there is a constant C_i with

$$\|u\|_s \le C_i(\|u\|_{s-k} + \|Pu\|_{s-k})$$

whenever $\operatorname{supp}(u) \subset V_i$. Let $C := \max_i C_i$. Pick a finite partition of unity $\mu_1, \dots, \mu_m$ subordinate to the cover by the V_i's. For general u with support in U, we conclude

$$\|u\|_s \le \sum_{i=1}^m \|\mu_i u\|_s \le \sum_{i=1}^m C(\|\mu_i u\|_{s-k} + \|P\mu_i u\|_{s-k}) \le C'\|u\|_{s-k} + C\sum_{i=1}^m \|P\mu_i u\|_{s-k}),$$

the constant C' depending on C and the $C^{|s-k|}$-norm of the functions μ_i. We can estimate

$$\|P\mu_i u\|_{s-k} \le \|[P,\mu_i]u\|_{s-k} + \|\mu_i Pu\|_{s-k} \le C_i''\|u\|_{s-1} + C_i'''\|Pu\|_{s-k}$$

because $[P,\mu_i]$ has order $k-1$ and thus

$$\|u\|_s \le C'\|u\|_{s-k} + C''\|u\|_{s-1} + C'''\|Pu\|_{s-k}.$$

By Peter and Paul, $C''\|u\|_{s-1} \le \epsilon\|u\|_s + C(\epsilon)\|u\|_{s-k}$ and picking $\epsilon < 1$ finishes the proof. □

3.4. **A smoothing procedure.** We will use Gardings inquality for the proof of the regularity theorem. It states that if $f \in W^s$ and $u \in W^s$ and if $Pu = f$, then in fact $u \in W^{s+k}$. By passage to the completion, Gardings inequality continues to hold; thus we have $\|u\|_{s+k} \le C(\|u\|_s + \|Pu\|_s)$ for all $u \in W^s$. Gardings inequality gives an *a priori estimate* for a solution u of $Pu = f$: $\|u\|_{s+k} \le C(\|u\|_s + \|f\|_s)$, suggesting that u is always in W^{s+k} if $f \in W^s$. If f is smooth, this should give that u is smooth. However, the assumption requires that u is already in W^{s+k}, and we are going around in a circle. To get around this problem, we introduce now the *Friedrichs mollifiers.*

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a function with $\phi \ge 0$, $\int \phi(x)dx = 1$ and $\phi(-x) = \phi(x)$. For $\epsilon > 0$, we let $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$.

Definition 3.4.1. The *Friedrichs mollifier* is the operator $F_\epsilon : \mathcal{S} \to \mathcal{S}$; $u \mapsto \phi_\epsilon * u$.

Proposition 3.4.2.

(1) *F_ϵ extends to a bounded operator $W^s \to W^s$, with operator norm ≤ 1.*
(2) *F_ϵ commutes with all differential operators with constant coefficients.*
(3) *For each $u \in W^s$, $F_\epsilon u$ is in $C^\infty \cap W^s$.*
(4) *For each $u \in W^s$, $F_\epsilon u \to u$ in the W^s-norm.*

Proof. For each $a \in C_c^\infty(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, one estimates

$$\|a * u\|_s^2 = \int |\widehat{a * u}(\xi)|^2 (1 - |\xi|^2)^s d\xi = \int |\hat{a}(\xi)\hat{u}(\xi)|^2 (1 - |\xi|^2)^s d\xi \leq \|\hat{a}\|_{L^\infty}^2 \|u\|^2 s.$$

But $\|\hat{a}\|_{L^\infty} \leq \|a\|_{L^1}$ and since $\|\phi_\epsilon\|_{L^1} = \|\phi\|_{L^1}$, the proof of (1) is complete. Part (2) is an easy consequence of Proposition 3.1.3.

For part (3), consider first the case $s \geq 0$. Since $W^s \subset L^2$, $F_\epsilon u$ is smooth by Proposition 3.1.3; one has to use that smoothness is a local property, and if $u \in L^2$ and $x \in \mathbb{R}^n$, then $F_\epsilon u(x) = F_\epsilon(\mu u)(x)$ for some cut off function μ with large support; but μu is L^1. For negative s, suppose that part (3) has been proven for the value s. Any $u \in W^{s-2}$ can be written uniquely as Lv, $v \in W^s$. Then

$$F_\epsilon(Lv) = LF_\epsilon v$$

and this is smooth.

For part (4), let $u \in W^s$ and pick $v \in C_c^\infty$ with $\|u - v\|_s < \delta/3$, so that

$$\|u - F_\epsilon u\|_s \leq \|u - v\|_s + \|v - F_\epsilon v\|_s + \|F_\epsilon(v - u)\|_s \leq \|v - F_\epsilon v\|_s + \frac{2}{3}\delta.$$

But

$$\|v - F_\epsilon v\|_s^2 = \int |(\hat{\phi}_\epsilon - 1)\hat{v}|^2 (1 + |\xi|^2)^s d\xi = \int |(\hat{\phi}(\epsilon\xi) - 1)|^2 |\hat{v}(\xi)|^2 (1 + |\xi|^2)^s d\xi$$

and the integrand is pointwise convergent to 0, and bounded by the L^1-function $2|\hat{v}|(1 + |\xi|^2)^s$, and so the integral tends to zero by the dominated convergence theorem. □

Proposition 3.4.3. *Let $U \subset \mathbb{R}^n$ be relatively compact, let $u \in W^r(U)$, $s > r$ and assume that $\|F_t u\|_s \leq C$ uniformly in t. Then $u \in W^s$.*

Proof. Let $F_n := F_{t_n}$ where $t_n \to 0$. Let $\Lambda_n : W^{-s} \to \mathbb{C}$ be the functional $v \mapsto \langle F_n u, v\rangle$. We have $|\Lambda_n(v)| \leq C\|v\|_{-s}$, by Proposition 3.2.9. So the family Λ_n is equicontinuous and bounded, by 1.4.5. Thus, by Arzela-Ascoli, there is a subsequence, also denoted Λ_n, such that Λ_n converges uniformly on all compact subsets of W^{-s}, to some linear functional $\Lambda : W^{-s} \to \mathbb{C}$ which is also bounded by C. By Proposition 3.2.9, there is $w \in W^s$ such that $\Lambda(v) = (w, v)$, for all $v \in W^{-s}$. We claim that the image of w in W^r is equal to u.

By Rellich, the image of $B_1(W^{-r}(U'))$ in W^{-s} is relatively compact for all relatively compact U', and so $\Lambda_n \to \Lambda$ in $(W^{-r}(U'))^*$. We assumed that $u \in W^r(U)$ and hence $F_n u \to u$ in W^r. So the restriction of Λ to $W^{-r}(U')$ must be given by pairing with u, for each relatively compact $U \subset U' \subset \mathbb{R}^n$. The union of the Sobolev spaces $W^{-r}(U')$ over all U' is dense in W^{-r}, and so the restriction of Λ to W^{-r} is given by pairing with u. Hence $u = w$ and the proof is complete. □

Remark 3.4.4. This proposition can be formulated more abstractly, using the notion of weak convergence.

Proposition 3.4.5. *(Friedrich's lemma) Let P a differential operator of order $k \geq 1$ with compact support and let F_t be a family of Friedrich mollifiers. Then the commutator $[F_t, P]$ is a bounded operator $W^s \to W^{s-k+1}$ for each $s \in \mathbb{R}$, and the operator norm is uniformly bounded (i.e. independent of t).*

For the proof, we need a useful estimate:

Lemma 3.4.6. *(Peetre inequality) Let $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$. Then*

$$\frac{(1+|x|^2)^s}{(1+|y|^2)^s} \leq 2^{|s|}(1+|x-y|^2)^{|s|}.$$

Proof. By switching the roles of x and y, it is enough to consider $s \geq 0$, and moreover $s = 1$. But

$$(1+|x|^2) = 1+|x-y|^2+|y|^2+2(x-y)x \leq 1+|x-y|^2+|y|^2+(|x-y|^2+|y|^2) \leq$$

$$2(1+|y|^2+|x-y|^2+|y|^2|x-y|^2) = 2(1+|y|^2)(1+|x-y|^2).$$

□

Proof of Friedrichs lemma. (This proof is the solution of an exercise in [28], p. 235). The result is proven by induction on the order k of P. The case $k = 1$ contains the core argument. For higher order, one argues by induction on k, in an algebraic way. Namely, let P be a differential operator of order k and ∂ be a constant coefficient operator of order 1. Then, using that ∂ commutes with F_t, one gets $[P\partial, F_t] = [P, F_t]\partial$, which easily implies the inductive step, for operators of the form $P\partial$. But by the very definition of a differential operator, each operator of order $k+1$ is the sum of such special operators.

Now consider the case $k = 1$. If the principal symbol of P is zero, the proof is trivial, since F_t itself is uniformly bounded and P has order zero. So we are let to study the operator cD^j, for some smooth, compactly supported function c (each order one operator can be written as a sum of such operators, plus an order 0 term). First, we need the estimate for $\xi, \eta \in \mathbb{R}^n$:

$$|(\xi_j+\eta_j)\hat{\phi}(t(\xi+\eta)) - \xi_j\hat{\phi}(t\xi)| \leq C|\eta| \tag{3.4.7}$$

for a constant C that does not depend on t. To see this, we write the term to be estimated as the absolute value of

$$(\xi_j+\eta_j)\hat{\phi}(t(\xi+\eta)) - \xi_j\hat{\phi}(t\xi) = \int_0^1 \frac{\partial}{\partial s}((\xi_j+s\eta_j)\hat{\phi}(t(\xi+s\eta)))ds =$$

$$= \int_{\mathbb{R}^n}\phi(x)\int_0^1 \frac{\partial}{\partial s}((\xi_j+s\eta_j)e^{-ixt(\xi+s\eta)})dsdx =$$

$$\int_{\mathbb{R}^n}\phi(x)\int_0^1 \eta_j e^{-ixt(\xi+s\eta)}dsdx + \int_{\mathbb{R}^n}\phi(x)\int_0^1(\xi_j+s\eta_j)(-itx\eta)e^{-ixt(\xi+s\eta)}dsdx.$$

The first integral is bounded by $\int \phi(x)dx|\eta| = |\eta|$. The second one equals

$$\int_0^1 \int_{\mathbb{R}^n} \phi(x)(x\eta) \frac{\partial}{\partial x_j} e^{-ixt(\xi+s\eta)} ds dx = -\int_0^1 \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(\phi(x)(x\eta)) e^{-ixt(\xi+s\eta)} ds dx$$

by partial integration. The absolute value can be estimated by

$$\int_0^1 \int_{\mathbb{R}^n} |\eta| |\frac{\partial}{\partial x_j}(\phi(x)x)| ds dx \le |\eta| \int_{\mathbb{R}^n} |\frac{\partial}{\partial x_j}(\phi(x)x)| dx.$$

So the proof of 3.4.7 is complete.

Now consider the differential operator $Pu = cD^j u$. The commutator is

$$[F_t, P] = F_t c D^j - cD^j F_t = F_t c D^j - F_t D^j c + F_t D^j c - cD^j F_t = F_t[c, D^j] + [F_t D^j, c]$$

using that F_t commutes with constant coefficient operators. As $[c, D^j]$ is of order 0, $F_t[c, D^j]$ is uniformly bounded, and we only have to take care of $[F_t D^j, c]$. For $u \in W^s$ and $v \in W^{-s}$, we have by Plancherel's theorem

$$\langle [F_t D^j, c]u, v\rangle = \langle ([F_t D^j, c]u)\hat{}, \widehat{v}\rangle.$$

This is equal to

$$\int_{\mathbb{R}^n} (\widehat{\phi_t}(\xi)\xi_j(\hat{c} * \hat{u})(\xi) - (\hat{c} * \widehat{\phi_t} M_j \hat{u})(\xi)\overline{\hat{v}(\xi)} d\xi,$$

by the rules for the Fourier transform, where M_j stands for multiplication by the function ξ_j. Writing the convolution out and using 3.1.11, we obtain

$$\int \int \left(\hat{\phi}(t\xi)\xi_j \hat{c}(\eta)\hat{u}(\xi-\eta) - \hat{c}(\eta)\hat{\phi}(t(\xi-\eta))(\xi_j - \eta_i)\hat{u}(\xi-\eta)\right) \bar{\hat{v}}(\xi) d\xi d\eta =$$

$$= \int \int \hat{c}(\eta)\bar{\hat{v}}(\xi)\hat{u}(\xi-\eta)\left(\hat{\phi}(t\xi)\xi_j - \hat{\phi}(t(\xi-\eta))(\xi_j - \eta_i)\right) d\xi d\eta.$$

By 3.4.7, the absolute value of this integral can be estimated by

$$C \int \int |\hat{c}(\eta)\bar{\hat{v}}(\xi)\hat{u}(\xi-\eta)||\eta| d\xi d\eta \overset{\eta \to \xi-\zeta}{=} C \int \int |\hat{c}(\xi-\zeta)\bar{\hat{v}}(\xi)\hat{u}(\zeta)||\xi-\zeta| d\xi d\zeta =$$

$$= C \int \int \left((1+|\xi-\zeta|^2)^{s/2}|\hat{c}(\xi-\zeta)||\xi-\zeta|\right)(1+|\xi-\zeta|^2)^{-s/2}|\bar{\hat{v}}(\xi)\hat{u}(\zeta)| d\xi d\zeta =:$$

$$C \int \int q(\xi-\zeta)(1+|\xi-\zeta|^2)^{-s/2}|\bar{\hat{v}}(\xi)\hat{u}(\zeta)| d\xi d\zeta$$

By the Peetre inequality, this is estimated by (since $s \ge 0$):

$$\le C \int \int q(\xi-\zeta)(1+|\zeta|^2)^{s/2}|\hat{u}(\zeta)|(1+|\xi|^2)^{-s/2}|\bar{\hat{v}}(\xi)| d\xi d\zeta.$$

(We use the symbol C for a constant that changes from line to line; the actual value of C is irrelevant for us) Using Cauchy-Schwarz, this is

$$\le C \left(\int \int q(\xi-\zeta)(1+|\zeta|^2)^s |\hat{u}(\zeta)|^2 d\zeta d\xi\right)^{1/2} \left(\int \int q(\xi-\zeta)(1+|\xi|^2)^{-s}|\hat{v}(\xi)| d\zeta d\xi\right)^{1/2}.$$

The first factor is

$$\leq \left(\|u\|_s^2 \int q(\xi) d\xi \right)^{1/2} \leq C\|u\|_s$$

and likewise the second factor is bounded by $C\|v\|_{-s}$. Altogether, we get that

$$|\langle [F_t D^j, c]u, v\rangle| \leq C\|u\|_s \|v\|_{-s}$$

and by duality, we conclude that $\|[F_t D^j, c]u\|_s \leq C\|u\|_s$, the constant C not depending on t and u. This concludes the proof.

□

3.5. **Local elliptic regularity.**

Theorem 3.5.1. *(The local regularity theorem) Let P be a differential operator of order k that is elliptic over $\bar{U}$, $U \subset \mathbb{R}^n$ relatively compact. Let k, l be integers, $f \in W^l$, and $u \in W^r$. Assume that $Pu = f$ (this equation takes place in W^{r-k}). Then for each function $\mu \in C_c^\infty(U)$, $\mu u \in W^{l+k}$.*

Corollary 3.5.2. *Let $u \in W^r$ and P elliptic over* $\mathrm{supp}(u)$*. Assume that Pu is smooth. Then u is smooth over U.*

This follows from the theorem by the Sobolev embedding theorem.

Proof of Theorem 3.5.1. By induction, we can assume that $\mu u \in W^{k+l-1}$ and we have to prove that $\mu u \in W^{k+l}$. By Garding's inequality

$$\|F_\epsilon \mu u\|_{k+l} \leq C(\|F_\epsilon \mu u\|_l + \|P F_\epsilon \mu u\|_l) \leq C(\|F_\epsilon \mu u\|_l + \|[P, F_\epsilon]\mu u\|_l + \|F_\epsilon [P, \mu]u\|_l + \|F_\epsilon \mu P u\|_l).$$

Now all four summands are uniformly bounded (independent of ϵ):

- $\|F_\epsilon \mu u\|_l \leq \|\mu u\|_l \leq C(\mu)\|u\|_l$ (by 3.4.2 and 3.3.3).
- $\|[P, F_\epsilon]\mu u\|_l \leq C\|\mu u\|_{l+k-1} \leq C'\|u\|_{l+k-1}$ by Friedrich's lemma and 3.3.3.
- $\|F_\epsilon [P, \mu]u\|_l \leq \|[P, \mu]u\|_l$ by 3.4.2. Moreover, $[P, \mu]$ is an operator of order $k-1$ with compact support and so, by 3.3.4, $\|[P, \mu]u\|_l \leq C\|u\|_{l+k-1}$.
- $\|F_\epsilon \mu P u\|_l \leq \|\mu f\|_l$ by 3.4.2.

Appealing to Proposition 3.4.3 concludes the proof. □

3.6. **Sobolev spaces on manifolds.** We now move on to globalize the theory so far developed. We have to define Sobolev spaces on manifolds, and we will do this only for integral indices, as this is everything we need. From now on, M will always be a *closed* n-dimensional manifold and $E \to M$ a complex vector bundle. We pick a finite cover of M by sets U_i with charts $h_i : U_i \cong \mathbb{R}^n$. Moreover, we pick bundle trivializations ϕ_i of $E|_{U_i}$ and a partition of unity μ_i subordinate to the cover $\{U_i\}$. We define the Sobolev norm of $u \in \Gamma(M, E)$ by

$$\|u\|_k^2 := \sum_i \|\mu_i \phi_i \circ u \circ (h_i)^{-1}\|_k^2.$$

As expected, the Sobolev space $W^s(M; E)$ is defined to be the completion of $\Gamma(M; E)$ with respect to that norm. We will then transfer the most important results from the previous sections to the manifold case. To get things started, one needs some pieces of information. The key is the following lemma.

Lemma 3.6.1. *Let $\phi : U' \to V'$ be a diffeomorphism of open subsets of $\mathbb{R}^n$ and let $U \subset U'$ and $V = \phi(U) \subset V'$ be relatively compact. Then $u \mapsto u \circ \phi$ extends to a bounded map $W^s(V) \to W^s(U)$, for all $s \in \mathbb{Z}$.*

Proof. Assume first $s = k \in \mathbb{N}$ and $u \in C_c^\infty(V)$. Compute

$$\|u \circ \phi\|_k^2 = \sum_{|\alpha| \leq k} \int |D^\alpha (u \circ \phi)(x)|^2 dx.$$

Now a qualitative version of the chain rule in several variables for higher order derivatives states that

$$D^\alpha (u \circ \phi) = \sum_{|\beta| \leq |\alpha|} ((D^\beta u) \circ \phi) P_{\alpha,\beta}(\phi)$$

where P is a universal polynomial in the derivatives of ϕ up to order $|\alpha|$ (an explicit formula is the so-called Faá di Bruno formula). Therefore, because U and V are relatively compact,

$$\int |D^\alpha (u \circ \phi)(x)|^2 dx \leq C \sum_{|\beta| \leq |\alpha|} \int |(D^\beta u) \circ \phi|^2 dx = \int C \sum_{|\beta| \leq |\alpha|} \int |(D^\beta u)|^2 |\det D\phi|^{-2} dy \leq C' \|u\|_k^2.$$

This settles the case of nonnegative k. For $-k$, we use duality. Let $U \subset U'' \subset U'$ be an intermediate relatively compact subset and $V'' = \phi(U'')$. Observe that

$$\int_{U''} u(\phi(x)) g(\bar{x}) dx = \int_{V''} u(y) g(\phi^{-1}(y)) |\det D\phi(y)|^{-1} dy$$

and therefore, by duality,

$$\|u \circ \phi\|_{-k} = \sup_{g \in C_c^\infty(U''), \|g\|_k \leq 1} \int_{U''} u(\phi(x)) \overline{g(x)} dx =$$

$$\sup_{g \in C_c^\infty(U''), \|g\|_k \leq 1} \int_{V''} u(y) g(\phi^{-1}(y)) |\det D\phi(y)|^{-1} dy \leq \|u\|_{-k} \|(g \circ \phi^{-1}) |\det D\phi|^{-1}\|_k \leq$$

$$\leq C \|u\|_{-k} \|g \circ \phi^{-1}\|_k \leq C' \|u\|_{-k} \|g\|_k.$$

This finishes the proof. □

Lemma 3.6.2.

(1) *The equivalence class of the norm $\|\|_k$ does not depend on the choices made.*
(2) *If M has a distinguished Riemann metric and the bundle E a distinguished hermitian bundle metric, then $\|\|_0$ is equivalent to the L^2-norm defined in 2.3.3.*
(3) *If u has support in a coordinate neighborhood U_i, then $\|\phi_i \circ u \circ h_i^{-1}\|_{k,\mathbb{R}^n} \leq C\|u\|_{k,M}$, where C depends on the choice of the trivializations and charts.*

Proof. The first claim follows from Lemma 3.6.1 and Proposition 3.3.3. The second part is similar, one uses local trivializations that respect the inner product. The third part is an easy exercise. □

Using this lemma, we can transfer the known results to manifolds. Here is the summary:

Theorem 3.6.3. *Let M be a closed manifold and $E \to M$ be a hermitian vector bundle. Then:*

(1) *The inclusion map* $W^l \to W^k$ *is injective for* $k > l$.
(2) *(Sobolev embedding) The elements of* W^l *are* C^k*-sections, provided that* $l > \frac{n}{2} + k$.
(3) *(Rellich compactness) The inclusion* $W^l \to W^k$ *is compact if* $l > k$.
(4) *Each differential operator* P *of order* k *induces a continuous operator* $W^{l+k} \to W^l$ *for all* l. *If* P *depends smoothly on a parameter* $t \in \mathbb{R}$, *then* $t \mapsto P_t$ *is a continuous map* $\mathbb{R} \to \mathrm{Lin}(W^{k+l}, W^l)$.
(5) *(Gardings inequality) If* P *is elliptic, then there is a constant* C *such that* $\|u\|_{k+l} \le C(\|u\|_l + \|Pu\|_l)$ *holds for all* $u \in W^{k+l}$.
(6) *(Duality) The map* $W^k(M, E) \to (W^{-k}(M; E))^*$ *given by* $u \mapsto (v \mapsto \langle u, v\rangle)$ *is an antilinear isomorphism of Hilbert spaces.*

No new ideas are required for the proof. As a sample, we show how to prove the Garding inequality. We use the notation from the beginning of this section. Denote $u_i := \phi_i \circ u \circ (h_i)^{-1}$, so that

$$\|u\|_{k+l} \le C \sum_i \|\mu_i u_i\|_{k+l} \le \sum_i CC_i(\|\mu_i u_i\|_l + \|P\mu_i u_i\|_l)$$

from Garding's inequality in $\mathbb{R}^n$. Using the third part of 3.6.2, we get

$$CC_i(\|\mu_i u_i\|_l \le C'\|u\|_l.$$

Let a_i be a compactly supported function in U_i with $a_i\mu_i = \mu_i$. Note that $\mu_i P a_i = a_i \mu_i P = \mu_i P$. We obtain

$$\|P\mu_i u_i\|_l \le \|[P, \mu_i]a_i u_i\|_l + \|\mu_i P a_i u_i\|_l \le C_i\|a_i u_i\|_{l+k-1} + C_i\|\mu_i P u_i\|_l$$

The second is at most $\|Pu\|_l$ by the definition of the Sobolev norm, and the first summand can be estimated by the Peter-Paul inequality (in $\mathbb{R}^n$) against $\epsilon\|a_i u\|_{l+k} + C\|a_i u\|_l$. By Lemma 3.6.2, both summands are bounded by the respective Sobolev norm.

Since the covering was finite (!!), the proof is completed by putting everything together.

3.7. **Global regularity and the Hodge theorem.** Now we finally come to the proof of the Hodge decomposition theorem and the Fredholm property of elliptic operators on closed manifolds. Let M be a closed manifold and let $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ be an elliptic differential operator. We first globalize the regularity theorem.

Theorem 3.7.1. *Let* $Pu = f$, $f \in W^l$, $u \in W^r$, *for some integer* r. *Then* $u \in W^{l+k}$.

Proof. Let $U \subset M$ a coordinate chart. Let $V \subset U$ be relatively compact. Pick functions $\mu, \lambda \in C_c^\infty(U)$ with $\mu \equiv 1$ on V and $\mu\lambda = \mu$. Since

$$\mu f = \mu P u = \mu P(\lambda u)$$

and μP is elliptic over V, the local regularity theorem 3.5.1 tells us that for each $\varphi \in C_c^\infty(V)$, the function $\varphi\lambda u = \varphi u$ belongs to W^{l+k}. Cover M by finitely many such sets V_i and let $(\varphi_i)_i$ be a partition of unity subordinate to this covering. Thus $u = \sum_i \varphi_i u$ belongs to W^{k+l}. □

For the proof of the Fredholm property, we need an abstract functional-analytic result.

Proposition 3.7.2. *Let U, V, W be Hilbert spaces, $P : U \to V$ bounded and $K : U \to W$ compact. Assume that there exists a constant C with*

$$\|u\|_U \leq C(\|Pu\|_V + \|Ku\|_W)$$

Then the kernel of P is finite dimensional and P has closed image.

Proof. Let u_n be a sequence with $Pu_n = 0$ and $\|u_n\|_U \leq 1$. Then

$$\|u_n - u_m\|_U \leq C\|K(u_n - u_m)\|_W$$

and by the compactness of K, Ku_n is subconvergent[3]. Therefore, a subsequence u_n is a Cauchy sequence. This shows that each bounded sequence in the kernel of P is subconvergent; and therefore $\ker(P)$ is finite-dimensional.

To prove that the image of P is closed, it is enough to consider $P|_{\ker(P)^\perp}$, in other words, we can assume that P is injective.

We claim that there exists a $c > 0$ with $c\|u\| \leq \|Pu\|$ for all $u \in U$. Suppose this is not the case; i.e. for each $b > 0$, there is u with $\|u\| = 1$ and $\|Pu\| \leq b$. We can then find a sequence u_n such that $\|u_n\| = 1$ and $\|Pu_n\| \to 0$.

Since K is compact, we can assume without loss of generality that Ku_n is convergent. Therefore

$$\|u_n - u_m\|_U \leq C(\|P(u_n - u_m)\|_V + \|K(u_n - u_m)\|_W)$$

which converges to 0. Thus u_n is a Cauchy sequence and the limit u must have $\|u\|_U = 1$ (since all u_n have norm 1) and $Pu = 0$ (since $Pu_n \to 0$ and P is continuous). This contradicts the assumption that P is injective, and this contradiction proves the existence of the constant c.

Now let $v \in \overline{\mathrm{Im}(P)}$ and let u_n be sequence with $Pu_n \to v$. Then $\|u_n - u_m\| \leq \frac{1}{c}\|Pu_n - Pu_m\| \to 0$, so u_n is a Cauchy sequence with limit u, and $Pu = v$, which is why P has closed image. □

Corollary 3.7.3. *Let M be a closed manifold and P an elliptic differential operator of order k. Then $P : W^{k+l} \to W^l$ has a finite dimensional kernel and closed image. Moreover, the dimension of the kernel of $P : W^{l+k} \to W^l$ does not depend on l.*

Proof. The first sentence is immediate from Gardings inequality, Rellichs Lemma and Proposition 3.7.2. The second follows from regularity. □

The proof of the Fredholm property is finished by a duality consideration. We consider $P : W^{k+l} \to W^l$, which has closed image. To prove that the image has finite codimension, it is therefore enough to prove that the space of all $\ell \in\in (W^l)'$ with $\ell \circ P = 0$ is finite dimensional. By duality, it has to be proven that

$$K = \{v \in W^{-l} | \langle Pu, v\rangle = 0 \forall u \in W^{k+l}\}$$

is finite dimensional. But

$$\langle Pu, v\rangle = \langle u, P^* v\rangle$$

and so $K = \ker(P^* : W^{-l} \to W^{-k-l})$ which is finite dimensional by Corollary 3.7.3. Note that we used at this point that P^* is elliptic if P is elliptic. This proves

[3]We say that a sequence is subconvergent if it has a convergent subsequence.

not only that $P : W^{k+l} \to W^l$ is a Fredholm operator, but also that the codimension of the image does not depend on l. We summarize.

Theorem 3.7.4. *Let $P : \Gamma(M; E_0) \to \Gamma(M, E_1)$ be an elliptic operator of order k on the closed manifold M. Then $P : W^{k+l} \to W^l$ is a Fredholm operator, with index not depending on l. The orthogonal complement of the image of $P : W^k \to L^2$ (taken with respect to the L^2-inner product induced by a Riemannian metric on M and hermitian bundle metrics on the vector bundles E_i) is the kernel of P^*. Thus we get an orthogonal decomposition $C^\infty = \ker(P^*) \oplus \mathrm{Im}(P)$.*

Proposition 3.7.5. *Let P_t, $t \in \mathbb{R}$ be a smooth family of elliptic differential operators on a compact manifold. Then the index of P_t does not depend on t. Moreover, if p_t is a smooth family of elliptic symbols on M, then for all operators P_i, $i = 0, 1$, with $\mathrm{smb}_k(P_i) = p_i$, the indices are the same.*

Proof. For the first part, use Theorem 3.6.3 to show that the Fredholm operator P_t depends continuously on t, and then use Theorem 1.2.7. For the second part, one has to show that there exists a smooth family of differential operators P_t with the symbol p_t (and then use the first part). This follows from the arguments given for Proposition 2.2.19. □

This last proposition is a smoking gun: it proves that the index of an elliptic differential operator *only depends on the homotopy class of the principal symbol*, where homotopy is to be understood through elliptic symbols. A suitable generalization of this will be one of the keys for the proof of the index theorem.

The last thing we want to get out of the analysis is the *Hodge decomposition theorem*. Let $\mathcal{E} = (E_*, P)$ be an elliptic complex on a smooth closed manifold M. Because $P_i \circ P_{i-1} = 0$, we can form the *cohomology* of the elliptic complex:

$$H^p(\mathcal{E}) := \frac{\ker(P_p : \Gamma(M, E_p) \to \Gamma(M, E_{p+1}))}{\mathrm{Im}(P_{p-1} : \Gamma(M, E_{p-1}) \to \Gamma(M, E_p))}.$$

For example, if $\mathcal{E}$ is the de Rham complex, then $H^p(\mathcal{E})$ agrees with the usual de Rham cohomology.

Equip M and the bundles with metrics, so that we can talk about the operator $D = P + P^*$, which is elliptic. Let $\Delta := D^2$ and observe that Δ maps $\Gamma(M, E_p)$ into itself. We let $\mathcal{H}^p(\mathcal{E})$ be the kernel of $\Delta : \Gamma(M, E_p) \to \Gamma(M; E_p)$. For elliptic complexes, the elliptic regularity theorem has the following formulation.

Theorem 3.7.6. *(The Hodge decomposition theorem) Let M be a closed Riemann manifold and $\mathcal{E}$ an elliptic complex on M. Then there are othogonal decompositions:*

(1) $\Gamma(M, E) = \ker(D) \oplus \mathrm{Im}(D)$. *The kernel* $\ker(D)$ *is finite-dimensional.*
(2) $\ker(D) = \ker(\Delta) = \ker(P) \cap \ker(P^*)$.
(3) $\mathrm{Im}(\Delta) = \mathrm{Im}(D) = \mathrm{Im}(PP^*) \oplus \mathrm{Im}(P^*P) = \mathrm{Im}(P) \oplus \mathrm{Im}(P^*)$.
(4) $\ker(P) = \mathrm{Im}(P) \oplus \ker(\Delta)$.
(5) *The natural map* $\ker(\Delta) \to H(\mathcal{E})$ *is an isomorphism.*

Proof. (1) This is clear from Theorem 3.7.4.
(2) The first equation is clear, and so is the $\supset$-inclusion of the second. Conversely, if $Pu + P^*u = 0$, then calculate $0 = \langle Pu + P^*u, Pu + P^*u\rangle = \langle Pu, Pu\rangle + \langle P^*u, P^*u\rangle$.

(3) $\mathrm{Im}(\Delta) = \mathrm{Im}(D)$ is clear by now. Because $\Delta = PP^* + P^*P$, it follows that $\mathrm{Im}(D) \subset \mathrm{Im}(PP^*) \oplus \mathrm{Im}(P^*P) \subset \mathrm{Im}(P) \oplus \mathrm{Im}(P^*)$, the orthogonality of all spaces is clear. To prove that $Pu + P^*v \in \mathrm{Im}(\Delta)$, we prove that $(Pu + P^*v) \perp \ker(\Delta)$ and use part (1). But if $w \in \ker(\Delta)$, then $\langle Pu + P^*v, w\rangle = \langle u + v, Dw\rangle = 0$.

(4) The $\supset$-inclusion is clear. If $Pu = 0$, then write $u = x + Py + P^*z$, $x \in \ker(\Delta)$, according to parts (1) and (3). Since $Pu = 0$, we find that $PP^*z = 0$. Thus $0 = \langle z, PP^*z\rangle = \langle P^*z, P^*z\rangle$, therefore $P^*z = 0$ and $u = x + Py$.

(5) This is clear from part (4)

$\square$

Remark 3.7.7. The elements of $\ker \Delta$ are called *harmonic*; this terminology comes from the de Rham complex. What is usually called Hodge theorem is the last part. It says that *each cohomology class of an elliptic complex over a closed manifold has a unique harmonic representative.*

3.8. **The spectral decomposition of an elliptic operator.** An important property of self-adjoint elliptic operators on closed manifolds is that they admit a spectral decomposition. Here is our goal.

Theorem 3.8.1. *Let M be a closed Riemann manifold, $E \to M$ be a hermitian vector bundle and $D : \Gamma(M, E) \to \Gamma(M, E)$ be a formally self-adjoint elliptic differential operator of order $k \geq 1$. For $\lambda \in \mathbb{C}$, let $V_\lambda := \ker(D - \lambda) \subset L^2(M, E)$ be the eigenspace of D to the eigenvalue λ. Then the following statements hold*

(1) *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $V_\lambda = 0$.*
(2) *If $\lambda \neq \mu$, then $V_\mu \perp V_\lambda$.*
(3) *For each $\Lambda \in \mathbb{R}$, the sum $U_\Lambda = \bigoplus_{|\lambda| \leq \Lambda} V_\lambda$ is finite-dimensional and consist of smooth sections.*
(4) *The direct sum $\bigoplus_\lambda V_\lambda$ is dense in $L^2(M, E)$.*

In particular, it follows from part (3) that the eigenvalues for a discrete subset of $\mathbb{R}$ and that each eigenvalue has finite multiplicity. The theorem is false if D has order zero (find a counterexample).

Proof of the easy parts of Theorem 3.8.1. The first two parts are proven exactly as the corresponding statements for selfadjoint endomorphisms of finite-dimensional Hilbert spaces. Namely if $\|x\| = 1$ and $Fx = \lambda x$, then

$$\lambda = \langle x, \lambda x\rangle = \langle x, Fx\rangle = \langle Fx, x\rangle = \bar{\lambda}\langle x, x\rangle = \bar{\lambda}.$$

If $Fx = \lambda x$, $Fy = \mu y$, then

$$(\lambda - \mu)\langle x, y\rangle = \langle \lambda x, y\rangle - \langle x, \mu y\rangle = \langle Fx, y\rangle - \langle x, Fy\rangle = 0$$

so if $\lambda \neq \mu$, then $\langle x, y\rangle = 0$.

The third part depends on elliptic regularity. Let $x \in U_\Lambda$. Since the operator $D - \lambda$ is elliptic, all eigenfunctions and hence x are smooth. We can write $x = \sum_{|\lambda| \leq \Lambda} x_\lambda$, with $x_\lambda \in V_\lambda$. Note that this is an orthogonal sum. Therefore

$$\|x\|_k^2 = \sum_{|\lambda| \leq \Lambda} \|x_\lambda\|_k^2 \leq \sum_{|\lambda| \leq \Lambda} C(\|x_\lambda\|_0^2 + \|Dx_\lambda\|_0^2) \leq$$

$$\leq C(1 + \Lambda^2) \sum_{|\lambda| \leq \Lambda} \|x_\lambda\|_0^2 = C(1 + \Lambda^2)\|x\|_0^2.$$

By Rellich's theorem, we conclude that the $\| \|_0$-unit ball in U_Λ is relatively compact, and hence U_Λ is finite-dimensional. $\square$

For the last part of the proof, we recall the spectral theorem for self-adjoint compact operators in an abtract setting. Recall that the spectrum of a bounded operator F is the set of all $\lambda \in \mathbb{C}$ such that $F - \lambda$ is not invertible. The spectrum is a compact, nonempty subset of $\mathbb{C}$, [13], Satz 23.5. Any eigenvalue is in the spectrum, but the converse does not need to hold.

Theorem 3.8.2. *Let H be a Hilbert space and $F : H \to H$ be a compact selfadjoint operator. Then the spectrum of F is a subset of $\mathbb{R}$ and has 0 as its only accumulation point. Any spectral value of F different from 0 is an eigenvalue. The eigenspaces H_λ are finite-dimensional unless $\lambda = 0$. The direct sum*

$$\ker(F) \oplus \bigoplus_{\lambda \neq 0} H_\lambda$$

is dense in H.

The proof is not very difficult, but would lead us too far away. See [13], Satz 26.5 and Satz 26.3.

Proof of the fourth part of Theorem 3.8.1. We look at the operator $L = 1 + D^2$, which is self-adjoint, elliptic and has order $2k$. If $Lu = 0$, then $0 = \langle u, u\rangle + \langle Du, Du\rangle$; thus L is injective, and by Theorem 3.7.4, $L : W^{2k} \to L^2$ is invertible. Let $S : L^2 \to W^{2k}$ be the inverse; note that since $L : \Gamma(M,E) \to \Gamma(M,E)$ is bijective, S maps smooth sections to smooth sections. Let $T : L^2 \xrightarrow{S} W^{2k} \to L^2$ be the composition, which is a compact operator by Rellich's theorem (and the open mapping theorem). We claim that T is self-adjoint. It is continuous, and $\Gamma(M,E) \subset L^2$ is dense, so it is enough to show that $\langle u, Tv\rangle = \langle Tu, v\rangle$ holds for smooth sections. However,

$$\langle Lx, TLy\rangle = \langle Lx, y\rangle = \langle x, Ly\rangle = \langle TLx, Ly\rangle$$

and L is surjective onto $\Gamma(M,E)$, so T is self-adjoint. Moreover

$$\langle Lx, TLx\rangle = \langle Lx, x\rangle = \langle x, Lx\rangle \geq 0$$

shows that $\langle Tu, u\rangle \geq 0$, i.e. that T is positive. Hence all eigenvalues of T are nonnegative. As $T : L^2 \to W^{2k}$ is bijective, 0 is not an eigenvalue of T. Therefore, by the spectral theorem for compact operators, the sum

$$\bigoplus_{\mu > 0} \ker(T - \mu)$$

lies dense in L^2.

Consider an eigenvector $Tx = \mu x$. Since $\langle Tx, Lu\rangle = \langle x, u\rangle$, we find that

$$\langle x, (1 - \mu L)u\rangle = \langle x, u\rangle - \mu\langle x, Lu\rangle = \langle x, u\rangle - \langle Tx, Lu\rangle = 0;$$

in other words, x is orthogonal to the image of $(1 - \mu L)$. As $\mu \neq 0$, $(1 - \mu L)$ is elliptic and so x is smooth, this means that all eigenfunctions of T are smooth. As T is the inverse to L, it follows that $\ker(T - \mu)$ is the $\frac{1}{\mu}$-eigenspace of L. Since D commutes with L, the space $W_\mu := \ker(L - \frac{1}{\mu})$ is D-invariant. The operator $D|_{W_\mu}$ satisfies the equation $(D|_{W_\mu})^2 + 1 = \frac{1}{\mu}$. Thus, W_μ decomposes into the eigenspaces of D to the two eigenvalues $\pm\sqrt{\frac{1}{\mu}-1}$ (note that $\|T\| \leq 1$, whence all μ are in $(0,1]$). This finishes the proof. □

3.9. **Guide to the literature.** The local regularity theorem is a very classical result and is - in version or another - covered in each introductory textbook on partial differential equations. I tried to combine arguments from different sources to achieve a "best-of". Later, we need some more analysis for the proof of the index theorem, and I designed this chapter so that the later arguments are supported by this approach.

Some sources try to avoid the Fourier transform in the definition of the Sobolev spaces, and use the norm given in Lemma 3.2.4 (which is perfectly possible, if one only uses W^k for natural numbers). Instead of using $\mathbb{R}^n$ as the model space, one could also take the torus T^n and patch pieces of the torus into the manifold. This approach replaces the Fourier transform by the (simpler) Fourier series. Fourier series are easier because the Pontrjagin dual of T^n is the *discrete* group $\mathbb{Z}^n$. This approach is pursued in several sources, and after initial changes, the overall argument is more or less isomorphic to the one using the Fourier transform. Warner [31] and Griffiths-Harris [11] only want to prove the Hodge theorem, not the index theorem. Higson and Roe [12], [21] give a different proof of the index theorem, but the proof of the index theorem we head to is not supported by this approach: the role of the symbol is more cleanly reflected by the self-duality of $\mathbb{R}^n$. Once the formalism of pseudodifferential operators is set up, the proof of the regularity theorem can be streamlined considerably [32].

The treatment of the Fourier transform is taken from Lang [17]. The basic theorems on Sobolev spaces (Sobolev embedding, Rellich, duality) are proven in [18], [10], [32], [2], with essentially the same argument that we gave. The proof of Gardings inequality is that from [31], with the changes needed to suit into our framework. In [32], [18], [10], Theorem 3.3.2 is derived using the calculus of pseudodifferential operators. In [11], [12], [21], the special structure of the operators studied is used heavily.

Friedrichs lemma is an "exercise" in [21] and a structured exercise in [28]. The proof we gave follows the outline in [28]. Sources as [31] and most PDE textbooks I looked into ([8], [28], [26]) replace the use of the mollfiers by a different smoothing procedure, which might be technically simpler. One can also use distribution theory.

The globalization of the Sobolev theory is a standard exercise (and done in most of the above sources). I have no specific source for the proof of the Fredholm property and the spectral secomposition.

4. Some interesting examples of differential operators

4.1. **The Euler number.** Let M^n be a closed smooth manifold of dimension n. Recall the de Rham complex $\mathcal{A}^*(M)$, which is an elliptic complex. To get an elliptic operator out of the de Rham complex, we need a Riemann metric on M and a hermitian bundle metric on the vector bundle $\Lambda^p T^*M$. There is a canonical choice of such a hermitian metric, depending on the Riemann metric on M.

Lemma 4.1.1. *Let V be an n-dimensional euclidean vector space, with an orthonormal basis $(e_1,\dots,e_n)$. For $I \subset \underline{n}$, $|I| = p$, consider the basis element $e^I \in \Lambda^p V^*$. Define a hermitian metric on $\Lambda^p V^*$ by declaring $(e^I)_{I\subset\underline{n};|I|=p}$ to be an orthonormal basis. This hermitian inner product does not depend on the choice of the orthonormal basis.*

Proof. We prove the following equivalent statement. Let $V = \mathbb{R}^n$ and use the standard basis to define the inner product on $\Lambda^p\mathbb{R}^n$. We claim that the action of the group $O(n)$ on $\Lambda^p\mathbb{R}^n$ is via isometries. The next lemma gives a system of generators of $O(n)$ and it is easy to check that the generators act by isometries. □

Lemma 4.1.2. *Let $G_n \subset O(n)$ be the subgroup that is generated by the permutation matrices and the matrices of the form*

$$\begin{pmatrix} \cos(t) & -\sin(t) & \\ \sin(t) & \cos(t) & \\ & & 1 \end{pmatrix}$$

($t \in \mathbb{R}$). Then $G_n = O(n)$.

Proof. This can be seen by induction on n. The case $n = 2$ is easy. For the induction step one uses the elementary fact that if a group G acts transitively on a set X and $H \subset G$ is a subgroup that acts transitively such that for a fixed $x \in X$ the isotropy groups H_x and G_x are equal, then $H = G$. By induction on n, one proves that G_n acts transitively on S^{n-1}. The isotropy group of $O(n)$ at e_n is $O(n-1)$, and $(G_n)_{e_n} \supset G_{n-1}$. By induction hypothesis, $G_{n-1} = O(n-1)$. □

If M^n is a closed Riemann manifold, we thus get a canonical bundle metric on Λ^*T^*M. The elliptic operator associated with the de Rham complex is the operator

$$D = d + d^* : \mathcal{A}^{ev}(M) \to \mathcal{A}^{odd}(M).$$

The Hodge decomposition theorem can be used to compute the index of D.

Theorem 4.1.3. *Let M be a closed manifold. The index of $D : \mathcal{A}^{ev}(M) \to \mathcal{A}^{odd}(M)$ is the Euler characteristic $\chi(M) := \sum_{p=0}^n (-1)^p \dim H^p(M)$ of M.*

Proof. Let $\mathcal{H}^p(M) \subset \mathcal{A}^p(M)$ be the space of harmonic forms on M. By the Hodge theorem, the natural map $\mathcal{H}^p(M) \to H^p(M)$ is an isomorphism. Moreover,

$$\mathrm{ind}(D) = \sum_{p=0}^n (-1)^p \dim \mathcal{H}^p(M) = \chi(M).$$

□

4.2. **The signature.** So far, we have completely computed the index of one differential operator that exists on any manifold M, namely the operator $d + d^*$. If this were the only interesting operator, there would be no "index theory". It turns out that we need more structure on a manifold to get new operators linked to that extra structure. The first such extra structure is an orientation. But let us go back to the operator $D = d + d^* : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ for a second. Observe that $D = d + d^* : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ is formally self-adjoint; therefore its index is zero and the operator itself is not very interesting from the perspective of index theory. In connection with the Euler characteristic, we studied the decomposition $\mathcal{A}^*(M) = \mathcal{A}^{ev}(M) \oplus \mathcal{A}^{odd}(M)$ and we obtained an interesting operator $D_0 : \mathcal{A}^{ev}(M) \to \mathcal{A}^{odd}$ (by restricting D). Let us formulate this a bit differently. Let $I : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ be the operator that is $(-1)^p$ on $\mathcal{A}^p(M)$. This is an involution, which is self-adjoint and comes from an involution of the vector bundle $\Lambda^* T^* M$ (important!). The spaces $\mathcal{A}^{ev}/\mathcal{A}^{odd}$ are the $+1/-1$-eigenspaces of I. The important fact that we used secretly is that

$$DI = -ID,$$

(both anticommute). If we decompose $\mathcal{A}^*(M)$ according to the eigenspaces of I, we get

$$I = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}; \; D = \begin{pmatrix} D_2 & D_1 \\ D_0 & D_3 \end{pmatrix}; \; D^* = D.$$

The equation $DI + ID = 0$ means (quick computation) that $D_2 = D_3 = 0$ and $D_1 = D_0^*$, i.e.

$$D = \begin{pmatrix} & D_0^* \\ D_0 & \end{pmatrix}$$

and

$$\ker(D) = \ker(D_0) \oplus \ker(D_0^*) = \ker(D_0) \oplus \operatorname{Im}(D_0)^\perp.$$

The involution I maps $\ker(D)$ to itself (if $Dx = 0$, then $DIx = -IDx = 0$) and we get the equalities

$$\operatorname{ind}(D_0) = \dim(\ker(D_0)) - \dim(\ker(D_1)) = \operatorname{Tr}(I|_{\ker(D)}).$$

If we take I as above, we get the Euler number of M. We refer to I as a *grading* of the de Rham complex. We solidify these observations in a definition.

Definition 4.2.1. Let M be a closed Riemannian manifold, $E \to M$ be a hermitian vector bundle and $D : \Gamma(M, E) \to \Gamma(M, E)$ be a formally self-adjoint elliptic differential operator. A *grading* of D is an orthogonal involutive vector bundle isomorphism $\iota : E \to E$ such that $D\iota = -\iota D$, with eigenbundles $E_\pm$. With respect to the decomposition $E = E_+ \oplus E_-$, write

$$D = \begin{pmatrix} & D_- \\ D_+ & \end{pmatrix}.$$

The index of (D, ι) is

$$\operatorname{ind}(D, \iota) = \operatorname{ind}(D_+) = -\operatorname{ind}(D_-) = \operatorname{Tr}(\iota|_{\ker(D)}) \in \mathbb{Z}.$$

Such a pair (D, ι) is called a *graded selfadjoint elliptic operator.*

If $P : \Gamma(E_0) \to \Gamma(E_1)$ is an arbitrary elliptic operator, we get a graded self-adjoint one, by setting $E = E_0 \oplus E_1$ (orthogonal sum), $\iota = (-1)^i$ on E_i and the operator is

$$\begin{pmatrix} & P^* \\ P & \end{pmatrix}$$

Thus ordinary elliptic operators and graded self-adjoint ones are essentially the same thing. The point is that the grading is often easier to describe than the eigenspaces! One can change the grading by a sign, the index changes by sign as well: $\operatorname{ind}(D, -\iota) = -\operatorname{ind}(D, \iota)$.

But for the de Rham complex on an oriented manifold, there is a more substantial change of the grading.

Lemma-Definition 4.2.2. *Let V be an n-dimensional oriented euclidean vector space. The* star operator *is the uniquely determined operator $\star : \Lambda^p V^* \to \Lambda^{n-p} V^*$ such that the identity*

$$\langle \omega, \eta \rangle \mathrm{vol} = \omega \wedge \star \eta$$

holds for all forms ω, η. For $\omega \in \Lambda^p V^*$, one has

$$\star \star \omega = (-1)^{n(n-p)} \omega.$$

Theorem 4.2.3. *Let M^n be an oriented Riemann manifold. Then the adjoint $d^* : \mathcal{A}^p(M) \to \mathcal{A}^{p-1}(M)$ is given by*

$$d^* \omega = (-1)^{pn+n+1} \star d \star \omega.$$

Moreover, $\star \Delta = \Delta \star$.

The first part is an easy (but tedious) consequence of the Stokes theorem and the second is a direct (tedious) consequence of the first part. Easy as it is, Theorem 4.2.3 has a profound consequence, namely:

Proposition 4.2.4. *If M is an oriented Riemann manifold and ω a harmonic form on M, then $\star\omega$ is harmonic.*

We restrict to *real-valued* forms. Let us denote by

$$\mathcal{H}^p(M) = \ker(\Delta : \mathcal{A}^p(M) \to \mathcal{A}^p(M))$$

the space of *harmonic forms*. Thus $\star$ induces an isomorphism $\star : \mathcal{H}^p(M) \to \mathcal{H}^{n-p}(M)$. By the Hodge decomposition theorem, we know that

$$\mathcal{H}^p(M) \cong H^p(M; \mathbb{R})$$

and hence the star operator induces an isomorphism

$$H^p(M) \cong H^{n-p}(M)$$

which *depends on the Riemann metric.* Thus we get a proof of a weak form of Poincaré duality. Let us remark that we killed a fly with a sledgehammer: there is a simple proof of Poincaré duality in the framework of de Rham cohomology which we present in a later chapter. Let us have a slightly closer look.

Theorem 4.2.5. *Let M be a closed oriented manifold. Then the pairing $H^p(M) \otimes H^{n-p}(M) \to \mathbb{C}$, $\alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta$ is a perfect pairing.*

Proof. By the Hodge theorem, it suffices to consider the pairing on $\mathcal{H}^*$. Compute, for $\omega \in \mathcal{H}^p$ and $\eta \in \mathcal{H}^{n-p}$:

$$\int_M \omega \wedge \eta = (-1)^{p(n-p)} \int_M \omega \wedge \star \star \eta = (-1)^{p(n-p)} \int_M \langle \omega; \star \eta \rangle \mathrm{vol} = \langle \omega; \star \eta \rangle.$$

and since $\star$ is an isomorphism, this is clearly a nondegenerate pairing (and perfect because all spaces involved are finite-dimensional). □

On *even-dimensional* manifolds, we get a finer structure. Recall that the spaces $\mathcal{A}^p(M)$ of complex-valued forms come with a natural real structure (i.e. a conjugation map) and that the operators $\star$, d and d^* are all real operators (commute with the conjugation). There are two cases of even dimensions: $n = 4k+2$ (moderately interesting) and $n = 4k$ (*very* interesting). Let us, in both cases, restrict to the middle dimension. Let $n = 2m$. Note that on $\mathcal{A}^m(M^{2m})$, one has

$$\star^2 = (-1)^m.$$

On $\mathcal{H}^m(M;\mathbb{R})$, we have *two* bilinear forms. One is given by

$$\Omega : (\omega, \eta) \mapsto \int_M \omega \wedge \eta,$$

and this has a *purely homological* meaning (no metric is used to define it). And there is the scalar product

$$\langle \omega; \eta \rangle : \int_M \omega \wedge \star \eta$$

which is defined on $\mathcal{H}^m(M)$ (and thus uses the metric). The two forms are related by

$$\langle \omega; \eta \rangle = \Omega(\omega, \star \eta).$$

In the case of odd m, the form Ω is symplectic (i.e. skew-symmetric and nondegenerate). On the other hand, $\star^2 = -1$ and thus it defines a *complex structure* on $H^m(M)$. If you are familiar with the terminology of symplectic linear algebra, then Ω is a symplectic form on $H^m(M)$, $\star$ is a compatible complex structure (depending on the metric on M).

The case $n = 4k$ is extremely interesting. Consider the symmetric, nondegenerate bilinear form Ω on $\mathcal{H} = H^{2k}(M)$. There exists an orthogonal basis of $\mathcal{H}$ (i.e. $\Omega(e_i, e_j) = \epsilon_i \delta_{ij}$, $\epsilon_i = \pm 1$). The Sylvester inertia theorem from linear algebra says that the number $\mathrm{sign}(\Omega) := \sharp\{i | \epsilon_i = 1\} - \sharp\{i | \epsilon_i = 1-\}$ does not depend on the choice of the basis.

Definition 4.2.6. Let M^{4k} be an oriented closed manifold. The *signature* of M is the signature of the bilinear form Ω on $H^{2k}(M)$.

This is a fundamental invariant in differential topology. One of its meanings is that the signature is a *bordism* invariant; if W^{4k+1} is an oriented compact manifold with boundary M, then $\mathrm{sign}(M) = 0$. The signature plays an important role in the classification theory of high-dimensional manifolds.

Problem 4.2.7. Express the signature in terms of characteristic classes.

This problem was solved by Hirzebruch in 1954, using topological methods developed by Thom. The Hirzebruch signature formula was one of the motivating examples for the search of the general index formula, and in this course, we will prove the signature formula as a special case of the index theorem. We have not yet stated the signature formula (the right-hand-side would not yet be understandable). But we can go the first step along its proof, and this is by identifying the signature as the index of a new elliptic operator (which exists only on oriented manifolds of dimension $4k$).

Lemma 4.2.8. *Let M^{4k} be oriented and closed. Then* $\mathrm{sign}(M)$ *is equal to the trace* $\mathrm{Tr}(\star|_{\mathcal{H}^{2n}(M)})$.

Proof. Since on $2k$-forms, $\star^2 = 1$, we see that $\Omega(\omega, \eta) = \langle \omega; \star\eta \rangle$. The rest of the proof is pure linear algebra. Let V be a finite-dimensional vector space, I an involution, B a symmetric bilinear form and $\langle ; \rangle$ and assume these are related by $\langle x; y \rangle = B(x, Iy)$. Since

$$B(x, Iy) = \langle x; y \rangle = \langle y; x \rangle = B(y, Ix) = B(Ix, y),$$

the involution I is selfadjoint, and the decomposition $V = V_+ \oplus V_-$ into the eigenspaces of I is orthogonal. If $x \in V_\pm$, we get $B(x, x) = \langle x; Ix \rangle = \pm \langle x; x \rangle$, and $\pm B$ is positive definite on $V_\pm$. If $x \in V_+$ and $y \in V_-$, then $B(x, y) = \langle x; Iy \rangle = -\langle x; y \rangle = 0$ because both spaces are orthogonal. Thus the signature is $\dim V_+ - \dim V_- = \mathrm{Tr}(I)$. □

4.3. **Complex manifolds and vector bundles.** We now investigate the refinement of the harmonic theory for *complex* manifolds. We first begin in arbitrary dimensions; but at a crucial point it turns out that one dimensional complex manifolds are much easier to treat. The index theorem on Riemann surfaces is a very classical result: the Riemann-Roch formula.

Definition 4.3.1. Let M be a smooth manifold of dimension $2n$. A smooth atlas (U_i, h_i) is *holomorphic* if all transition functions are holomorphic. A *complex structure* is a maximal holomorphic atlas, and a *complex manifold* is a manifold M, together with a complex structure. A 1-dimensional complex manifold is called *Riemann surface.*

Examples: $\mathbb{C}$, $\mathbb{CP}^1$, tori. Moreover, one can prove that each differentiable surface of genus g has a complex structure (by no means unique, and this is by no means an easy result).

Definition 4.3.2. Let M be a complex manifold and $V \to M$ be a complex vector bundle. A bundle atlas (U_i, h_i) of V is holomorphic if the transition functions $h_{ij} : U_{ij} \to \mathrm{GL}_r(\mathbb{C})$ are holomorphic.

Examples 4.3.3. *The tautological line bundle on $\mathbb{CP}^n$ is a holomorphic vector bundle. The tangent bundle of a complex manifold has a natural holomorphic structure. On the dual vector bundle E^*, there is a holomorphic structure. Likewise, tensor products and hom-bundles have holomorphic structures.*

We discuss the tangent bundle to a complex manifold in a little more detail. Let $x \in M$ and let $x \in U \xrightarrow{h} h(U) \subset \mathbb{C}^n$ be a holomorphic chart. The composition

$$J_x : T_x M \overset{Th}{\to} T_{h(x)}\mathbb{C}^n \overset{can}{\cong} \mathbb{C}^n \overset{i\cdot}{\to} \mathbb{C}^n \overset{can}{\cong} T_{h(x)} \overset{Th^{-1}}{\to} T_x M$$

satisfies $J_x^2 = -1$, does not depend on the choice of h (since the atlas is holomorphic). J is a smooth bundle endomorphism and satisfies $J^2 = -1$. This turns TM (which is a priori only a real vector bundle) into a complex vector bundle.

Definition 4.3.4. Let M be a real manifold. An almost-complex structure on M is a complex structure on the vector bundle TM, in other words, a smooth endomorphism J of TM with $J^2 = -1$.

One can prove that on surfaces, each almost complex structure is induced from a complex structure. This requires an amount of analysis (not directly related to index theory). The corresponding fact in higher dimensions is false (but there is an additional condition on J that guarantees this).

4.4. **Multilinear algebra of complex vector spaces.** We now have to delve into (multi)linear algebra of complex vector spaces. Let V be a real vector space of dimension $2n$, equipped with a complex structure J. This defines the structure of a complex vector space on V, namely $(a+ib)v := av + bJv$. We consider $\Lambda^* V^*$, the algebra of *complex-valued*, $\mathbb{R}$-*multilinear* alternating forms. This is a complex vector space, the piece $\Lambda^p V^*$ has complex dimension $\binom{2n}{p}$. There is a conjugation map $\omega \mapsto \bar{\omega}$ on $\Lambda^* V^*$, defined by

$$\bar{\omega}(v_1, \ldots, v_p) := \overline{\omega(v_1, \ldots, v_p)}.$$

If $e_1, \ldots, e_n$ is a $\mathbb{C}$-basis of V and $e^1, \ldots, e^n$ the dual basis of $V^* \subset \Lambda^1 V^*$, then $(e^1, \bar{e^1}, \ldots, e^n, \bar{e^n})$ is an $\mathbb{C}$-basis of $\Lambda^1 V^*$. Then the set

$$\{e^{i_1} \wedge e^{i_p} \wedge \bar{e^{j_1}} \wedge \bar{e^{j_q}} | p+q = r,\ i_1 < \ldots < i_p;\ j_1 < \ldots < j_q\}$$

is a $\mathbb{C}$-basis of $\Lambda^r V^*$. We define subspaces

$$\Lambda^{p,q} V^* := \operatorname{span}\{e^{i_1} \wedge e^{i_p} \wedge \bar{e^{j_1}} \wedge \omega \bar{e^{j_q}} | i_1 < \ldots < i_p;\ j_1 < \ldots < j_q\}.$$

It is clear that $\bigoplus_{p+q=r} \Lambda^{p,q} V^* = \Lambda^r V^*$ and that $\dim(\Lambda^{p,q} V^*) = \binom{n}{p}\binom{n}{q}$ and $\overline{\Lambda^{p,q} V^*} = \Lambda^{q,p} V^*$. In basis-free terms, $\Lambda^{p,q} V^*$ is the subspace of all $\omega \in \Lambda^{p+q} V^*$ such that for all $v_1, \ldots, v_r \in V$ and all $z \in \mathbb{C}^\times$, one has

$$\omega(zv_1, \ldots, zv_r) = z^p \bar{z}^q \omega(v_1, \ldots, v_r).$$

Complex vector spaces are naturally oriented; if $(e_1, \ldots, e_n)$ is a $\mathbb{C}$-basis, then the $\mathbb{R}$-basis $(e_1, ie_1, \ldots, e_n, ie_n)$ is said to be positively oriented.

If V is a complex vector space, a *compatible* scalar product is a $\mathbb{R}$-valued scalar product $\langle ; \rangle$ such that J is an orthogonal map. A compatible scalar product extends to a complex scalar product by

$$h(v, w) = \langle v; w \rangle - i \langle v; Jw \rangle.$$

However, when V is the tangent space to a complex manifold, we use only $\mathbb{R}$-valued scalar products. If $\langle ; \rangle$ is a compatible scalar product, one can find an orthonormal basis of the form $(e_1, ie_1, \ldots, e_n, ie_n)$. This is seen by taking a complex orthonormal basis with respect to h.

The Hodge star operator is most usefully not extended as a $\mathbb{C}$-linear operator, but as a $\mathbb{C}$-antilinear operator $\bar{\star}$ (the usual Hodge star, followed by complex conjugation). The extension of the inner product on the real valued forms to complex valued forms is given by the formula

$$\langle \omega ; \eta \rangle \mathrm{vol} = \omega \wedge \bar{\star} \eta .$$

The volume form is an element of $\Lambda^{n,n} V^*$.

Lemma 4.4.1.

(1) *The spaces $\Lambda^{p,q}$ for different values of (p,q) are orthogonal.*
(2) *The Hodge operator $\bar{\star}$ takes $\Lambda^{p,q} V^*$ to $\Lambda^{n-p,n-q} V^*$.*

Proof. The group $S^1 \subset \mathbb{C}^\times$ acts by isometries on V, since the metric is compatible. Thus the induced action on $\Lambda^* V^*$ is by isometries. The subspace $\Lambda^{p,q} V^*$ is the subspace on which S^1 acts by the character $z \mapsto z^{p-q}$. Thus the spaces $\Lambda^{p,q} V^*$ are orthogonal (by the same argument that proves that the eigenspaces of a unitary matrix are orthogonal). The second statement follows by the formula for the scalar product. $\square$

The next lemma, easy as it is, is nothing short of a miracle, it allows us to describe the interplay between compatible metrics and the complex structure for Riemann surfaces. In higher dimensions, the situation is much more complicated.

Lemma 4.4.2. *Let V be a 1-dimensional complex vector space with a compatible metric. For all $\omega \in \Lambda^1 V^*$, the identity*

$$\omega \circ J = - \star \omega$$

holds.

Proof. A straightforward check on a basis: Let $(e_1, e_2 = J e_1)$ be an oriented $\mathbb{R}$-basis of V. Then

$$e^1 \circ J(e_1) = 0;\ e^1 \circ J(e_2) - 1;\ e^2 \circ J(e_1) = 1;\ e^2 \circ J(e_2) = 0$$

and

$$\star e^1(e_1) = e^2(e_1) = 0;\ \star e_1(e_2) = e^2(e_2) = 1;\ \star e^2(e_1) = -e^1(e_1) = -1;\ \star e^2(e_2) = -e^1(e_2) = 0.$$

$\square$

These notions generalize to forms with values in a fixed (finite-dimensional) hermitian vector space (E,h). The natural map $\tau : E \to E^*$, $e \mapsto h(e, \_)$ is a $\mathbb{C}$-antilinear isomorphism. We define

$$\bar{\star}_E : \Lambda^{p,q} V^* \otimes E \to \Lambda^{n-p,n-q} V^* \otimes E^*$$

by

$$\omega \otimes e \mapsto \bar{\star} \omega \otimes \tau(e),$$

an antilinear isometry.

4.5. **The Dolbeault (Cauchy-Riemann) operator.** Let M be a complex manifold. From the linear algebra in the previous section, we get a decomposition $\mathcal{A}^r(M) = \bigoplus_{p+q=r} \mathcal{A}^{p,q}(M)$, $\mathcal{A}^{p,q}(M) := \Gamma(M, \Lambda^{p,q} T^* M)$. In local holomorphic charts, we can write forms in $\mathcal{A}^{p,q}(M)$ as sums of forms of the form

$$a dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_q}; \; a \in C^\infty.$$

If we define, in local coordinates, the operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}); \; \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}),$$

we get that for functions a:

$$da = \sum_{j=1}^n \frac{\partial}{\partial z_j} a dz_j + \frac{\partial}{\partial \bar{z}_j} a d\bar{z}_j.$$

This implies that

$$d\mathcal{A}^{p,q}(M) \subset \mathcal{A}^{p,q+1}(M) \oplus \mathcal{A}^{p+1,q}(M)$$

and we set, for $\omega \in \mathcal{A}^{p,q}(M)$; $d\omega = \partial\omega + \bar{\partial}\omega$, with $\partial\omega \in \mathcal{A}^{p+1,q}$ and $\bar{\partial}\omega \in \mathcal{A}^{p,q+1}$.

Remark 4.5.1. The holomorphic functions are the solutions of $\bar{\partial} f = 0$. Moreover $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

Lemma 4.5.2. *Let $E \to M$ be a holomorphic vector bundle and let $\mathcal{A}^{p,q}(M, E)$ be the space of (p,q)-forms with values in E. Let s be a local holomorphic section of E. If $\omega \in \mathcal{A}^{p,q}(M)$, we define $\bar{\partial}_E(\omega \otimes s) := (\bar{\partial}\omega) \otimes s$. Then*

1. *$\bar{\partial}_E$ is a well-defined differential operator $\mathcal{A}^{p,q}(M,E) \to \mathcal{A}^{p,q+1}(M,E)$ of order 1.*
2. *The symbol is $\mathrm{symb}_{\bar{\partial}_E}(\xi) e = i\xi^{0,1} \wedge e$, where $\xi \in \Lambda^{0,1} T^* M$ denotes the projection of ξ.*
3. *$0 \to \mathcal{A}^{0,0}(M,E) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(M,E) \xrightarrow{\bar{\partial}} \ldots \mathcal{A}^{0,n}(M,E) \to 0$ is an elliptic complex.*
4. *$\Lambda^{p,0}(T^* M) \to M$ is a holomorphic vector bundle, and the diagram*

$$\begin{tikzcd}
\mathcal{A}^{p,q}(M) \arrow[r, "\bar{\partial}"] \arrow[d, "\cong"] & \mathcal{A}^{p,q+1}(M) \arrow[d, "\cong"] \\
\mathcal{A}^{0,q}(M; \Lambda^{p,0} T^* M) \arrow[r, "\bar{\partial}"] & \mathcal{A}^{0,q+1}(M; \Lambda^{p,0} T^* M)
\end{tikzcd}$$

commutes.

Proof. (1) follows because we used a holomorphic local section. (4) is easily seen in local coordinates. The symbol is computed as follows. Let f be a function with $d_x f = \xi$ and s a local holomorphic section with $s(x) = e$ and ω a (p,q)-form. Then

$$\mathrm{symb}_{\bar{\partial}}(\xi)(\omega \otimes e) = i[\bar{\partial}_E, f](\omega \otimes s)(x) = i\bar{\partial}_E(f\omega \otimes s) - if\bar{\partial}_E(\omega \otimes s) = i\bar{\partial} f \wedge \omega \otimes s$$

by the Leibniz rule for the $\bar{\partial}$-operator. But $\bar{\partial} f(x) = \xi^{0,1}$. For (3), it is clear that $\bar{\partial}_E^2 = 0$. The exactness of the symbol sequence is proven exactly as in the real case; the additional argument needed is that $\xi \mapsto \xi^{0,1}$ is an isomorphism $T^*_{\mathbb{R}} M \to \Lambda^{0,1} T^* M$ $(dx_i \mapsto 1/2 d\bar{z}_i,\; dy_i \mapsto i/2 d\bar{z}_i)$. $\square$

Note that there is no canonical possibility to extend the operators ∂ to vector valued forms (a suitable connection on E will give such a possibility).

Problem 4.5.3. (The Riemann-Roch problem) Compute the index of the Dolbeault complex for a holomorphic vector bundle $E \to M$ on a complex closed manifold.

We undertake the first steps in this lecture. Then we specialize to the case of dimension 1. This is the classical Riemann-Roch problem, as we will see. Then we take the first steps towards the solution of the index problem on a Riemann surface. This will motivate the introduction of two major players: characteristic classes and K-theory. In higher dimensions, the index formula for the Dolbeault complex goes under the name Hirzebruch-Riemann-Roch theorem, and we will prove this as a special case of the general index formula.

We now assume that our complex manifold M comes with a compatible Riemann metric.

Proposition 4.5.4. *The adjoint of $\bar\partial : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$ is given by $\bar\partial^* = -\bar\star\bar\partial\bar\star$. More generally, the adjoint of $\bar\partial_E$ is $\bar\partial_E^* = -\bar\star\bar\partial_{E^*}\bar\star$. (See* [32] *p. 168).*

If $E \to M$ is a holomorphic vector bundle, we let $H^p(M,E)$ be the p th cohomology of the elliptic complex ∂_E.

Theorem 4.5.5. *(Serre duality) There is a conjugate linear isomorphism $H^p(M,E) \cong H^{n-p}(M, \Lambda^{n,0}T^*M \otimes E^*)$.*

Proof. There is a diagram that commutes

$$\begin{tikzcd}
\mathcal{A}^{0,p-1}(M,E) \arrow[d, "\bar\star"] & \mathcal{A}^{0,p}(M,E) \arrow[l, "\bar\partial_E^*"'] \arrow[r, "\bar\partial_E"] \arrow[d, "-\bar\star"] & \mathcal{A}^{0,p+1}(M,E) \arrow[d, "\bar\star"] \\
\mathcal{A}^{n,n-p+1}(M,E^*) & \mathcal{A}^{n,n-p}(M,E^*) \arrow[l, "\bar\partial_{E^*}"] \arrow[r, "\bar\partial_{E^*}^*"] & \mathcal{A}^{n,n-p-1}(M,E^*).
\end{tikzcd}$$

The respective cohomology groups are, by the Hodge theorem $H^p(M,E) = \ker(\Delta_E)$, equal to the intersection of the kernels of the horizontal maps. The result follows (note that the vertical arrows are antilinear). □

4.6. **The Hodge decomposition on a Riemann surface.** Let M be a closed connected Riemann surface.

Definition 4.6.1. The *genus* of M is the number $g := \frac{1}{2}\dim H^1(M,\mathbb{R})$.

The genus is an integer by Poincare duality. We study the de Rham complex of M:

$$\begin{tikzcd}
 & \mathcal{A}^{0,1} \arrow[dr, "\partial"] & \\
\mathcal{A}^{0,0} \arrow[ur, "\bar\partial"] \arrow[dr, "\partial"'] & & \mathcal{A}^{1,1} \\
 & \mathcal{A}^{1,0} \arrow[ur, "\bar\partial"'] &
\end{tikzcd}$$

and define $\mathcal{H}^{p,q} = \mathcal{H}^{p+q} \cap \mathcal{A}^{p,q}$, the space of d-harmonic p,q-forms. Since d is real, we have $\star d \star = \bar\star d \bar\star$.

Theorem 4.6.2. *For a closed Riemann surface, the following hold:*

1. $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$.
2. $\mathcal{H}^r = \bigoplus_{p+q=r} \mathcal{H}^{p,q}$.
3. $\mathcal{H}^{0,0} = \ker(\bar{\partial})$.
4. $\mathcal{H}^{1,0} = \ker(\bar{\partial})$
5. $\mathcal{H}^{0,1} = \ker(\bar{\partial}^*) = \mathrm{Im}(\bar{\partial})^\perp$.
6. $\mathcal{H}^{1,1} = \ker(\bar{\partial}^*) = \mathrm{Im}(\bar{\partial})^\perp$.

Proof. (1) is clear. (2) is easy if $r \neq 1$ (in these cases, there is only one summand). Let $\omega \in \mathcal{H}^1$ be harmonic. Then $\star\omega$ is harmonic and thus $\omega \circ J$ as well, by Lemma 4.4.2. But $\mathcal{A}^{1,0}$ and $\mathcal{A}^{0,1}$ are the $\pm i$-eigenspaces of $\circ J$, and the projection onto these is therefore still harmonic. (2) follows.

For the other four part, we first consider the easy inclusions.

(3i) $\omega \in \mathcal{H}^{0,0} \Rightarrow 0 = d\omega = \partial\omega + \bar{\partial}\omega$, and both summands have to be zero.

(4i) If $\omega \in \mathcal{H}^{1,0}$, then $d\omega = 0 = \partial\omega + \bar{\partial}\omega = \bar{\partial}\omega$ ($\partial\omega \in \mathcal{A}^{2,0} = 0$).

(5i) $\omega \in \mathcal{H}^{0,1} \Rightarrow 0 = d^*\omega = \partial^*\omega + \bar{\partial}^*\omega = \bar{\partial}^*\omega$ for degree reasons. The second equality follows from the main regularity theorem.

(6i) $\omega \in \mathcal{H}^{1,1} \Rightarrow 0 = d^*\omega = \partial^*\omega + \bar{\partial}^*\omega = \bar{\partial}^*\omega$ for degree reasons. The second equality follows from the main regularity theorem.

(4ii) $\bar{\partial}\omega = 0$. Then, for degree reasons, $\partial\omega = 0$ and ω is closed. On the other hand, $d^*\omega = -\bar{\star}d\bar{\star}\omega = -\bar{\star}d \star \bar{\omega}$. As $\bar{\omega} \in \mathcal{A}^{0,1}$, $\star\omega = -\omega \circ J = i\omega$. Therefore $-\bar{\star}d \star \bar{\omega} = -\bar{\star}di\omega = i\bar{\star}d\omega = 0$, and so ω is harmonic.

(3ii) $\bar{\partial}f = 0$, then $0 = ddf = \partial\bar{\partial}f + \bar{\partial}\partial f = \bar{\partial}\partial f$. Thus ∂f is exact and in $\ker(\bar{\partial})$. By (4), ∂f is also harmonic, and thus harmonic and exact, therefore zero.

(5ii) $0 = \bar{\partial}^*\omega = -\bar{\star}\bar{\partial}\bar{\star}\omega$. Therefore, $\bar{\star}\omega = \star\bar{\omega} \in \mathcal{A}^{1,0}$ is $\bar{\partial}$-closed and therefore harmonic by (4).

(6ii) $\omega \in \mathcal{A}^{1,1}$, $\bar{\partial}^*\omega = 0$. Then $0 = d^*d^*\omega = \partial^*\bar{\partial}^*\omega + \bar{\partial}^*\partial^*\omega = \bar{\partial}^*\partial^*\omega$. By (5), $\partial^*\omega$ is harmonic and coexact, therefore 0. ◻

Remark 4.6.3. Part (1) of Theorem 4.6.2 holds for all compact complex manifolds, with the same proof. Parts (3)-(6) are specific to the complex dimension 1: in higher dimensions, the individual operators $\bar{\partial}$ are not elliptic, and you should not expect the kernels/cokernels to be finite-dimensional. For complex manifolds of higher dimensions, one would part (2) of the above theorem to be true, i.e.

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M).$$

This is not true in general. For example, if it is true, then $\dim H^1(M) = \dim \mathcal{H}^1(M) = 2\dim \mathcal{H}^{1,0}(M)$ has to be even. But $S^1 \times S^3$ does not have this property, and still has a complex structure: let $\mathbb{Z}$ act on $\mathbb{C}^2 \setminus 0$ by $n \cdot z := \lambda^n z$, $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$. This action is properly discontinuous and by biholomorphic maps, so the quotient $(\mathbb{C}^2 \setminus 0)/\mathbb{Z} \cong S^1 \times S^3$ inherits a complex structure. These complex manifolds are called *Hopf surfaces*.

The additional condition for the decomposition to hold is that the metric on M is *Kähler*, meaning that the 2-fom $\omega(X,Y) = g(X,JY)$ is closed. The complex projective space $\mathbb{CP}^n$ has an essentially unique (up to multiplication by a constant) Riemann metric, the *Fubini-Study metric*, so that the canonical action of $U(n+1)$ is by isometries. This metric is obtained by "averaging" an arbitrary compatible metric using the invariant integral on $U(n+1)$. Hence the form ω is $U(n+1)$-invariant as well.

Complex submanifolds of Kähler manifolds are Kähler. This means that each complex submanifold of $\mathbb{CP}^n$, i.e. a *complex projective variety*, is Kähler. Therefore, Kähler metrics play an important role in complex algebraic geometry.

The key to the Hodge decomposition for Kähler manifolds are the *Kähler identities* relating the Laplace operator Δ to the operators $\bar{\partial}$ and ∂. The proof of the Kähler identities is only differential calculus, but far from trivial and lies beyond the scope of this lecture.

Suggested further reading: The classic text on complex algebraic geometry is [11]. A more terse exposition of the Hodge decomposition in the complex case is in [32]. More recent sources are [1] (more to the differential-geometric side of the story) and [29], [30], more to algebro-geometric side.

Theorem 4.6.4. *Let M be a compact connected Riemann surface. Then*

(1) $\bar{\partial}: \mathcal{A}^{0,0}(M) \to \mathcal{A}^{0,1}(M)$ *has index* $1-g$.
(2) $\bar{\partial}: \mathcal{A}^{1,0}(M) \to \mathcal{A}^{1,1}(M)$ *has index* $g-1$.

Proof. By the previous theorem, one sees that

$$\operatorname{ind}(\bar{\partial}_{\Lambda^{0,0}}) = \dim \mathcal{H}^{0,0} - \dim \mathcal{H}^{0,1};\ \operatorname{ind}(\bar{\partial}_{\Lambda^{1,0}}) = \dim \mathcal{H}^{1,0} - \dim \mathcal{H}^{1,1}.$$

But $\dim \mathcal{H}^{1,0} + \mathcal{H}^{0,1} = \dim \mathcal{H}^1 = \dim H^1(M) = 2g$ by Theorem 4.6.2 (2) and $\dim \mathcal{H}^{1,0} = \dim \mathcal{H}^{0,1}$, so both numbers equal g. A harmonic 0-form is closed, hence constant, whence $\dim \mathcal{H}^{0,0} = 1$. Finally $\dim \mathcal{H}^{1,1} = \dim \mathcal{H}^1 = 1$. □

The above index computation turns out to be enough for the computation of $\operatorname{ind}(\bar{\partial}_E)$ for a general holomorphic vector bundle $V \to M$ over a Riemann surface. We have accumulated enough knowledge to take one further step towards the general case.

Proposition 4.6.5. *Let M be a Riemann surface and $V \to M$ be a holomorphic vector bundle. Then the index $\operatorname{ind}(\bar{\partial}_V) \in \mathbb{Z}$ depends only on the vector bundle V, not on the holomorphic structure.*

Proof. We have seen that the symbol of $\bar{\partial}_E$ is $\operatorname{symb}_{\bar{\partial}_E}(\xi) = i\xi^{0,1} \wedge \_$, and this means that *the symbol only depends on the complex structure of M, not on the holomorphic structure on E.* If two holomorphic structures are given on E, denote the Cauchy-Riemann operators by $\bar{\partial}_E^0$ and $\bar{\partial}_E^1$. For each $t \in [0,1]$, the operator $D_t : (1-t)\bar{\partial}_E^0 + t\bar{\partial}_E^1$ is elliptic and has the same symbol.

Thus we get a path $[0,1] \to \operatorname{Fred}(W^1(M,E); L^2(M,E))$, $t \mapsto D_t$, and this path is continuous. Since the index is homotopy invariant, we see that $\operatorname{ind}_{\bar{\partial}_E^0} = \operatorname{ind}_{\bar{\partial}_E^1}$. □

Proposition 4.6.6. *Let $E \to M$ be a complex vector bundle on the Riemann surface. Then $\sigma(\xi) = i\xi^{0,1} \wedge \_$ is an elliptic symbol, and there is an elliptic operator D_E with that symbol.*

This is clear (existence of differential operators with given symbol).

Definition 4.6.7. Denote by $\operatorname{Vect}(M)$ the set of isomorphism classes of complex vector bundles. It becomes a commutative semigroup by taking direct sums of vector bundles.

The previous two propositions show that

$$[V] \mapsto \mathrm{ind}(D_V)$$

is a well-defined homomorphism map $\mathrm{Vect}(M) \to \mathbb{Z}$. It is trivial that $\mathrm{ind}(D_{V\oplus W}) = \mathrm{ind}(D_V) + \mathrm{ind}(D_W)$, and we have a semigroup homomorphism.

Proposition 4.6.8. *Let $(A, \oplus)$ be a commutative semigroup. Let $F(A)$ be the quotient of the free abelian group $\mathbb{Z}A$, by the subgroup generated by the elements*

$$a \oplus b - a - b$$

and let $\iota : A \to F(A)$, $a \mapsto a$, be the natural homomorphism. Then if B is any abelian group and $f : A \to B$ a homomorphism of semigroups, then there is a unique group homomorphism $g : F(A) \to B$ such that $g \circ \iota = f$. If A is already a group, the ι is an isomorphism.

Proof. This is clear (universal property formal nonsense). □

Definition 4.6.9. Let X be a compact Hausdorff space. The K-theory group of X is $K^0(X) := F(\mathrm{Vect}(X))$.

Let us summarize what we have achieved so far.

Proposition 4.6.10. *Let M be a Riemann surface. There is a unique homomorphism $I : K^0(M) \to \mathbb{Z}$ such that for each holomorphic vector bundle $V \to M$, the identity $I(V) = \mathrm{ind}(\bar{\partial}_V)$ holds.*

The rest of the proof of the index theorem for Riemann surfaces will now be:

- Find numbers that one can attach to complex vector bundles on a surface (one will be of course the rank, the other will be the *Chern number*).
- Prove that the bundles $\mathbb{C}$ and $\Lambda^{1,0}$ generate $K^0(X)$ in a suitable way and find the right linear combination of the numerical invariants.

4.7. **Relation to the classical theory.** In the literature on Riemann surfaces, the Riemann-Roch theorem is typically not stated as an index theorem for an elliptic operator. Let us briefly describe the classical outlook of the theorem. Let X be a compact Riemann surface. A formal linear combination $D = \sum_{i=1}^r n_i p_i$, $p_i \in X$ points, $n_i \in \mathbb{Z}$, is called a *divisor*. We identify the relation $0p_i = 0$ and $np + mp = (m+n)p$; with these conventions the set of divisors becomes an abelian group $\mathrm{div}(X)$. A divisor is nonnegative if $n_i \geq 0$ and we say $D_1 \geq D_0$ if $D_1 - D_0$ is nonnegative.

For example, consider a meromorphic function f on X. Let $p_1, \ldots, p_s$ be the zeroes and let n_i be the order of f at p_i. Moreover, let $p_{s+1}, \ldots, p_r$ be the poles, with order $-n_i$. We denote by $(f) = \sum_{i=1}^r n_i p_i$ the divisor of f. More generally, if f were a meromorphic section of a line bundle, we can apply the same idea and get a divisor (f) on X.

The *degree* of the divisor D is the sum $\sum_i n_i \in \mathbb{Z}$. A divisor is *principal* if there exists a meromorphic *function* f with $D = (f)$.

To any divisor, one can construct a line bundle L_D, in the following canonical way (this uses the cocycle description of vector bundles). Let $D = \sum_i n_i p_i$ be a divisor, written in minimal form. Let $U_0 = X - \{p_i\}$, and let U_i be a disc neighborhood of p_i. Pick holomorphic charts $h_i : U_i \to \mathbb{E}$, $h_i(p_i) = 0$ and assume that the U_i are disjoint for $i \geq 1$. Let $L_D = \coprod_i U_i \times \mathbb{C}/\sim$; the equivalence relation is that

$U_0 \times \mathbb{C} \ni (x, z) \sim (x, zh_i(x)^{n_i})$ whenever $x \in U_0 \cap U_i$. With the obvious projection to X, this becomes a holomorphic line bundle L_D.

This line bundle comes equipped with a meromorphic section; namely, take $s_D(x) = 1$ over U_0. Inspection shows that $(s_D) = D$ holds. The construction satisfies $L_{D_0+D_1} \cong L_{D_0} \otimes L_{D_1}$. If $D = (f)$ for a meromorphic function, the bundle L_D is trivial, because $f^{-1} s_D$ is a meromorphic section without zeroes or poles. More generally, if s is a meromorphic section of a line bundle, then $L_{(s)} \cong L$. It follows, by the Poincaré-Hopf theorem, that the degree of D equals the Chern number $\int_X c_1(L_D)$.

Let $\mathcal{M}^\times(X)$ be the multiplicative group of nonzero meromorphic functions and $H^1(X, \mathcal{O}^\times)$ the group of isomorphism classes of holomorphic line bundles, we get an exact sequence

$$0 \to \mathbb{C}^\times \to \mathcal{M}^\times(X) \to \operatorname{div}(X) \to H^1(X, \mathcal{O}^\times)$$

(it is indeed a cohomology sequence of a sequence of sheaves). The image of the last map is the group of all line bundles that have a meromorphic section. These are all line bundles (and so the sequence is exact at the end).

Lemma 4.7.1. *Each holomorphic line bundle over a Riemann surface has a nonzero meromorphic section.*

Proof. Let the Chern number c of L be at least $2g - 1$. Then, by Serre duality $\dim \operatorname{coker} \bar{\partial}_L = \dim \ker \bar{\partial}_{\Lambda^{1,0} \otimes L^*}$. By Poincaré-Hopf, this is zero, since the Chern number of $\Lambda^{1,0} \otimes L^*$ equals $2g - 2 - c < 0$. By Riemann-Roch, we conclude that $\dim \ker \bar{\partial}_L = \operatorname{ind} \bar{\partial}_L = 1 - g + d > 0$. Thus each line bundle of large degree has a homlomorphic section.

For a given line bundle L, pick a line bundle L' such that $c(L') - c(L)$ and $c(L')$ are both at least $2g - 1$. We get a holomorphic section s of L' and t of $L' \otimes L^*$. Then st^{-1} is the desired meromorphic section. □

Now suppose that s is a holomorpic section of L_D. In the chart over U_0, we get simply a holomorphic function g. At a point p_i with $n_i < 0$, g must have a zero, of order at least $-n_i$, while if $n_i > 0$, g has at worst a pole of order n_i.

So we see:

Lemma 4.7.2. *The space* $\ker(\bar{\partial}_{L_D})$ *of holomorphic sections of* L_D *is isomorphic to the space of meromorphic functions* g *such that* $(g) \geq D$.

The bundle $\Lambda^{1,0}$ is called the *canonical bundle* in the classical theory. Any divisor associated with a meromorphic section of K is called *canonical divisor* and denoted K. Using Serre duality, we might now restate the Riemann-Roch theorem:

Theorem 4.7.3. *(Riemann-Roch, classical version) For each divisor* D *on a compact Riemann surface of genus* g*, of degree* d*, we have*

$$\dim \ker(\bar{\partial}_{L_D}) - \dim \ker(\bar{\partial}_{L_{K-D}}) = 1 - g + d.$$

It is worth to work out everything for $\mathbb{CP}^1$. For the proof of the Riemann-Roch theorem, we also need to look at the case $g = 1$.

Proposition 4.7.4. *Let* X *be a Riemann surface of genus* $g = 1$ *and let* $x \in X$*. Then the index of* $\bar{\partial}_{L_{(x)}}$ *is equal to* 1.

Proof. First we show that $\Lambda^{1,0}$ is the trivial holomorphic line bundle. This is obvious if we use the fact that X must be a complex torus $\mathbb{C}/\Gamma$, but we do not wish to rely on that. Instead, by Theorem 4.6.2, the space of holomorphic sections of $\Lambda^{1,0}$ is one-dimensional. Pick a nonzero holomorphic section ω. As the Chern number of $\Lambda^{1,0}$ is zero, by the topological Gauss-Bonnet theorem, and because all local indices of holomorphic sections are positive, we find that ω has no zeroes; in other words, the bundle $\Lambda^{1,0}$ is holomorphically trivial.

The space $\ker(\bar{\partial}_{L_{(x)}})$ is the space of meromorphic functions on X which have at worst a simple pole at x. It contains the constant functions, and a meromorphic function on X with a single simple pole at x can be viewed as a map $f : X \to \mathbb{CP}^1$. As ∞ is a regular value, f must have degree 1. It follows that for all regular values z, $f^{-1}(z)$ must be a single point (this uses the holomorphicity of f). Near a critical point of f of order k, f assumes each value k times, so we conclude that f has no critical values and therefore is a diffeomorphism, contradicting the assumption that $g = 1$. We conclude that $\ker(\bar{\partial}_{L_{(x)}})$ is one-dimensional.

The space $\ker(\bar{\partial}_{L_{(-x)}})$ is the space of holomorphic functions on X which have a zero at x. As each holomorphic function on X is constant, $\ker(\bar{\partial}_{L_{(-x)}}) = 0$. □

Exercise 4.7.5. *Find a canonical divisor of* $\mathbb{CP}^1$. *Prove that each divisor is linearly equivalent to* $n \cdot 0$*, for a unique* $n \in \mathbb{Z}$*. Compute* $\dim H^0(X, D)$ *by hands and verify the Riemann-Roch theorem by hands.*

The proof of Riemann-Roch that we gave used the main theorem on elliptic regularity and the theory of characteristic classes as the main ingredients. While the characteristic class theory was overkill (in fact, we only needed the first Chern class only, and that can be done in an easier way), the use of the regularity theorem is, most emphatically, neccessary. In Riemann surface texts, the analysis going into the Riemann-Roch theorem is a version of the general theory (which can be somehow simplified, but is still difficult).

If one knows in advance that X is a projective variety (i.e. a complex submanifold of $\mathbb{CP}^n$ for some n), then it is known (Chow's theorem) that X is algebraic and in this case, there is a purely algebraic proof of Riemann-Roch (GAGA). In fact, each Riemann surface can be embedded into projective space, and this is a consequence of Riemann-Roch!

5. Some bundle techniques

5.1. **Vector bundles.** The definition of a vector bundle won't be repeated here. We work with vector bundles over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. There are two categories: differentiable vector bundles over smooth manifolds, and topological vector bundles over Hausdorff spaces. All theorems will hold in both categories. We formulate everything for topological bundles, replacing the words "topological space" by "manifold", "vector bundle" by "smooth vector bundle" and "continuous" by "smooth" gives a valid argument. When $\pi : V \to X$ is a vector bundle, we denote the fibres by $V_x := \pi^{-1}(x)$.

Definition 5.1.1. Let $V \to X$ be a vector bundle. A *subbundle* $W \subset V$ is a union of sub vector spaces of the fibres $W = \coprod_{x \in X} W_x$, such that W is locally trivial in the subspace topology.

There are several types of bundle maps. Unfortunately, there is no suggestive terminology. The most general notion is when $W \to Y$, $V \to X$ are vector bundles and $f : X \to Y$ is a continuous map. One considers maps $\varphi : V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes and such that φ is fibrewise linear (and continuous). We call such an φ a *bundle morphism over* f. Two special cases are important enough to deserve a name on its own:

(1) If f is the identity map on X, we call φ a *vector bundle homomorphism.*
(2) If f is arbitrary, but $\varphi : V_x \to W_{f(x)}$ is an isomorphism for each $x \in X$, then φ is called a *bundle map over* f.

If you know a better name for these things, please let me know. The pullback of vector bundles has the following universal property. Let $f : X \to Y$ and $\pi : W \to Y$ be a vector bundle. Then there is a bundle map $\hat{f} : f^*W := \{(x, w) \in X \times W | f(x) = \pi(w)\} \to W$ over f, defined by $\hat{f}(x, w) = w$. Assume that $V \to X$ is another bundle and $\phi : V \to W$ be a bundle morphism oder f. Then there is a unique bundle homomorphism $\varphi : V \to f^*W$ such that $\hat{f} \circ \varphi = \phi$.

Lemma 5.1.2. *Any vector bundle over a paracompact base space admits a bundle metric.*

Lemma 5.1.3. *Let $F : V \to W$ be a vector bundle homomorphism which is bijective. Then F is an isomorphism of vector bundles, i.e. F^{-1} is continuous.*

Proof. This follows from the fact that the inversion map on $\mathrm{GL}_n(\mathbb{K})$ is differentiable. □

Lemma 5.1.4. *A subbundle $W \subset V$ has adapted charts, i.e for each $x \in X$, there is a neighborhood U and a bundle chart $V|_U \cong U \times \mathbb{K}^n$ that sends $W|_U$ to $U \times \mathbb{K}^m$.*

Proof. The problem is a local one, which is why we can assume that $V = X \times \mathbb{K}^n$. Let $o \in X$, and let $U_o \subset \mathbb{K}^n$ be a complement of W_o. Consider $F : W \oplus U_o \to V$, $(w, u) \mapsto w + u$; a bundle homomorphism. F is an isomorphism at o, and so it is for all x in a neighborhood U of o. By Lemma 5.1.3, F gives an isomorphism $W|_U \oplus U_o \cong V|_U$. The inverse is the desired adapted chart. □

Corollary 5.1.5. *Let $V \subset X \times \mathbb{K}^n$ be a subbundle and let P_x be the orthogonal projection onto V_x. Then $X \mapsto \mathrm{Mat}_{n,n}(\mathbb{K})$, $x \mapsto P_x$ is continuous.*

Proof. The problem is local, so assume $V \cong X \times \mathbb{K}^m$. This isomorphism defines sections $s_1, \ldots, s_m$ of V which are everywhere linear independent. Applying the Gram-Schmidt process to $s_1, \ldots, s_m$ defines an orthonormal basis $t_1, \ldots, t_m$ of V. The inclusion $V \to X \times \mathbb{K}^m$ is given by a continuous function $A : X \to \mathrm{Mat}_{n,m}(\mathbb{K})$ that takes values in the matrices A such that $A^*A = 1_m$. The orthogonal projection is $P = AA^*$. □

With the same technique, one can prove.

Lemma 5.1.6. *Let $F : V \to W$ be a vector bundle homomorphism, and assume that $x \mapsto \mathrm{rank}(F_x)$ is constant. Then $\ker(F)$ is a vector bundle.*

Proof. Again, the problem is local, so we can assume that V and W are trivial and F is given by a continuous function $F : X \to \mathrm{Mat}_{m,n}(\mathbb{K})$. Let $o \in X$ and P the orthogonal projection onto $\ker(F_o)$. Consider $G = F^*F + P$. At o, G is an isomorphism, so it is for nearby $x \in U \subset X$. By definition, G_x maps $\ker(F_x)$ to $\ker(F_o)$. Since G_x is an isomorphism and the dimensions of $\ker(F_o)$ and $\ker(F_x)$ agree, $G_x : \ker(F_x) \to \ker(F_o)$ is an isomorphism. It follows that G is a bundle isomorphism over U that maps $\ker(F)$ to $X \times \ker(F_o)$ and hence it reveals $\ker(F)$ as a subbundle. □

Corollary 5.1.7.
1. *The orthogonal complement of a subbundle is again a vector bundle.*
2. *The image of a vector bundle homomorphism with constant rank is a vector bundle.*

Proof. Let $W \subset V$ be a subbundle. Equip V with a bundle metric. Let $P : V \to V$ be the orthogonal projection onto W. By looking at adapted charts, one sees that P is a continuous bundle homomorphism. The orthogonal complement $W^\perp$ is $\ker(P)$, which by Lemma 5.1.6 is a subbundle. For the second part, pick bundle metrics and observe that $\mathrm{Im}(F) = \ker(F^*)^\perp$, which by the first part and Lemma 5.1.6 is a vector bundle. □

The most important vector bundle is the tautological bundle.

Definition 5.1.8. The Grassmann manifold of k-dimensional subspaces of $\mathbb{K}^n$ $\mathrm{Gr}_k(\mathbb{K}^n)$ is the set of all k-dimensional subspaces of $\mathbb{K}^n$. We identify $\mathrm{Gr}_k(\mathbb{K}^n)$ with the set $\{P \in \mathrm{Mat}_{n,n}(\mathbb{K}) | P^2 = P; P^* = P, \mathrm{rank}(P) = k\}$, by sending a subspace $V \subset \mathbb{K}^n$ to the orthogonal projection onto it (and by sending a projection onto its image). Let $V_{k,n} \subset \mathrm{Gr}_k(\mathbb{K}^n) \times \mathbb{K}^n$ be the set of all pairs (V, v), $V \in \mathrm{Gr}_k(\mathbb{K}^n)$, $v \in V$.

The Grassmann manifold is compact (it is a closed bounded subset of the space of matrices), and we will see soon that it is indeed a manifold.

Lemma 5.1.9. *$V_{k,n}$ is a subbundle of the trivial vector bundle.*

Proof. $V_{k,n}$ is the image of the canonical homomorphism $(P, v) \mapsto (P, Pv)$ of the bundle $\mathrm{Gr}_k(\mathbb{K}^n) \times \mathbb{K}^n$. □

Proposition 5.1.10. *Let X be a space. Then there is a bijection between the set of k-dimensional subbundles of $X \times \mathbb{K}^n$ and continuous maps $X \to \mathrm{Gr}_k(\mathbb{K}^n)$. The bijection sends a bundle $V \subset X \times \mathbb{K}^n$ to the map $x \mapsto V_x$; the other direction is $f \mapsto f^* V_{n,k}$.*

If $V \to X$ is a vector bundle, then bundle monomorphisms $V \to X \times \mathbb{K}^n$ are in bijection with pairs (f, a), $f : X \to \mathrm{Gr}_k(\mathbb{K}^m)$ and $a : V \cong f^* V_{k,n}$.

Proof. Let V be a subbundle. The map $f_V : X \to \mathrm{Gr}_k(\mathbb{K}^n)$, $x \mapsto V_x$ is continuous, by Lemma 5.1.5. Clearly both bijections are mutually inverse. □

One should think of a vector bundle as a family of vector spaces that depends continuously on a space X. This proposition gives some first credibility that this way of thinking is indeed accurate. There are two steps missing: first we need to show that any vector bundle is indeed isomorphic to a subbundle of a trivial vector bundle, and this isomorphism needs to be canonical in a reasonable way. Second, we want to prove that homotopic maps give rise to isomorphic vector bundles.

Theorem 5.1.11. *Let $F : [0,1] \times X \to Y$ be a homotopy from F_0 to F_1 and $V \to Y$ be a vector bundle. If X is paracompact, then $F_0^* V \cong F_1^* V$.*

Before we give the proof, let us collect a technical lemma.

Lemma 5.1.12. *Let $V_i \to X$, $i = 0, 1$, be vector bundles over a paracompact Hausdorff space and let $A \subset X$ be closed. Assume that $V_0|_A \cong V_1|_A$. Then there exists a neighborhood U of A and a bundle isomorphism $V_0|_U \cong V_1|_U$.*

Proof. Let $\phi : V_0|_A \to V_1|_A$ be an isomorphism. We can find open sets $U_i \subset X$, $i \in I$, $0 \notin I$ that cover A, such that $V_j|_{U_i}$ is trivial and take $U_0 = X - A$ as another open set. Let λ_i be a partition of unity subordinate to the covering. Since paracompact spaces are normal, the space U_i is normal. The restriction of ϕ to U_i is given by a function $U_i \cap A \to \mathrm{GL}_n(\mathbb{K})$, with respect to some unnamed bundle charts.

By Tietze's extension theorem, we can find extension ϕ_i of $\phi|_{A \cap U_i}$ over U_i. ϕ_i is only a bundle homomorphism, not an isomorphism. Put $\psi = \sum_{i \in I} \lambda_i \phi_i$. This is a bundle homomorphism, and an isomorphism over A. Since being an isomorphism is a local condition, ψ is an isomorphism over some neighborhood U of A. □

Remark 5.1.13. In the differentiable case, there are two types of extension problems one could consider. If $A \subset X$ is an arbitrary closed subset, one calls a function $A \to \mathbb{R}$ smooth if it is the extension of a smooth function on some neighborhood of A. In this case, the statement of Lemma 5.1.12 is vacuous. The other relevant case is when $A \subset X$ is also a submanifold. In that case, one has to use a tubular neighborhood of A in X.

Proof, under the additional assumption that X is compact. Let $j_t : X \to [0,1] \times X$ be the inclusion $x \mapsto (t, x)$. Since $F_t = F \circ j_t$, it is enough to prove the theorem when F is the identity, viewed as a homotopy from j_0 to j_1. In other words, we assume that $V \to [0,1] \times X$ is a vector bundle, and let $V_t := j_t^* V$. We want to show that $V_0 \cong V_1$. To this end, we introduce an equivalence relation $\sim$ on $[0,1]$: $t \sim s$ iff $V_t \cong V_s$. Of course, this is an equivalence relation. Once we prove that the equivalence classes are open, we are done, since $[0,1]$ is connected. Fix $t \in [0,1]$ and consider the bundles $V_t \times [0,1]$ and V over $X \times [0,1]$. By definition, their restrictions to $X \times \{t\}$ are isomorphic. By Lemma 5.1.12, we find a neighborhood

$X \times t \subset U \subset X \times [0,1]$ over which these two bundles are isomorphic. By compactness, U contains a strip $X \times (t-a, t+b)$. Hence if $s \in (t-a, t+b)$, then $V_t \cong V_s$. ☐

The general case needs similar ideas, but with more care. I recommend to read the proof in [27] in greater generality.

Theorem 5.1.14. *Let X be a compact space and $\pi : V \to X$ be a vector bundle of rank k. Then there exists $n >> 0$ and an injective bundle homomorphism $\phi : V \to X \times \mathbb{K}^n$. If moreover $A \subset X$ is closed and $\psi : V|_A \to A \times \mathbb{K}^m$ is an already given bundle monomorphism, then we can pick ϕ to coincide on A with $i_{n,m} \circ \psi$, where $i_{n,m} : X \times \mathbb{K}^m \to X \times \mathbb{K}^n$ (the price one has to pay is that m is potentially very large).*

Proof. Let U_i, $i = 1, \dots, r$, be an open cover, $(\pi; h_i) : V|_{U_i} \cong U_i \times \mathbb{K}^k$ bundle trivializations and λ_i be a partition of unity subordinate to this cover. Let $n = rk$ and define $\phi :: V \to X \times (\mathbb{K}^k)^r$ by

$$\phi(v) := (\pi(x), \lambda_1(\pi(v))\phi_1(v), \dots, \lambda_r(\pi(v))\phi_r(v)).$$

This is a bundle injection, as one checks easily. For the relative case, let U_0 be a neighborhood of A and $(\pi, \phi_0) : V|_{U_0} \to X \times \mathbb{K}^n$ be an extension of ψ to a bundle homomorphism, as guaranteed by Lemma 5.1.12. Since being injective is an open condition, we can assume that (π, ϕ_0) is injective (after making U_0 smaller). Let $(\pi, \phi_1) : V \to X \times \mathbb{K}^m$ be an embedding as just constructed. Let μ be a function which is equal to 1 on A and has support in U_0. Let $\phi(v) := (\pi(v), \mu(\pi(v))\phi_0(v), (1 - \mu(\pi(v)))\phi_1(v))$, which is the desired extension. ☐

Corollary 5.1.15. *For each vector bundle $V \to X$ over a compact space, there is a bundle $V^\perp \to X$ such that $V \oplus V^\perp \cong X \times \mathbb{C}^n$.*

Remark 5.1.16. For compact manifolds, we can use the same argument. It is a little surprising that in the smooth case, the compactness of X is not necessary. More precisely, if $V \to M$ is a smooth vector bundle, we can embed V into $M \times \mathbb{R}^m$, for some large m. The reason is the *Whitney embedding theorem.* Since V is among other things a manifold, we can find an embedding of manifolds $j : V \to \mathbb{R}^m$. The differential $dj : TV \to V \times \mathbb{R}^m$ is an everywhere injective homomorphism of vector bundles over V. But the restriction of TV to the zero section is nothing else than $TM \oplus V$, so we can produce the desired embedding.

If V is complex, we first take a real embedding $f : V \to X \times \mathbb{R}^m \subset \mathbb{C}^m$ and define a $\mathbb{C}$-linear embedding by $\hat{f}(v) = f(v) - if(iv)$.

We can now formulate and prove the classification theorem for vector bundles. Let X be a compact space and $[X, \mathrm{Gr}_k(\mathbb{K}^n)]$ be the set of homotopy classes. Moreover, $\mathrm{Vect}^k_{\mathbb{K}}(X)$ is the set of isomorphism classes of rank k vector bundles over X. If $f : X \to \mathrm{Gr}_k(\mathbb{K}^n)$, we can form $f^* V_{k,n} \to X$. The isomorphism class of this vector bundles does not depend on f, by Theorem 5.1.11. So we get a well-defined map

$$[X, \mathrm{Gr}_k(\mathbb{K}^n)] \to \mathrm{Vect}^k_{\mathbb{K}}(X).$$

There is no n on the right hand side. In fact, there is an inclusion $i : \mathrm{Gr}_k(\mathbb{K}^n) \to \mathrm{Gr}_k(\mathbb{K}^{n+1})$, and $i^* V_{k,n+1} \cong V_{k,n}$. Thus, by making n larger and larger, we obtain a map

$$\mathrm{colim}_n[X, \mathrm{Gr}_k(\mathbb{K}^n)] \to \mathrm{Vect}^k_{\mathbb{K}}(X).$$

Theorem 5.1.17. *For each compact Hausdorff space X, the above map is a bijection.*

Proof. Since any vector bundle can be embedded into $X \times \mathbb{K}^n$, the map is surjective. Let $f_i : X \to \mathrm{Gr}_k(\mathbb{K}^{n_i})$ be two maps, which represent two elements in the colimit that go to the same vector bundle V. We can take both f_i to go to $\mathrm{Gr}_k(\mathbb{K}^n)$, $n \geq n_0, n_1$. What this means is that there are isomorphisms $a_i : V \cong f_i^* V_{k,n}$. In other words, we have two bundle maps $j_i : V \to X \times \mathbb{K}^n$, which is the same as a bundle map of $V \times \{0,1\} \to X \times [0,1] \times \mathbb{K}^n$. By Theorem 5.1.14, we can extend this to a bundle map of j, after increasing n. This means that f_0 and f_1 are homotopic, after increasing n. □

If X is compact, then $\mathrm{colim}_n[X, \mathrm{Gr}_k(\mathbb{K}^n)] \cong [X, \mathrm{colim}_n \mathrm{Gr}_k(\mathbb{K}^n)]$ (a property of the colimit topology). Thus we have shown that there is a bijection

$$[X, \mathrm{Gr}_k(\mathbb{K}^\infty)] \to \mathrm{Vect}_k(X).$$

5.2. **Principal bundles.** A more flexible jargon to talk about bundles is provided by the theory of *principal bundles*. Let us briefly recall the notion of fibre bundle.

Definition 5.2.1. A *fibre bundle* over a space X is a map $\pi : E \to X$ so that for each $x \in X$, there exists a neighborhood U of x and a homeomorphism $\pi^{-1}(U) \cong U \times \pi^{-1}(x)$ over U. The space $\pi^{-1}(x) =: E_x$ is called the fibre over x.

At least if X is connected, then all fibres are homeomorphic. Sometimes, we say that $\pi : E \to X$ is a fibre bundle with fibre F, if all fibres are homeomorphic to F. But there is a danger in this notion, because it invites the reader to *identify* all fibres with each other, which will inevitably get you into hot water. This is because there are several ways of identifying the fibres E_x with F. The theory of principal bundles provides a precise calculus to keep track of all identifications and if you understand it, you have taken a big psychological hurdle when dealing with bundles.

Definition 5.2.2. Let G be a topological group and X a space. A *G-principal bundle* consists of a right G-space E and a continuous map $\pi : E \to X$, such that the following condition holds: For each $x \in X$, there is a neighborhood U and a homeomorphism $\phi : \pi^{-1}(U) \to U \times G$, such that $\mathrm{pr}_G \circ \phi = \pi$. Moreover, ϕ is G equivariant when $U \times G$ is equipped with the action $(x,h)g := (x, hg)$.

A bundle map $E \to E'$ of G-principal bundles (possibly over different spaces) is a G-equivariant map. Any bundle map covers a map $f : X \to X'$. If the bundle map covers the identity, it is bijective and we say that the bundle map is a bundle isomorphism.

It can be shown that a bijective bundle map is a homeomorphism, and this justifies our usage of the word "isomorphism".j In our applications, G will be a Lie group. Requiring that E and X are smooth manifolds, the action of G and π and ϕ being smooth, one arrives at the notion of a smooth principal bundle. There is a notion of pullback: if $f : Y \to X$ is a map and $\pi : E \to X$ a G-principal bundle, then $f^*E := \{(y,e) \in Y \times E | f(y) = \pi(e)\}$ has the natural structure of a G-principal bundle. If you are a novice in bundle theory, you are invited to provide the details as an exercise.

Exercise 5.2.3. *Let X be a connected space that has a universal covering $\tilde{X} \to X$. Equip $\tilde{X}$ with the structure of a $\pi_1(X,x)$-principal bundle (this is irrelevant for index theory).*

The most substantial example of a principal bundle is the *frame bundle* of a vector bundle.

Example 5.2.4. Let $V \to X$ be an n-dimensional $\mathbb{K}$-vector bundle. Let $\mathrm{Fr}(V) := \coprod_x \mathrm{Iso}(\mathbb{K}^n, V_x)$, $\mathrm{Iso}(\mathbb{K}^n; V_x)$ is the set of all vector space isomorphisms. There is an obvious map $\pi : \mathrm{Fr}(V) \to X$. The $\mathrm{GL}_n(\mathbb{K})$-action is by precomposition: if $f : \mathbb{K}^n \to V_x$ is an isomorphism and $g \in \mathrm{GL}_n(\mathbb{K})$, then $f \cdot g := f \circ g$. If $U \subset X$ is open and $\phi : U \times \mathbb{K}^n \to V|_U$ be a trivialization, we get a bijection $U \times \mathrm{GL}_n(\mathbb{K}) \to \pi^{-1}(U) = \mathrm{Fr}(V|_U)$, namely

$$(x, g) \mapsto \phi_x \circ g.$$

The topology on $\mathrm{Fr}(V)$ is the finest one so that all these maps are continuous, and they are all homeomorphisms.

Exercise 5.2.5. *Provide the details of the proof that the above construction gives indeed a $\mathrm{GL}_n(\mathbb{K})$-principal bundle. If V is a smooth vector bundle, equip $\mathrm{Fr}(V)$ with the structure of a smooth principal bundle.*

Remark 5.2.6. A point in $\mathrm{Fr}(V)$ is by definition an isomorphism $f : \mathbb{K}^n \to V_x$ for some x. If $e_i \in \mathbb{K}^n$ denotes the ith basis vector, we get a basis $(v_1, \ldots, v_n)$ of V_x, $v_i := f(e_i)$. Now let $g = (g_{ij}) \in \mathrm{GL}_n(\mathbb{K})$. Note that $ge_i = \sum_{k=1}^n g_{ki} e_k$ (sic!). Therefore $f \circ g(e_i) = \sum_{k=1}^n g_{ki} v_i$. So if we view frames as bases of the fibres, the $\mathrm{GL}_n(\mathbb{K})$-action becomes $(v_1, \ldots, v_n) \cdot g := (\sum_{k=1}^n a_{k1} v_k, \ldots \sum_{k=1}^n a_{kn} v_k)$. This might be confusing, but it is not bundle theory that is to be blamed, but linear algebra.

Exercise 5.2.7. *Prove that local trivializations of a vector bundle $V \to X$ are in bijective correspondance with local cross-sections of $\mathrm{Fr}(V)$. More generally, local trivializations of a principal bundle are in bijection with cross-sections. A principal bundle has a global cross-section iff it is trivial.*

Exercise 5.2.8. *Let $V \to M$ be a rank n vector bundle with Riemannian bundle metric. Define the $O(n)$-principal bundle $\mathrm{Fr}^O(V)$ of orthonormal frames. Similarly, let $V \to M$ be oriented. Define the $\mathrm{GL}_n(\mathbb{R})^+$-principal bundle of oriented frames.*

A principal bundle is, among other things, a G-space E, and the base space X is the quotient E/G. The G-action is free. The next result is a basic fact in the theory of Lie groups. The proof can be found in [25]. If G is linear, then you can find an easier proof in [4], but the details are still quite subtle.

Theorem 5.2.9. *Let G be a Lie group and $H \subset G$ be a closed subgroup. Then H is a Lie group, the quotient space G/H has the unique structure of a smooth manifold such that $G \to G/H$ is smooth, and the quotient map $G \to G/H$ is a H-principal bundle.*

The power of this result can be explained by some examples. First a lemma.

Lemma 5.2.10. *Let M be a smooth manifold and let G be a Lie group that acts transitively from the left on M. Let $x \in M$ and $H \subset G$ be the isotropy group of x. Then $G/H \to M$, $gH \to gx$ is a diffeomorphism; $G \to M$, $g \mapsto gx$ is a H-principal bundle.*

Proof. The map $p : G \to M$, $g \mapsto gx$ is smooth and H-invariant, $p(gh) = p(g)$. Therefore it descends to a smooth map $f : G/H \to M$ which is moreover bijective. We claim that this is a diffeomorphism. Since f is G-equivariant (G acting from the left!) and the action on both G/H and M is transitive, the rank of df is constant. By Sard's theorem, f has a regular value, and so f must be a submersion. Since f is injective, the dimensions have to agree, and so f is a bijective map which has everywhere full rank, i.e. a diffeomorphism. □

Examples 5.2.11.

(1) *$O(n+1)$ acts on S^n, by rotations. The isotropy group of the vector e_{n+1} is $O(n)$. This shows that $O(n+1)/O(n) \cong S^n$ is a diffeomorphism. In fact, we can identify $O(n+1)$ with the total space of the orthogonal frame bundle of TS^n (with the usual metric of the sphere).*
(2) *In a similar way, $U(n) \to S^{2n-1}$ is a $U(n-1)$-principal bundle.*
(3) *$\mathbb{CP}^n$ is $U(n+1)/U(n)\times U(1)$. More generally, the Grassmannian is $\mathrm{Gr}_k(\mathbb{C}^n) \cong U(n)/U(k) \times U(n-k)$.*
(4) *The quotient map $S^{2n+1} \to \mathbb{CP}^n$ is the unitary frame bundle of the tautological line bundle.*

Corollary 5.2.12. *Let G be a Lie group and let G (as a group) act transitively on a set S. Then S has a unique topology and smooth structure so that the G-action is smooth.*

Besides pullbacks, there are a couple of useful constructions with principal bundles.

Definition 5.2.13. Let $E \to X$ be a G-principal bundle and $F \to Y$ be an H-principal bundle. Then $E \times F \to X \times Y$ is a $G \times H$-principal bundle with the product action.

The next one is what we call "change of fibre", which is *very* important. If you want to be comfortable with bundles, you have to absorb this construction.

Definition 5.2.14. Let $\pi : P \to X$ be a G-principal bundle and F a left G-space. The group G acts on the space $P \times F$ diagonally, $(p, f) \cdot g := (pg, g^{-1}f)$. We define $P \times_G F := (P \times F)/G$. The projection map $[(p, f)] \mapsto \pi(p)$ is a well-defined map $P \times_G F$. Then $P \times_G F$ is a fibre bundle with typical fibre F.

The definition comes with a companion.

Definition 5.2.15. A fibre bundle with structural group G and fibre F on X consists of a fibre bundle $E \to X$, a G-principal bundle $P \to X$ and an isomorphism of fibre bundles $P \times_G F \cong E$.

It is time to convince ourselves with the use of this construction; there are many useful examples.

Example 5.2.16. If V is a vector bundle, then $V \cong \mathrm{Fr}(V) \times_{\mathrm{GL}_n(\mathbb{K})} \mathbb{K}^n$. Hence we could define the notion of a vector bundle by saying that it is a fibre bundle with structural group $\mathrm{GL}_n(\mathbb{R})$ and fibre $\mathbb{R}^n$. More generally, if a representation of a Lie group is given (this is a smooth homomorphism $G \to \mathrm{GL}(V)$ for some vector space) and if P is a G-principal bundle, then $P \times_G V$ is a vector bundle. One can formulate the notion of oriented vector bundle, vector bundle with metric etc. etc. using this calculus.

Example 5.2.17. If $V \to X$ and $W \to Y$ are two vector bundles of rank n, m, then we can form the $\mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_m(\mathbb{K})$-principal bundle $\mathrm{Fr}(V) \times \mathrm{Fr}(W) \to X \times Y$. The Lie group $\mathrm{GL}_n \times \mathrm{GL}_m$ acts on $\mathbb{K}^n \oplus \mathbb{K}^m$. The resulting bundle $\mathrm{Fr}(V) \times \mathrm{Fr}(W) \times_{\mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_m(\mathbb{K})} \mathbb{K}^{m+n} \to X \times Y$ is called the external direct sum. If $X = Y$, we can pull back the external direct sum to X with the diagonal $X \to X \times X$ and obtain the direct sum.

Exercise 5.2.18. *Along the lines of this example, define $V \otimes W$ for two vector bundles,* $\mathrm{Hom}(V, W)$*, the dual bundle V^*, the bundle of alternating and symmetric multilinear forms and so on.*

Example 5.2.19. Consider the principal bundle $S^{2n+1} \to \mathbb{CP}^n$. Let $S^1 \subset \mathbb{C}$ act on $\mathbb{C}$ by multiplication. The vector bundle $S^{2n+1} \times_{S^1} \mathbb{C} \to \mathbb{CP}^n$ is the tautological line bundle on $\mathbb{CP}^n$.

These examples suggest that when the group action of G on the fibre F preserves some kind of structure on F, we find that the bundle $P \times_G F$ has this structure, but now in families. Caution: even though the fibres of a principal bundle look like groups, they are not. They are right-G-spaces, and do, most emphatically, *not* have a multiplication.

Exercise 5.2.20. *Formulate the notion of a bundle of: topological groups, finite-dimensional* $\mathbb{R}$*-algebras, finite-dimensional Lie algebras.*

Example 5.2.21. Let V be a complex vector bundle of rank n. The group $\mathrm{GL}_n(\mathbb{C})$ acts on the complex projective space, in exactly one meaningful way. The bundle $\mathbb{P}V := \mathrm{Fr}(V) \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{CP}^n$ is called the projective bundle to V. Moreover, the group action of $\mathrm{GL}_n(\mathbb{C})$ lifts to an action on the universal line bundle $H \to \mathbb{CP}^n$ and hence gives rise to a line bundle on $\mathbb{P}V$. Prove that the pullback of V to $\mathbb{P}V$ splits off a one-dimensional line bundle.

Example 5.2.22. If $G \to H$ is a group homomorphism and $P \to X$ a G-principal bundle, then $P \times_G H$ is in a natural way an H-principal bundle.

Definition 5.2.23. Let $G \to H$ be a group homomorphism and $Q \to X$ be an H-principal bundle. A *reduction of the structural group* from H to G consists of a G-principal bundle $P \to X$ and an isomorphism of H-principal bundles $P \times_G H \cong Q$.

Most additional structures that exist on bundles can be expressed using this notion. We discuss one example in great detail.

Example 5.2.24. Let $V \to X$ be a real n-dimensional vector bundle and let $P = \mathrm{Fr}(V) \to X$ be the frame bundle. We want to explain that a bundle metric on V is "the same" as a reduction of the structural group of P from $\mathrm{GL}_n(\mathbb{R})$ to $O(n)$.

Let $Q \to X$ be an $O(n)$-principal bundle and $\eta : Q \times_{O(n)} \mathbb{R}^n \cong V$ be an isomorphism. Then η induces an isomorphism

$$Q \times_{O(n)} \mathrm{GL}_n(\mathbb{R}) \cong P. \tag{5.2.25}$$

Namely, by considering the n columns of a matrix, we get an embedding $\mathrm{GL}_n(\mathbb{R}) \subset (\mathbb{R}^n)^n$. This embedding is $O(n)$-equivariant, with $O(n)$ acting on $\mathrm{GL}_n(\mathbb{R})$ by left-multiplication and on $(\mathbb{R}^n)^n$ by acting on each factor separately. Thus, a point in $(Q \times_{O(n)} \mathrm{GL}_n(\mathbb{R}))_x$ ($x \in X$) gives rise to n vectors in the vector space $(Q \times_{O(n)} \mathbb{R}^n)_x$, and these vectors are of course linearly independent. Vice versa, from an isomorphism as in 5.2.25, we obtain an isomorphism

$$Q \times_{O(n)} \mathbb{R}^n \cong Q \times_{O(n)} \mathrm{GL}_n(\mathbb{R}) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n \cong P \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n \cong V.$$

Thus a reduction of the structure group of P to $O(n)$ is "the same" as an $O(n)$-principal bundle Q and a vector bundle isomorphism $Q \times_{O(n)} \mathbb{R}^n \cong V$.

If V has a bundle metric, then we let Q be the bundle of orthogonal frames of V. We can describe Q_x either as the set of all orthonormal frames of V_x or as the set of all isometries $\mathbb{R}^n \to V_x$. This is an $O(n)$-principal bundle. There is a bundle isomorphism

$$Q \times_{O(n)} \mathbb{R}^n \to V$$

sending an equivalence class $[f, v]$ ($(f, v) \in P_x \times \mathbb{R}^n$) to $f(v) \in V$. Therefore, a bundle metric gives rise to a reduction of the structural group. On the other hand, the bundle $Q \times_{O(n)} \mathbb{R}^n$ carries a bundle metric. Let $[f, v], [f, w] \in Q \times_{O(n)} \mathbb{R}^n$. Then, as $O(n)$ preserves the standard scalar product on $\mathbb{R}^n$, $\langle [f, v], [f, w] \rangle := (v, w)$ does not depend on the choice of representative.

We can describe the whole correspondance even more abstractly. Let $Q \to X$ be an $O(n)$-principal bundle and $P = Q \times_{O(n)} \mathrm{GL}_n(\mathbb{R})$. Let $\mathcal{M}$ be the set of all positive symmetric bilinear forms on $\mathbb{R}^n$; this is an open subset of a finite-dimensional vector space and hence a manifold which has a $\mathrm{GL}_n(\mathbb{R})$-action. In fact, the $GL_n(\mathbb{R})$-action is transitive. The group $O(n)$ is (by definition) the isotropy group of the element $A_0 \in \mathcal{M}$ (the standard inner product). Therefore $\mathcal{M} = \mathrm{GL}_n(\mathbb{R})/O(n)$ as $\mathrm{GL}_n(\mathbb{R})$-space. Moreover, since A_0 is $O(n)$-invariant, it defines a fibre-preserving map

$$X = Q \times_{O(n)} * \to Q \times_{O(n)} \mathcal{M}.$$

The bundle $Q \times_{O(n)} \mathcal{M} \to X$ is the bundle whose fibre over x is the space of all inner products on the fibre of $Q \times_{\mathbb{R}} \mathbb{R}^n$ over X. Therefore, a section of $\mathcal{M} \to X$ is a bundle metric.

Example 5.2.26. Let V be a vector bundle and $W \subset V$ be a subbundle, of ranks $n < m$. Let $G_{m,n}$ be the group of all linear transformations of $\mathbb{K}^m$ that map $\mathbb{K}^n$ to itself. Show that the frame bundle $\mathrm{Fr}(V)$ admits the reduction of the structure group to $G_{m,n}$ and show how to construct the subbundle W, the bundle V and the quotient bundle V/W out of this reduction.

Example 5.2.27. A reduction of the structural group from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})^+$ is the same as an orientation, to $O(n+1)$ is the same as a bundle metric, and so on.

Finally, we briefly indicate how the classification theory of principal bundles works.

Theorem 5.2.28. *Let $F : X \times [0,1] \to Y$ be a homotopy and $P \to Y$ be a G-principal bundle. If X is paracompact, then $F_0^* P \cong F_1^* P$.*

Theorem 5.2.29. *Let G be a topological group. Then there exists a "universal G-principal bundle" $EG \to BG$, such that for each paracompact space X, there is a bijection $[X, BG] \cong \mathrm{Prin}_G(X)$. The bundle $EG \to BG$ is unique up to homotopy equivalence, and it is characterized by the property that EG is contractible.*

Examples 5.2.30. *Let $G = \mathrm{GL}_n(\mathbb{K})$. The frame bundle of the tautological vector bundle over $\mathrm{Gr}_n(\mathbb{K}^k)$ is the* Stiefel manifold $\mathrm{St}_{n,k}(\mathbb{K})$. *The colimit* $\mathrm{colim}_k \mathrm{St}_{n,k}(\mathbb{K})$ *is contractible and this shows that $E\,\mathrm{GL}_n(\mathbb{K}) = \mathrm{colim}_k \mathrm{St}_{n,k}(\mathbb{K})$, in accordance to our previous classification theory.*

If $G \subset \mathrm{GL}_n(\mathbb{R})$ is a closed subgroup, we could take $EG := E\,\mathrm{GL}_n(\mathbb{R})$ and $BG := E\,\mathrm{GL}_n(\mathbb{R})(G) = E\,\mathrm{GL}_n(\mathbb{R}) \times_{GL_n(\mathbb{R})} \mathrm{GL}_n(\mathbb{R})/G$.

6. More on de Rham cohomology

The remaining goal for this term is the proof of the Gauß-Bonnet-Chern theorem and the Riemann-Roch theorem, which are the role models for the general index theorem.

In both cases, the index theorem will take the following form. The Gauß-Bonnet-Chern theorem states that there is, for an oriented manifold, a specific cohomology class $e(TM) \in H^n(M)$ such that $\mathrm{ind}(D) = \chi(M) = \int_M e(TM)$. The class $e(TM)$ is the *Euler class*. Riemann-Roch will be a similar formula.

For a while, we will abandon the differential operators, Sobolev spaces and estimates and focus on the right-hand side of the index theorem. We develop the theory of characteristic classes. We do this in the framwork of de Rham cohomology, and we will actually do it *twice*. There is the *global* theory of characteristic classes, which is an offspring of Poincaré duality. And there is the local theory (curvature).

We reveal a close connection between differential forms and the global geometry of a manifold. The tools we develop will be crucial to the proof of the Gauß-Bonnet-Chern theorem and also for the later translation of the index formula from K-theory to cohomological terms. There is also an inherent beauty! But as often in mathematics, beauty and elegance needs support by strong workhorses.

In singular (co)homology theory, there are two main technical workhorses: *relative cohomology* and the pairing between cohomology and homology. We have to replace these pillars by something.

6.1. **Technical prelimiaries.** The fundamental property of the de Rham cohomology is its *homotopy invariance*. We recall how the proof works because we will need to see that it can be modified to prove homotopy invariance of some versions of de Rham cohomology. Let M be a manifold. One defines an operator

$$P : \mathcal{A}^p(M \times [0,1]) \to \mathcal{A}^{p-1}(M)$$

by setting

$$P\omega := \int_0^1 j_t^*(\iota_{\partial_t}\omega)dt.$$

Explanations: ∂_t is the vector field pointing in the $[0,1]$-direction; $j_t : M \to M \times [0,1]$ is $j_t(x) = (x,t)$. The form $j_t^*(\iota_{\partial_t}\omega)$ is a $p-1$-form on M, and $t \mapsto j_t^*(\iota_{\partial_t}\omega)$ is a smooth curve in the vector space $\mathcal{A}^{p-1}(M)$, and the integral is the usual Lebesgue integral for functions with values in topological vector spaces. In local coordinates, one shows that $Pd + dP = j_1^* - j_0^*$, and therefore P is a chain homotopy.

We need to fix an *orientation convention*, once and for all.

Convention 6.1.1. If M and N are oriented manifolds, we orient the product by the following requirement. If $(v_1, \ldots, v_m)$ is an oriented basis for T_xM, and $(w_1, \ldots, w_m)$ an oriented basis for T_yN, the $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is an oriented basis of $T_{(x,y)}M \times N$.

If $V \to M$ is an oriented vector bundle over an oriented manifold, then we orient the total space by saying that an oriented chart $V|_U \cong U \times \mathbb{R}^n$ is an orientation preserving diffeomorphism of manifolds.

If $W, V \to X$ are two oriented vector bundles, we orient $V \oplus W$ by saying that an oriented basis of V_x, followed by an oriented basis of W_x, is an oriented basis of $V_x \oplus W_x$.

If $N \subset M$ is a submanifold, then there is an almost natural isomorphism $TM|_N \cong TN \oplus \nu_N^M$, where $\nu_N^M = TM|_N / TN$ denotes the normal bundle of N in M. So if M is oriented, then, according to the convention for sums of vector bundles, an orientation of N determines an orientation of the normal bundle and vice versa.

Definition 6.1.2. Let M be a manifold. By the symbol $\mathcal{A}_c^*(M)$, we denote the space of compactly supported differential forms. It is clear that this is a chain complex and an ideal in $\mathcal{A}^*(M)$. $H_c^*(M)$ is the cohomology of this chain complex, the *compactly supported cohomology* of M.

Using this new cohomology, we obtain a new level of flexibility when dealing with cohomology, but there are some pitfalls. If $f : M \to N$ is a smooth map, then we do *not* have in general a map $f^* : \mathcal{A}_c^*(N) \to \mathcal{A}_c^*(M)$ (let $f : \mathbb{R} \to *$ to see what goes wrong). But if f is a *proper* map, we have a pullback f^*.

There is another functoriality: if $U \subset M$ is an open subset, we get a map $\mathcal{A}_c^*(U) \to \mathcal{A}_c^*(M)$, and of course there is a map $\mathcal{A}_c^*(M) \to \mathcal{A}^*(M)$, which sometimes carries important information as well. Recall that in singular (co)homology, *relative cohomology* is a central technical tool. Here is our replacement for it:

Definition 6.1.3. Let M be a manifold and let $A \subset M$ be a closed subset. We define $\mathcal{A}^*(M)_A := \operatorname{colim}_{A \subset U} \mathcal{A}^*(U)$, the space of germs of forms near A.

The technical environment that makes cohomology theory breathe is homological algebra, and in particular exact sequences. There are three important exact sequences in de Rham theory. First, there are two Mayer-Vietoris-sequences. Let U, V be open. Then there are sequences

$$0 \to \mathcal{A}^*(U \cup V) \to \mathcal{A}^*(U) \oplus \mathcal{A}^*(V) \to \mathcal{A}^*(U \cap V) \to 0$$

$$\omega \mapsto (\omega|_U, \omega|_V);\ (\omega, \eta) \mapsto \omega|_{U \cap V} - \eta|_{U \cap V}$$

and

$$0 \to \mathcal{A}_{cpt}^*(U \cap V) \to \mathcal{A}_{cpt}^*(U) \oplus \mathcal{A}_{cpt}^*(V) \to \mathcal{A}_c^*(U \cup V) \to 0$$

$$\omega \mapsto (\omega, -\omega);\ (\omega, \eta) \mapsto \omega + \eta.$$

That the second is exact is obvious (!). The first one is slightly more complicated and involves partitions of unity, see [4], p. 287 f. In cohomology, we obtain two Mayer-Vietoris sequences.

Lemma 6.1.4. *Assume that $M \setminus A$ is relatively compact in M. Then there is an exact sequence*

$$0 \to \mathcal{A}_c^*(M \setminus A) \to \mathcal{A}^*(M) \to \mathcal{A}^*(M)_A \to 0.$$

If A is either a submanifold or a codimension 0 submanifold (with boundary), then the restriction $\mathcal{A}^*(M)_A \to \mathcal{A}^*(A)$ is a quasiisomorphism.

Proof. It is clear that the first map is injective and that the composition is zero. If $\omega \in \mathcal{A}^*(M)$ maps to zero, it means that there is a neighborhood $A \subset U$ such that $\omega|_U = 0$, whence ω has support in $\overline{M - U}$ and this is compact. An element in $\mathcal{A}^*(M)_A$ is represented by a form ω on $U \supset A$. Pick a function μ that is 1 near A

and has support in U, then we can extend $\mu\omega$ to a form on M, which represents the same form in the colimit.

For the second part, first note that for an open neighborhood U of A, $\mathcal{A}^*(U)_A = \mathcal{A}^*(M)_A$. We use the *tubular neighborhood theorem* [5]. Let U be a tubular neighborhood and $r : U \to A$ be the projection. The map $\mathcal{A}^*(M)_A \to \mathcal{A}^*(A)$ is surjective since for any η, the form $r^*\eta$ represents a preimage. Also, since r is a homotopy equivalence, the composition $\mathcal{A}^*(U) \to \mathcal{A}(U)_A \to \mathcal{A}^*(A)$ is a quasiisomorphism, and therefore $\mathcal{A}(U)_A \to \mathcal{A}^*(A)$ is surjective in cohomology. Thus it remains to prove that the first map $\mathcal{A}^*(U) \to \mathcal{A}^*(U)_A$ is surjective in cohomology, and it is enough to show that for each cohomology class of $\mathcal{A}^*(U)_A$, there is a smaller tubular neighborhood W such that the class comes from $\mathcal{A}^*(W)$, because all tubular neighborhoods are homotopy equivalent. So let $\omega \in H^*\mathcal{A}^*(M)_A$. It is represented by a *closed* form on a certain $V \supset A$. Take a tubular neighborhood $U \subset V$ and a cut-off function η which is 1 on U and has support in V. The form $\eta\omega$ is not closed on V, but its restriction to U is closed. □

Lemma 6.1.5. *For $k \neq n$, $H^k_c(\mathbb{R}^n) = 0$. The integration homomorphism $H^n_c(\mathbb{R}^n) \to \mathbb{R}$, $\omega \mapsto \int_{\mathbb{R}^n} \omega$ is an isomorphism.*

Proof. The case $n = 0$ is trivial, and we assume $n > 0$. We consider the 1-point compactification S^n of $\mathbb{R}^n$. From Lemma 6.1.4, we get the exact sequence $0 \to \mathcal{A}^*_c(\mathbb{R}^n) \to \mathcal{A}^*(S^n) \to \mathcal{A}^*(S^n)_\infty \to 0$ and the fact that $\mathcal{A}^*(S^n)_\infty$ is quasiisomorphic to $\mathcal{A}^*(*)$. The long exact cohomology sequence has the following outlook

$$\begin{tikzcd} H^*_c(\mathbb{R}) \arrow[rr] & & H^*(S^n) \arrow[dl, "\mathrm{epi}"] \\ & H^*(*) \arrow[ul, "{[+1]}"] & \end{tikzcd}$$

The symbol [+1] reminds you of the degree shift, and the map $H^*(S^n) \to H^*(*)$ is surjective, since it is induced from the inclusion $* = \infty \to S^n$, and splits via the constant map $S^n \to *$. Thus the cohomology sequence falls apart as

$$0 \to H^k_c(\mathbb{R}^n) \to H^k(S^n) \to H^k(*) \to 0.$$

Together with the known computation of $H^*(*)$ and $H^*(S^n)$, this proves that $H^n(\mathbb{R}^n)$ is one-dimensional (and $H^k_c(\mathbb{R}^n) = 0$ for $k \neq n$). For the statement about the integration homomorphism, it is enough to find a single compactly supported closed n-form on $\mathbb{R}^n$ with integral 1, which is easy: take a suitable bump function a and let $\omega = a(x)dx_1 \wedge \ldots \wedge dx_n$. □

We need another preliminary, namely a flexible local-to-global principle.

Proposition 6.1.6. *(The bootstrap lemma) Let M be a manifold, and let $P(U)$ be a statement about open subsets of M. Suppose:*

- *$P(\emptyset)$ is true,*
- *There is a cover $(U_i)_{i\in I}$ of M, so that if V that is contained in one of the U_i and is diffeomorphic to an open convex subset of $\mathbb{R}^n$, then $P(V)$ is true*[4].
- *If $P(U)$, $P(V)$ and $P(V \cap U)$ are true, then so is $P(U \cup V)$.*

[4] This precise formulation is quite useful in later applications

- *If U_n are disjoint open subsets and if $P(U_i)$ is true for all i, then $P(\cup_i U_i)$ is true.*

The proof can be found in [4], Lemma V.9.5, with a slightly weaker assumption. It is easy to generalize the proof.

6.2. **Poincaré duality - again! -, Künneth theorem and the Thom isomorphism.** The Poincaré duality theorem was the main result of the class "Topology II" and if you attended the class, you remember that among all the results in this class, this was by far the most difficult one. We gave a proof for closed manifolds using elliptic regularity theory. In de Rham theory, there is a slick and short proof. What should the theorem - for a noncompact manifold - look like? Let M^n be an *oriented* manifold. If $\omega \in \mathcal{A}^p(M)$ and $\eta \in \mathcal{A}_c^{n-p}(M)$, we form

$$I(\omega,\eta) := \int_M \omega \wedge \eta \in \mathbb{R}.$$

The integral is well-defined because ω has compact support. It is easy to see that $I(\omega,\eta)$ depends, when the forms are closed, only on the cohomology classes in $H_c^p(M)$ and $H^{n-p}(M)$. Thus we get a bilinear map

$$I : H^p(M) \times H_c^{n-p}(M) \to \mathbb{R}$$

and therefore two maps

$$D : H^p(M) \to H_c^{n-p}(M)^\vee;\ E : H_c^{n-p}(M) \to H^p(M)^\vee.$$

We used the symbol $\vee$ for the dual space, in order to avoid having to many symbols $*$ floating around.

Theorem 6.2.1. *(Poincaré duality - de Rham version) For any oriented n-manifold M, the map $D : H^{n-p}(M) \to H_c^p(M)^\vee$ is an isomorphism.*

We *explicitly do not* assert that the other map $E : H_c^p(M) \to H^{n-p}(M)^\vee$ is an isomorphism. There is an asymmetry between cohomology and compactly supported cohomology, and this has a real reason, rooted in - set theory.

Here is a counterexample. Let M be a countably infinite discrete set (a manifold of dimension 0). It is easy to see that $H_c^0(M) \cong \mathbb{R}^\infty$ (the dimension is $\aleph_0$), while $H^0(M)$ has dimension $2^{\aleph_0}$. Under a suitable finiteness condition, the other map is an isomorphism as well, see below.

Proof. The first step consists of sign conventions. Let $\mathcal{A}_c^*(M)^\vee$ be the dual chain complex. The differential is

$$\delta : \mathcal{A}_c^{n-p}(M)^\vee \to \mathcal{A}_c^{n-p-1}(M)^\vee;\ \delta(\ell)(\eta) := (-1)^{p+1}\ell(d\eta)$$

and the sign is chosen so that $D : \mathcal{A}^*(M) \to \mathcal{A}^{n-*}(M)^\vee$ is a chain map.

We use the bootstrap lemma. If $M = \emptyset$, there is not much to show. For $M = \mathbb{R}^n$, we know $H^*(\mathbb{R}^n)$ and $H_c^*(\mathbb{R}^n)$ by Lemma 6.1.5. Clearly, the constant form 1 goes under D to the integration homomorphism, which is a nonzero element in a 1-dimensional vector space.

If $U \subset V$ is an open subset, then the diagram

$$\begin{tikzcd}
\mathcal{A}^p(V) \arrow[r] \arrow[d, "D"] & \mathcal{A}^p(U) \arrow[d, "D"] \\
\mathcal{A}_c^{n-p}(V)^\vee \arrow[r] & \mathcal{A}_c^{n-p}(U)^\vee
\end{tikzcd}$$

(the horizontal maps are induced by restriction) is commutative, which is trivial to verify. Therefore, for two open sets $U_0, U_1 \subset M$, the following diagram of chain complexes and chain maps (with exact rows) commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}^*(U_0 \cup U_1) \arrow[r] \arrow[d, "D_{U_0 \cup U_1}"] & \mathcal{A}^*(U_0) \oplus \mathcal{A}^*(U_1) \arrow[r] \arrow[d, "D_{U_0} \oplus D_{U_1}"] & \mathcal{A}^*(U_0 \cap U_1) \arrow[r] \arrow[d, "D_{U_0 \cap U_1}"] & 0 \\
0 \arrow[r] & \mathcal{A}^{n-*}(U_0 \cup U_1)^\vee \arrow[r] & \mathcal{A}^{n-*}(U_0)^\vee \oplus \mathcal{A}^{n-*}(U_1)^\vee \arrow[r] & \mathcal{A}^{n-*}(U_0 \cap U_1)^\vee \arrow[r] & 0.
\end{tikzcd}$$

If D_{U_0}, D_{U_1} and $D_{U_0 \cap U_1}$ are quasiisomorphisms, then, using the long exact cohomology sequence of the above sequences and the 5-lemma, it follows that $D_{U_0 \cup U_1}$ is a quasiisomorphism.

The last hypothesis of the bootstrap lemma is to verify that if U_i, $i \in I$, are disjoint open subsets of M and D_{U_i} is a quasiisomorphism for all $i \in I$, then D_U is a quasiisomorphism where we put $U = \coprod_{i \in I} U_i$. This is because $H^*(U) = \prod_{i \in I} H^*(U_i)$ and $H_c^*(U) = \bigoplus_{i \in I} H_c^*(U_i)$ and because taking dual spaces converts direct sums into direct products (this is also the reason for the asymmetry). $\square$

Proposition 6.2.2. *The following conditions on a manifold M are equivalent:*

1. *$H^*(M)$ is finite dimensional.*
2. *$H_c^*(M)$ is finite dimensional.*
3. *The other duality homomorphism E is an isomorphism as well.*

Here we take $H^* = \bigoplus_{p \geq 0} H^p$, which is a finite sum. A manifold with these properties is called of finite type*. Each closed manifold is of finite type.*

We need a fact from linear algebra:

Lemma 6.2.3. *Let V be a real vector space. Then V is finite dimensional if and only if the natural map $\iota : V \to V^{\vee\vee}$ to the dual space is an isomorphism.*

This is proven in [3], chapter II, §7.5 Theorem 6.

Proof of Proposition 6.2.2. The easy parts of the Proposition are:

- If M is compact, then $H^*(M) = H_c^*(M)$, and therefore E agrees with D. So for compact M, E is an isomorphism.
- If $H^*(M)$ is finite-dimensional, then so is the dual space of $H_c^*(M)$, and hence $H_c^*(M)$ itself.
- If $H_c^*(M)$ is finite-dimensional, then so is $H^*(M) \cong H_c^*(M)^\vee$.

It remains to be shown that E is an isomorphism iff $H^*(M)$ is finite dimensional. To see this, consider the diagram

$$\begin{tikzcd} H^*(M) \arrow[r, "\iota"] \arrow[dr, "D"'] & H^*(M)^{\vee\vee} \arrow[d, "E^\vee"] \\ & H_c^{n-*}(M)^\vee, \end{tikzcd}$$

which is commutative, as one checks easily. If $H^*(M)$ is finite-dimensional, then ι is an isomorphism and hence so is $E^\vee$. But then E has to be an isomorphism (taking duals is an exact functor). Vice versa, if E is an isomorphism, then so is ι, which means, by the lemma, that $H^*(M)$ is finite-dimensional. □

Corollary 6.2.4. *If M is compact, then $H^k(M)$ is finite dimensional.*

Now we pass to the Künneth theorem, which expresses the cohomology of a product in terms of the cohomologies of the product. The Künneth theorem only holds under a finiteness assumption or for compactly supported cohomology. Let M, N be two manifolds and let $\mathrm{pr}_M : M \times N \to M$ and $\mathrm{pr}_N : M \times N \to N$ be the projections. We define the exterior product of two forms ω on M and η on N by

$$\omega \times \eta := \mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \eta$$

(one can recover $\omega \wedge \eta := \Delta^*(\omega \times \eta)$, using the diagonal $\Delta_M : M \to M \times M$). One the tensor product $\mathcal{A}^*(M) \otimes \mathcal{A}^*(N)$, we introduce the differential $\partial(\omega \otimes \eta) := (d\omega) \otimes \eta + (-1)^{|\omega|} \omega \otimes d\eta$, so that

$$\mathcal{A}^*(M) \otimes \mathcal{A}^*(N) \to \mathcal{A}^*(M \times N) \text{ and } \mathcal{A}_c^*(M) \otimes \mathcal{A}_c^*(N) \to \mathcal{A}_c^*(M \times N)$$

are chain maps. The second one induces a map

$$\bigoplus_{p+q=k} H_c^p(M) \otimes H_c^q(N) \to H_c^k(M \times N). \tag{6.2.5}$$

Theorem 6.2.6. *(Künneth theorem) The map 6.2.5 is an isomorphism, for all manifolds M and N.*

If M and N are of finite type, one can derive that

$$\bigoplus_{p+q=k} H^p(M) \otimes H^q(N) \to H^k(M \times N)$$

is an isomorphism (we do not try to state the most general assumption here).

Proof of the Künneth theorem. We use the bootstrap lemma, but proceed in two steps. First assume that $N = \mathbb{R}^n$ and let M vary. For the case $M = \mathbb{R}^m$, use Lemma 6.1.5 and Fubini's theorem. The other hypotheses of the bootstrap lemma are verified by the same ideas as in the proof of the Poincaré duality theorem. For the Mayer-Vietoris property, use that taking tensor products is an exact functor (because we work over a field). For the countable disjoint union property, use that tensor products commute with direct sums.

The second step is the general case and uses the same argument. □

The next fundamental result we need is the *Thom isomorphism* theorem. Let $\pi : V \to M$ be a vector bundle. Let $\mathcal{A}^p_{cv}(V)$ be the space of forms with *vertically compact support* (i.e the map $\pi : \operatorname{supp}(\omega) \to M$ is proper). This is clearly a chain complex and we have a Mayer-Vietoris sequence, in the sense that if $U_0, U_1 \subset M$ are open and $V_i = V|_{U_i}$, then there is an exact sequence of chain complexes

$$0 \to \mathcal{A}^*_{cv}(V_0 \cup V_1) \to \mathcal{A}^*_{cv}(V_0) \oplus \mathcal{A}^*_{cv}(V_1) \to \mathcal{A}^*_{cv}(V_0 \cap V_1) \to 0.$$

Moreover, the complex of vertically compactly supported forms is contravariant for bundle maps $V \to W$, and homotopic bundle maps induce chain homotopic maps.

Exercise 6.2.7. *Prove these assertions. Hint: go to the proof of homotopy invariance of de Rham cohomology and modify the details.*

Definition 6.2.8. Let $V \to M$ be an oriented vector bundle of rank n. A *Thom form* is a closed form $\tau \in \mathcal{A}^n_{cv}(V)$ such that

$$\int_{V_x} \tau = 1. \tag{6.2.9}$$

holds for each $x \in M$. A *Thom class* is the cohomology class of a Thom form.

Theorem 6.2.10. *Each oriented vector bundle $\pi : V \to M$ has a Thom form.*

Proof. First, we assume that the base space M is compact, k-dimensional and oriented. Then V is oriented as a manifold. Moreover, V is homotopy equivalent to M, and thus it has finite type. Moreover, $\mathcal{A}^*_{cv}(V) = \mathcal{A}^*_c(V)$. By Poincaré duality, there are isomorphisms

$$H^n_{cv}(V) = H^n_c(V) \overset{E}{\cong} H^k(V)^\vee \cong H^k(M)^\vee.$$

A closed form $\alpha \in \mathcal{A}^*_{cv}(V)$ is mapped, under these isomorphisms, to the linear form

$$\eta \mapsto \int_V \pi^*\eta \wedge \alpha.$$

On the other hand, we have a distinguished element in $H^k(M)^\vee$, the integration homomorphism $J : H^k(M) \to \mathbb{R}$. We pick τ so that it maps to J. In other words, for each closed k-form η on M, we have

$$\int_V \pi^*\eta \wedge \tau = \int_M \eta. \tag{6.2.11}$$

We claim that τ is a Thom form, in other words, the integral $\int_{V_x} \tau$ has value 1 for each $x \in M$. We pick an oriented coordinate chart $x : U \to \mathbb{R}^k$ on M and an arbitrary k form η on U with integral 1 and compact support in U. Moreover, we pick oriented bundle coordinates ξ on V. The form η can be written as $a(x)dx_1 \wedge \ldots \wedge dx_k$, and τ can be written as

$$b(x,\xi)d\xi_1 \wedge \ldots d\xi_n + \zeta,$$

with ζ a form that is a linear combination each term of which involves at most $n-1$ of the $d\xi_i$'s and hence at least one dx_j. Thus $\pi^*\eta \wedge \tau$ is, in these coordinates,

$$a(x)b(x,\xi)dx_{\underline{k}} \wedge d\xi_{\underline{n}}.$$

We compute

$$\int_V \pi^*\eta \wedge \tau = \int a(x)b(x,\xi)dx_{\underline{k}} \wedge d\xi_{\underline{n}} = \int \int a(x)b(x,\xi)d\xi dx,$$

in the last step we replaced the integral of forms by the Lebesgue integral (the usual normalization) and used Fubini. Furthermore, this equals

$$\int \left(a(x) \int b(x,\xi)d\xi\right) dx = \int a(x)c(x)dx \overset{!}{=} \int a(x)dx,$$

where $c(x) \coloneqq \int_{V_x} \tau$ and the last equality is 6.2.11. Since the last equation holds for each compactly supported function a, it follows that $c(x) = 1$, and this finishes the proof that τ is indeed a Thom form.

The case of a general base (nonoriented, noncompact) is reduced to this case by the following trick. We know, by classification of vector bundles, that there is an orientation preserving bundle map $f : V \to \tilde{V}_{n,r}$, where $\tilde{V}_{n,r}$ is the tautological oriented bundle over the Grassmannian $\tilde{\mathrm{Gr}}_{n,r}$ of oriented n-planes in $\mathbb{R}^r$, for some large r. The Grassmannian is a 2-fold cover of the ordinary Grassmannian $\mathrm{Gr}_{n,r}$, and because $\mathrm{Gr}_{n,r}$ is compact, so is the oriented Grassmannian. We have to argue why the oriented Grassmannian is an orientable manifold (this is not a tautology: the tautological vector bundle is not at all the tangent bundle of the Grassmann manifold). But $\tilde{\mathrm{Gr}}_{n,r}$ is simply connected and hence orientable: the oriented Grassmannian is $SO(r)/SO(n) \times SO(r-n)$. The long exact homotopy sequence

$$\pi_1(SO(n) \times SO(r-n)) \to \pi_1(SO(r)) \to \pi_1(\tilde{\mathrm{Gr}}_{n,r}) \to \pi_0(SO(n) \times SO(r-n)) = 0$$

becomes

$$(\mathbb{Z}/2)^2 \to \mathbb{Z}/2 \to \pi_1(\tilde{\mathrm{Gr}}_{n,r}) \to 0$$

and the first map is surjective (at least if $r \geq 2$, which can be assumed without loss of generality). What this argument proves is that each oriented bundle has a bundle map f to an oriented vector bundle over a compact oriented base. If σ is a Thom form for the tautological bundle (provided by the first part of the proof), then $f^*\sigma$ is a Thom form for V.

□

Theorem 6.2.12. *(The Thom isomorphism theorem) Let M be a manifold and $\pi : V \to M$ be a smooth oriented vector bundle, of rank n. Let τ be a Thom form on V. Then the chain maps*

$$\mathrm{th} : \mathcal{A}^*(M) \to \mathcal{A}^{*+n}_{cv}(V); \ \mathrm{th} : \mathcal{A}^*_c(M) \to \mathcal{A}^{*+n}_c(V),$$

defined by $\alpha \mapsto \pi^*\alpha \wedge \tau$, are quasiisomorphisms.

Proof. This is by the bootstrap lemma. There is not much to say if $M = \emptyset$. Each point $x \in M$ has a chart neighborhood $U \cong \mathbb{R}^m$ such that $V|_U$ is trivial. So we have to show that for the trivial vector bundle on $\mathbb{R}^n$, the theorem holds. Let us do the compactly supported case first. By the computation of $H^*_c(\mathbb{R}^k)$, all that remains to be done is that $\mathrm{th} : H_c(\mathbb{R}^m) \to H^{*+n}_c(\mathbb{R}^n)$ is nonzero. But if $\phi = a(x)dx_1 \wedge \ldots \wedge dx_m \in \mathcal{A}^m_c(\mathbb{R}^m)$, then

$$\int_{\mathbb{R}^{m+n}} \pi^*\phi \wedge \tau = \int_{\mathbb{R}^m} a(x) \int_{\{x\}\times\mathbb{R}^n} \tau dx = \int_{\mathbb{R}^m} \phi$$

by the computation in the proof of Theorem 6.2.10. In the noncompactly supported case, we have to show:

- $H^k_{cv}(\mathbb{R}^m \times \mathbb{R}^n) = 0$ unless $k = n$.
- $\dim H^n_{cv}(\mathbb{R}^m \times \mathbb{R}^n) = 1$ and
- $\mathrm{th} : H^0(\mathbb{R}^m) \to H^n_{cv}(\mathbb{R}^m \times \mathbb{R}^n)$ is injective.

We have already seen that integration over $\{x\} \times \mathbb{R}^n$ defines a map $H^n_{cv}(\mathbb{R}^m \times \mathbb{R}^n) \to \mathbb{R}$, and it takes the Thom class to 1, so the third property holds. By the homotopy invariance of H^*_{cv} in the base space, we find that

$$H^*_{cv}(\mathbb{R}^m \times \mathbb{R}^n) \cong H^*_{cv}(\mathbb{R}^n)$$

which implies the first two properties.

The proof of the Mayer-Vietoris and disjoint union property is completely analogous to the proof of Theorem 6.2.1. The bootstrap lemma applies to conclude the proof. □

Corollary 6.2.13. *The cohomology class of a Thom form (the Thom class) is uniquely determined by the orientation of V.*

Proof. If M is a disjoint union $\coprod_i U_i$ and $V_i := V|_{U_i}$, then $H^*_{cv}(V) \cong \prod_{i\in I} H^*_{cv}(V_i)$. Therefore, it is enough to check the case when M is connected. By Theorem 6.2.12, $H^n_{cv}(V) \cong H^0(M) \cong \mathbb{R}$. By Stokes theorem, for each $x \in M$, we get a homomorphism $J_x : H^n_{cv}(V) \to \mathbb{R}$, $\eta \mapsto \int_{V_x} \eta$. A Thom form maps to 1, and therefore J_x is a nonzero map between 1-dimensional vector spaces, hence an isomorphism. Thus an element in $H^n_{cv}(V)$ is uniquely determined by its integral over V_x. □

Lemma 6.2.14. *Let $p : V \to M$, $q : W \to N$ be two oriented vector bundles, with Thom forms τ_V, τ_W. Then a Thom form of the product bundle $V \times W \to M \times N$ is given by $\tau_V \times \tau_W = p^*\tau_V \wedge q^*\tau_W$.*

Let $f : M \to N$ be a smooth map, which is covered by an orientation preserving bundle map $\hat{f} : V \to W$. Then $\hat{f}^*\tau_W = \tau_W$.

The proof is trivial: you just have to check that the product satisfies the axioms for a Thom class, which requires little more than Fubinis theorem. If F is a bundle automorphism of V, we can talk about the determinant $\det(F_x)$. The sign of $\det(F_x)$ is a locally constant function of x, and we can pull $\mathrm{sign}(\det(F))$ back to V; this is a locally constant function, in other words, an element $\sigma(F)$ of $H^0(V)$. It is easy to verify that

$$F^*\tau_V = \sigma(F)\tau_V. \tag{6.2.15}$$

Let $V \to M$ be equipped with a bundle metric and let $\epsilon : M \to (0,\infty)$. Then we can find a Thom form which has support in $D_\epsilon V := \{v \in V \| v| < \epsilon(\pi(v))\}$, by the following procedure. For each positive function $a : M \to \mathbb{R}$, we take the bundle automorphism $h_a(v) := \frac{1}{a(\pi(v))}v$, and by picking a small enough, the form $h_a^*\tau$ is a Thom form and has the desired property.

Definition 6.2.16. Let $V \to M$ be an oriented vector bundle. The *Euler class* of V is $\iota^*\tau \in H^n(M)$, where $\iota : M \to V$ is the zero section.

Since two sections are homotopic, one could use any other section instead of the zero section.

The Euler class satisfies some easily verified properties:

Proposition 6.2.17.

(1) *The Euler class is natural, i.e. if $f : N \to M$ is smooth and $V \to M$ an oriented vector bundle, the $f^* e(V) = e(f^* V)$.*
(2) *Reversing the orientation of V reverses the sign of the Euler class.*
(3) $e(V \oplus W) = e(V) \wedge e(W)$.
(4) *If V has a section that is nowhere zero, then $e(V) = 0$.*
(5) *If the rank of V is odd, then $e(V) = 0$.*

The first four statements are straightforward to prove, but the last requires an idea. Any vector bundle has the automorphism $F = -1$. If the rank is odd, then F is orientation reversing, and thus

$$e(V) = \iota^* \tau = \iota^* F^* F^* \tau \overset{1}{=} -\iota^* F^* \tau \overset{2}{=} -\iota^* \tau = -e(V).$$

The equation 1 is from 6.2.15 and 2 is because $F \circ \iota = \iota$.

6.3. **Geometric interpretation of the Thom class and the Poincaré-Hopf theorem.** Let $N^n \subset M^m$ be a submanifold. We assume that M and N are oriented, which induces an orientation on the normal bundle. We assume that N is *compact*. Under these circumstances, we get a linear map

$$\ell_N : H^n(M) \to \mathbb{R};\ \omega \mapsto \int_N \omega.$$

If M *has finite type*, then there exists, by Poincaré duality, a unique $\delta \in H_c^{m-n}(M)$ such that $E(\delta) = \ell_N$, or

$$\int_M \omega \wedge \delta = \int_N \omega$$

holds for all closed forms ω on M. If M is the total space of the oriented vector bundle $V \to N$, then the Thom form has this property. We now present a little geometric argument to get rid of the finite type assumption and which gives a nice geometric interpretation of Poincaré duality. The main idea is that each closed submanifold of a manifold sits inside the large manifold just as the zero section lies inside a vector bundle. The precise technical ingredient from differential topology that we need is the *tubular neighborhood theorem.*

Theorem 6.3.1. *(The tubular neighborhood theorem) Let $N \subset M$ be a compact submanifold. Choose a Riemann metric on M, so that the normal bundle E of N in M is just the orthogonal complement of TN inside $TM|_N$. Then each open neighborhood O of N contains a smaller open neighborhood $N \subset U \subset O$ such that there is a diffeomorphism $e : E \to U$, having the following properties:*

(1) *The restriction of e to $N \subset E$ is the identity.*
(2) *Under the natural splitting $(TE)|_N \cong TN \oplus E$, induced by the metric, and $TM|_N \cong E \oplus TN$, the differential of e at points of N is the identity.*

For further reference, let us note that e is orientation preserving if M and N are oriented and E is equipped with an orientation by the orientation convention.

The idea of the tubular neighborhood theorem is simple, but the details are highly nontrivial. An unabridged proof can be found in [5], §12. Now let $\tau \in$

$\mathcal{A}_c^{m-n}(E)$ be a Thom form (since N was assumed to be compact, this has compact support). We let $\delta \in \mathcal{A}^{m-n}(M)$ be the form $(e^{-1})^*\tau$, extended by zero to all of M. If $\omega \in \mathcal{A}^n$, then

$$\int_M \omega \wedge \delta = \int_U \omega \wedge (e^{-1})^*\tau = \int_E e^*\omega \wedge \tau = \int_N \omega$$

(use that e is orientation-preserving), by the way the Thom class over a compact manifold was constructed. So the form δ represents the functional ℓ_N. Note that this shows that ℓ_N therefore lies in the image of E, regardless of finiteness assumptions and therefore does not lie in the outlandish part of the dual space. A geometric picture shows that the class of δ is supported in a small neighborhood of N.

Theorem 6.3.2. *Let $N^n \subset M^m$ be a compact oriented submanifold of an oriented manifold. Then there exists a unique $\delta \in H_c^{m-n}(M)$ such that for all $\omega \in H^n(M)$, we have*

$$\int_M \omega \wedge \delta = \int_N \omega. \tag{6.3.3}$$

Rest of the proof. The uniqueness of δ remains to be proven. Let $\epsilon \in H_c^{m-n}(M)$ have the property that

$$\int_M \omega \wedge \epsilon = 0$$

holds for all $\omega \in H^{m-n}(M)$. This can be reformulated by saying that for all $\omega \in H^{m-n}(M)$: $D(\omega)(\epsilon) = 0$, or, by Poincaré duality that $\ell(\epsilon) = 0$ for all $\ell \in H_c^{m-n}(M)^\vee$, in other words, $\epsilon = 0$. □

We call the class $\delta = \delta_N$ the *Poincaré dual* to N. It is represented by forms that are supported in a tubular neighborhood of N, and the support can be made arbitrarily small. The defining equation 6.3.3 is the nonlinear analog of the relation 6.2.11. Now we look at the behaviour of the Poincaré duals under smooth maps, which can be viewed as a vast generalization of the defining property of the Thom class 6.2.9. For this, we recall the notion of *transversality*.

Assume that $f : L^l \to M^m$ is a smooth map of oriented manifolds. Assume that f is *transverse* to N, $f \pitchfork N$, which means by definition that $Tf(T_xL) + T_{f(x)}N = T_{f(x)}M$ holds for all $x \in L$, $f(x) \in N$. It follows that $K := f^{-1}(N)$ is a submanifold of L of the same codimension as N, and that there is a natural bundle map $\hat{f} : \nu_K^L \to \nu_N^M$ over $f|_K$. An orientation on K is induced by this bundle map and the orientation convention. The important *transversality theorem* from differential topology states that for each map $g : L^l \to M$, there is a map $f : L \to M$, arbitrarily close to g which is transverse to N. For the proof, see [5]. Our goal is:

Theorem 6.3.4. *Let $f : L \to M$ be a proper map of oriented manifolds which is transverse to the oriented closed submanifold $N \subset M$ and let $K := f^{-1}(N)$. Let $\delta_N \in H_c^{m-n}(M)$ be the Poincaré dual to N. Then $f^*\delta_N$ is the Poincaré dual to K.*

We begin with a characterization of a form representing δ_N. Using the tubular map $e : E \to U \subset M$, we can and will identify the normal bundle of N in M with a neighborhood of N (also relatively compact).

Definition 6.3.5. Let $N^n \subset M^m$ be a closed oriented submanifold of the oriented manifold M. Let $N \subset U \subset E \subset M$ be the unit disc bundle in the normal bundle. A map $f : \mathbb{R}^{m-n} \to M$ is a *simple cut for* N if the following conditions hold:

(1) $f^{-1}(N) = \{0\}$, $f(0) = x_0 \in N$,
(2) $f^{-1}(U) \subset \mathbb{R}^{m-n}$ is relatively compact,
(3) $f \pitchfork N$ (thus f is an immersion near 0),
(4) The vector space isomorphism $\mathrm{id} + T_0 f : T_{x_0} N \oplus T_0 \mathbb{R}^{m-n} \to T_{x_0} M$ is orientation preserving.

For example, the composition $\mathbb{R}^{m-n} \cong E_x \to E$ of an orientation-preserving isomorphism with the fibre inclusion is a simple cut through x.

Proposition 6.3.6. *A closed form $\delta \in \mathcal{A}_c^*(U)$ is a Poincaré dual to N if and only if for each simple cut f to N, the integral $\int_{\mathbb{R}^{m-n}} f^*\delta = 1$.*

Proof. If the integral is 1 for each simple cut, then the integral $\int_{E_x} \delta = 1$ for each $x \in N$. This proves that δ is a Thom form of E, transplanted to M. Vice versa, we have to prove that for each simple cut, the integral is 1. Let $f : \mathbb{R}^{m-n} \to M$ be a simple cut through $x_0 \in N$. Pick an oriented coordinate chart x for N around x_0, with $x(x_0) = 0$ and oriented bundle coordinates y for E. Altogether, we get an orientation preserving diffeomorphism $(x, y) : W \to \mathbb{R}^m$, sending x_0 to 0. By the remarks preceeding Definition 6.2.16, we can rechoose δ to have arbitrarily small support. In particular, we can choose the support so small that if $y \in \mathbb{R}^{m-n}$ satisfies $f(y) \in \mathrm{supp}(\delta)$, then $f(y) \in W$. With these manipulation, we have completely localized the situation.

The map f is represented by a map

$$(f_1, f_2) : \mathbb{R}^{m-n} \to \mathbb{R}^n \times \mathbb{R}^{m-n}.$$

The function f_2 has only one zero, namely at zero, and the transversality condition says that 0 is a regular value of f_2. Moreover, the orientation assumption means that $D_0 f_2$ has positive determinant. Furthermore, we can assume that $f_2^{-1}(D^l)$ is compact and that $\delta \in \mathcal{A}^{m-n}(\mathbb{R}^n \times \mathbb{R}^{m-n})$ is a closed form with support in $\mathbb{R}^n \times D^{m-n}$, such that $\int_{x \times \mathbb{R}^{m-n}} \delta = 1$ for all $x \in \mathbb{R}^n$.

We have to evaluate the integral $\int_{\mathbb{R}^{m-n}} (f_1, f_2)^*\delta$, which is difficult without a trick. The trick is to consider the family $f_t = (t f_1, f_2)$, $t \in [0,1]$. By our assumption on the support of δ and f_2, the smooth family $\tau_t := f_t^* \tau$ has compact support for all t. Therefore the integral $\int_{\mathbb{R}^{m-n}} \tau_t$ does not depend on t. We want to know the value for $t = 1$. But the computation is easier for $t = 0$. Namely, we have to compute $\int_{\mathbb{R}^{m-n}} f_2^* \delta$. By further shrinking the support of δ, we can arrange that f_2 is an orientation-preserving diffeomorphism $f^{-1}(\mathrm{supp}(\delta)) \to \mathrm{supp}(\delta)$. Therefore, the integral is 1. $\square$

Proof of Theorem 6.3.4. Let $K \subset V \subset L$ be a tubular neighborhood. Fix $y \in K$, let $g : \mathbb{R}^{m-n} \cong V_y \subset V$ be the composition of an orientation-preserving diffeomorphism with the fibre inclusion. View $\mathbb{R}^{m-n}$ as the interior of D^{m-n} and if V is chosen sufficiently small, then g extends to a smooth map $g : D^{m-n} \to \overline{V} \subset L$. The map $f \circ g : \mathbb{R}^{m-n} \subset V \to M$ is a cut through N, except that the second condition of Definition 6.3.5 might fail, which is corrected as follows.

Pick a Riemann metric on M; there is an $\epsilon > 0$ such that $\mathrm{dist}(f(z), N) \geq 2\epsilon$ for all $z \in S^{m-n-1}$. Pick a tubular neighborhood $N \subset U \subset M$ that is contained in the

ϵ-neighborhood on N in M and choose the form $\delta \in \mathcal{A}_c^{m-n}(M)$ to be supported in this tubular neighborhood. Then $f \circ g$ is a cut, and to finish the proof, we compute

$$\int_{\mathbb{R}^{m-n}} g^* f^* \delta = \int_{\mathbb{R}^{m-n}} (f \circ g)^* \delta.$$

Since fg is a cut, the right-hand side is 1. □

Theorem 6.3.4 has some interesting applications.

Theorem 6.3.7. *Let M^m be a closed oriented manifold and $\pi : V \to M$ be an oriented vector bundle of rank n. Let s be a section of V which is transverse to the zero section and let $Z^{m-n} := s^{-1}(0) \subset M$ be the zero set. This submanifold has an induced orientation, because $\nu_Z^M \cong V|_Z$. Then the Euler class $e(V) \in H^n(M)$ is Poincaré dual to Z.*

Proof. Let s_0 be the zero section and let τ be a Thom class for V. Then $e(V) = s_0^* \tau = s^* \tau$. Since τ is the Poincar'e dual of M in V, it follows that $s^*\tau$ is the Poincaré dual of Z in M. □

The case $m = n$ is of particular interest. In this case, Z is a finite set $Z = \{x_1, \dots, x_r\}$; each point x_i comes equipped with a sign $\epsilon_i \in \pm 1$ that determines its orientation. By Theorem 6.3.7, we compute

$$\int_M e(V) = \int_M \delta Z \overset{6.3.2}{=} \int_Z 1 = \sum_{i=1}^r \epsilon_i.$$

The signs ϵ_i are called the *local indices* of the section s and denoted $I_{x_i} s$. We have proven

Theorem 6.3.8. *(Poincaré-Hopf theorem) If $V \to M$ is an oriented rank n vector bundle over an oriented closed n-manifold and s a cross-section of V that is transverse to 0, then $\int_M e(V) = \sum_{s(x)} I_x s$.*

Another interesting application is:

Theorem 6.3.9. *Let M^m be an oriented manifold of dimension and let $L^l, N^n \subset M$ be two closed oriented submanifolds. Assume that L and N intersect transversally, in other words $\iota \pitchfork N$, where $\iota : L \to M$ is the inclusion. Then $\delta_N^M \wedge \delta_L^M$ is a Poincaré dual to $K = L \cap N$.*

Proof. Let ω be a closed form on M. Then

$$\int_K \omega|_K = \int_L \omega|_L \wedge \delta_K^L = \int_L (\omega \wedge \delta_N^M)|_L = \int_M \omega \wedge \delta_N^M \wedge \delta_L^M.$$

□

We are sloppy about signs here; ultimately, we are interested in even-dimensional manifolds only, where all sign questions disappear.

Exercise 6.3.10. *Determine the signs in the previous theorem.*

We now turn to a fundamental computation. Let $H \to \mathbb{CP}^n$ be the *dual* to the tautological line bundle. Here, it is useful to do the computation in the most invariant way. Let V be a complex vector space of dimension $n+1$. Why do we use the dual? Let $f \in V^*$ be a linear form on V. By restriction, f induces a linear form on each line $\ell \in \mathbb{P}V$, in other words, a section s_f of the bundle L^*, which is a holomorphic section. One could try the dual thing, but the only way to product a section of L out of a vector $v \in V$ is by projecting v to ℓ. This is fine, but the projection involves conjugates which is why the induced section is not holomorphic. In fact, L does not have any nonzero holomorphic section.

The zero set of s_f is the set of all lines such that $f|_\ell = 0$, or the projective space of $\ker(f)$. Let us assume that $f \neq 0$; then it can be shown without difficulty that s_f is transverse to the zero section.

Theorem 6.3.11. *Let $H \to \mathbb{CP}^n$ be the dual of the tautological bundle. Then*

$$\int_{\mathbb{CP}^n} e(H)^n = 1.$$

Proof. If $n = 1$, then the section s_f $(f \neq 0)$ has a unique zero, since $\ker(f)$ is 1-dimensional. Because the section is holomorphic, the local index must be $+1$, not -1, and the claim follows from the Poincaré-Hopf theorem 6.3.8. For higher n, we use that $e(H)^n = e(H^{\oplus n})$. Take linearly independent $f_1, \ldots, f_n \in V^*$. They induce a holomorphic section $s = (s_{f_1}, \ldots, s_{f_n})$ which has a unique zero, namely the line $\cap_{i=1}^n \ker(f_i)$. Again, the local index must be $+1$ by holomorphicity, and Poincaré-Hopf finishes the proof. $\square$

Now we can easily compute the cohomology ring of $\mathbb{CP}^n$.

Theorem 6.3.12. *Put $x := e(H)$. Then*

$$H^*(\mathbb{CP}^n) = \mathbb{R}[x]/(x^{n+1})$$

and $\int_{\mathbb{CP}^n} x^n = 1$.

Proof. We have just computed the integral, which shows that $x^k \neq 0$ for all $k \leq n$. It remains to compute the dimensions of $H^i(\mathbb{CP}^n)$. For that, we use that $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} \cong \mathbb{C}^n$ and the sequence 6.1.4:

$$\begin{tikzcd}
H^*(\mathbb{CP}^n) \arrow[rr] & & H^*(\mathbb{CP}^{n-1}) \arrow[dl, "{[+1]}"] \\
& H_c^*(\mathbb{C}^n) \arrow[ul] &
\end{tikzcd}$$

which gives an exact sequence

$$0 = H^{2n-1}(\mathbb{CP}^{n-1}) \to \mathbb{R} \cong H_c^{2n}(\mathbb{C}^n) \to H^{2n}(\mathbb{CP}^n) \to H^{2n}(\mathbb{CP}^{n-1}) = 0$$

and that the restriction maps are isomorphisms

$$H^k(\mathbb{CP}^n) \cong H^k(\mathbb{CP}^{n-1})$$

for $k \leq 2n-1$. Together, this makes an inductive proof, starting from the case $n = 0$. $\square$

6.4. **The topological Gauß-Bonnet theorem.**

Theorem 6.4.1. *Let M^n be a closed oriented manifold. Then*

$$\int_M e(TM) = \chi(M).$$

As a corollary, we obtain the first nontrivial instance of the Atiyah-Singer index theorem:

Corollary 6.4.2. *Let M be a closed oriented Riemann manifold and $d + d^* : \mathcal{A}^{ev}(M) \to \mathcal{A}^{odd}(M)$ be the Euler characteristic operator. Then*

$$\operatorname{ind}(d + d^*) = \int_M e(TM).$$

Lemma 6.4.3. *Let M be a manifold and $\Delta : M \to M \times M$ be the diagonal embedding, with image $\Delta(M)$. Then the normal bundle of $\Delta(M)$ in $M \times M$ is naturally isomorphic to the tangent bundle TM; and the isomorphism can be chosen to preserve orientations.*

Let U be a tubular neighborhood of $\Delta(M)$ and let V be the normal bundle of $\Delta(M)$. Let τ be a Thom form of TM and let $\rho \in \mathcal{A}^n(M \times M)$ be the result of grafting the Thom form τ into $M \times M$ (in other words, ρ is the Poincaré dual to $\Delta(M)$). Note that

$$e(TM) = \Delta^* e(V) = \Delta^* \iota^* \rho$$

where ι is the inclusion. Thus $e(TM) = \Delta^* \rho$. Now let $\{\alpha\}$ be a homogeneous basis of $H^*(M)$ and let $\{\beta^\#\}$ be the dual basis, i.e.

$$\int_M \alpha^\# \wedge \beta = \delta_{\alpha,\beta}.$$

By the Künneth theorem, $\{\alpha^\# \times \beta\}$ is a basis for $H^*(M \times M)$. There are unique $c_{\alpha,\beta} \in \mathbb{R}$ with

$$\rho = \sum_{\alpha,\beta} c_{\alpha,\beta} \alpha \times \beta^\#.$$

For two basis elements, γ, ϵ, we compute $\int_{M \times M} (\gamma^\# \times \epsilon) \wedge \rho$ in two ways. First of all

$$\int_{M \times M} (\gamma^\# \times \epsilon) \wedge \rho \overset{1}{=} \int_M \Delta^*(\gamma^\# \times \epsilon) = \int_M \gamma^\# \wedge \epsilon = \delta_{\gamma,\epsilon}.$$

In the first equation, we used the fact that the Thom class is Poincare dual to the diagonal. The other way to compute is

$$\int_{M \times M} (\gamma^\# \times \epsilon) \wedge \rho = \sum_{\alpha,\beta} c_{\alpha,\beta} \int_{M \times M} (\gamma^\# \times \epsilon) \wedge (\alpha \times \beta^\#). \tag{6.4.4}$$

But

$$\int_{M \times M} (\gamma^\# \times \epsilon) \wedge (\alpha \times \beta^\#) = (-1)^{|\epsilon||\alpha|} \int_{M \times M} (\gamma^\# \wedge \alpha) \times (\epsilon \wedge \beta^\#) =$$
$$= (-1)^{|\epsilon||\alpha|} \int_M (\gamma^\# \wedge \alpha) \int_M (\epsilon \wedge \beta^\#) = (-1)^{|\epsilon||\alpha|} \delta_{\gamma,\alpha} (-1)^{(n-|\beta|)|\epsilon|} \delta_{\beta,\epsilon}$$

and therefore

$$\delta_{\gamma,\epsilon} = (6.4.4) = c_{\gamma,\epsilon} (-1)^{|\epsilon||\gamma|} (-1)^{(n-|\epsilon|)|\epsilon|} = (-1)^{n|\gamma|}.$$

In other words

$$\rho = \sum_\alpha (-1)^{n|\alpha|} \alpha \times \alpha^\#$$

and therefore

$$\int_M e(TM) = \int_M \Delta^* \rho = \sum_\alpha (-1)^{n|\alpha|} \int_M \alpha \wedge \alpha^\# = \sum_\alpha \underbrace{(-1)^{n|\alpha||\alpha|(n-|\alpha|)}}_{=(-1)^{|\alpha|}} \underbrace{\int_M \alpha^\# \wedge \alpha}_{=1} = \chi(M).$$

6.5. **The Gysin map and the splitting principle for complex vector bundles.**

Definition 6.5.1. Let $f : M^{n+d} \to N^n$ be a map between closed oriented manifolds (we allow d to be negative). The *Gysin map* $f_! : H^k(M) \to H^{k-d}(N)$ is defined to be the composition

$$H^k(M) \overset{D_M}{\to} H^{n+d-k}(M)^* \overset{(f^*)^*}{\to} H^{n+d-k}(N)^* \overset{D_N^{-1}}{\to} H^{k-d}(N).$$

The two stars in the symbol $(f^*)^*$ mean two different things: the inner $*$ is the induced map on cohomology, and the outer $*$ is the dual in the sense of linear algebra. Some remarks on terminology: often the Gysin map is called *umkehr map* (also in the English literature). I want to warn against two misnamings that occur quite often. The first misnaming is "pushforward". This is casual terminology, and I do not want to advertise this. The second misnaming is "transfer", and using this word for the Gysin map is an outright mistake - the true use of the word "transfer" is for something closely related, but different.

Let us unwind the definition of the Gysin map. Let $\omega \in H^k(M)$ and $\eta \in H^{n+d-k}(N)$. Then we compute $(D_N f_!(\omega))(\eta) = \int_N f_!(\omega) \wedge \eta$ by the definition of D_N. On the other hand

$$(D_N f_!(\omega))(\eta) = ((f^*)^* D_M(\omega))(\eta) = D_M(\omega)(f^*\eta) = \int_M \omega \wedge f^*\eta.$$

Thus we arrive at the equation

$$\int_N f_!(\omega) \wedge \eta = \int_M \omega \wedge f^*\eta \tag{6.5.2}$$

which characterizes $f_!$ and can be expressed by saying that $f_!$ is adjoint to f^* with respect to the duality pairing. We will use equation 6.5.2 to derive all properties of the Gysin map. First

$$\int_N f_!(\omega \wedge f^*\zeta) \wedge \eta = \int_M \omega \wedge f^*\zeta \wedge f^*\eta = \int_M \omega \wedge f^*(\zeta \wedge \eta) = \int_N f_! \wedge \zeta \wedge \eta.$$

Since this holds for all η, we find - using duality - that

$$f_!(\omega \wedge f^*\zeta) = f_!(\omega) \wedge \zeta. \tag{6.5.3}$$

For another consequence, consider the constant map $f : M \to *$. Then $f_!(\omega) = \int_M \omega$. Also

$$(f \circ g)_! = f_! \circ g_!$$

is an immediate consequence of the functoriality of cohomology.

Proposition 6.5.4. *Let $f : M^{n+d} \to N^n$ be a smooth map of closed oriented manifolds, let $z \in N$ be a regular value of f. Assume that N is connected and let $\omega \in H^d(M)$. Then $f_!(\omega) = \int_{f^{-1}(z)} \omega \in \mathbb{R} = H^0(N)$.*

Proof. Let $\tau \in H^n(N)$ be the Poincaré dual to $z \subset N$ (this is just a class with $\int_N \tau = 1$). By Theorem 6.3.4, $f^*\tau$ is the Poincaré dual to $f^{-1}(z)$ in M. Then

$$\int_N f_!(\omega) \wedge \tau \overset{6.5.2}{=} \int_M \omega \wedge f^*\tau \overset{6.3.2}{=} \int_{f^{-1}(z)} \omega.$$

□

A nice situation appears if all $z \in N$ are regular values, in other words, if f is a submersion. A not so hard theorem from differential topology, the Ehresmann fibration lemma, says that a proper submersion is a fibre bundle. In this situation, one can give an explicit differential form representative of $f_!(\omega)$, obtained by integration over the fibres. We do not need to consider this refinement of the definition of the Gysin homomorphism here.

If $f : M \to N$ is a submersion, we denote by $T_vM := \ker(df)$ the kernel of the differential of f. This is a vector bundle by Lemma 5.1.6 and called the *vertical tangent bundle*; because it consists of all tangent vectors that are tangent to the fibres of f. There is a vector bundle splitting

$$TM \cong f^*TN \oplus T_vE. \tag{6.5.5}$$

If M and N are both oriented, then T_vE acquires an orientation. The topological Gauss-Bonnet theorem has two interesting consequences.

Definition 6.5.6. Let $f : M \to N$ be a proper submersion of closed oriented manifolds. The *transfer* $\mathrm{trf}_f : H^*(M) \to H^*(N)$ is the map $\mathrm{trf}_f(\omega) := f_!(e(T_vM)\omega)$.

Theorem 6.5.7. *Let $f : M \to N$ be a proper submersion of closed oriented manifolds. Assume that M is connected and that the Euler characteristic $\chi(F)$ of the fibres $F := f^{-1}(x)$ is nonzero. Then $f^* : H^*(N) \to H^*(M)$ is injective.*

Proof. We calculate

$$\mathrm{trf}_f(f^*\omega) = f_!(e(T_vE)f^*\omega) \overset{6.5.3}{=} f_!(e(T_vE))\omega \overset{6.5.4}{=} \chi(F)\omega.$$

Since $\chi(F) \neq 0$, this implies that f^* is injective.

□

Theorem 6.5.8. *Let $f : M^{n+d} \to N^n$ be a proper submersion of closed oriented manifolds, with $F := f^{-1}(z)$. Then $\chi(M) = \chi(N)\chi(F)$.*

Proof. This is a straightforward consequence of the topological Gauss-Bonnet theorem:

$$\chi(M) = \int_M e(TM) = \int_M f^*e(TN) \wedge e(T_vE) = (-1)^{dn} \int_M e(T_vE) \wedge f^*e(TN) =$$

$$= (-1)^{nd} \int_N f_!(e(T_vE)) \wedge e(TN) = (-1)^{nd}\chi(F) \int_N e(TN) = (-1)^{nd}\chi(F)\chi(N).$$

If nd is odd, then both F and N are odd-dimensional and thus have zero Euler numbers, and so does M by the above equation, and the sign does not matter. In all other cases, $\chi(M) = \chi(F)\chi(N)$, as claimed. □

7. Connections, curvature and the Chern-Weil construction

Now we develop the local theory of characteristic classes.

7.1. Covariant derivatives.

Definition 7.1.1. Let V be a vector bundle on a manifold M. A *covariant derivative* alias *connection* is a map $\nabla : \mathcal{A}^0(M;V) \to \mathcal{A}^1(M;V)$ such that $\nabla(fs) = df \otimes s + f\nabla s$.

We will use the two terms interchangingly; later, we will describe the same object in a different language, and then we distinguish the names properly. We see that $[\nabla, f] = df\cdot$. In other words, a covariant derivative is a first order differential operator whose principal symbol is given by $\mathrm{smb}_\nabla(\xi) = i\xi$ for all cotangent vectors ξ. Therefore, covariant derivatives exist on any vector bundle (Proposition 2.2.19). We want some more concrete examples.

Proposition 7.1.2. *The exterior derivative $\mathcal{A}^0 \to \mathcal{A}^1$ is a connection. Let $V \subset M \times \mathbb{R}^n$ and p the orthogonal projection onto V. Then $\nabla := pd$ is a covariant derivative.*

The proof is trivial. On a Riemann manifold, there is a special covariant derivative on the tangent bundle.

Definition 7.1.3. Let $V \to M$ be a Riemannian vector bundle. A covariant derivative ∇ is called *metric* if $X\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$ holds for all vector fields X and all sections s, t.

Theorem 7.1.4. *(The fundamental lemma of Riemannian geometry) On any Riemann manifold, there is a unique connection on TM, which is metric and torsion-free, $\nabla_X Y - \nabla_Y X = [X,Y]$. This connection is also called Levi-Civita connection.*

The proof can be found in any textbook on Riemann geometry.

Lemma 7.1.5. *Let $V \to M$ be a vector bundle and ∇ a connection on V. Then there is a unique sequence of linear maps (differential operators)*

$$\mathcal{A}^0(M;V) \xrightarrow{\nabla} \mathcal{A}^1(M;V) \xrightarrow{\nabla} \mathcal{A}^2(M;V) \xrightarrow{\nabla} \ldots$$

such that ∇ coincides with the connection for $p = 0$ and such that $\nabla(\omega \wedge s) = d\omega \wedge s + (-1)^{|\omega|}\omega \wedge \nabla s$ holds.

Proof. Locally, each $s \in \mathcal{A}^p(M;V)$ can be written as a linear combination of terms of the form $\omega \otimes t$, where ω is a scalar-valued form and t a section of V. The product rule prescribes the value of ∇ on these elements. This shows uniqueness.

Choosing a local frame $e_1, \ldots, e_r$ of V, each section can uniquely written as $s = \sum_i \omega_i \otimes e_i$, for forms ω_i. We set

$$\nabla s = \sum_i d\omega_i \otimes e_i + \omega_i \wedge \nabla e_i.$$

This has the desired property locally, and uniqueness proves that it is coordinate independent. □

Note the similarity to the definition of the exterior derivative. But we have to sacrifice the condition $\nabla^2 = 0$, for a very substantial reason, as we shall see.

Proposition 7.1.6. *There exists a unique 2-form Ω with values in* $\mathrm{End}(V)$ *such that $\nabla^2 = \Omega$. This form is called the* curvature.

Proof. First we prove that ∇^2 is of order 0. This is because

$$\nabla\nabla(fs) = \nabla(df \wedge s) + \nabla(f\nabla s) = -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s.$$

More generally, if ω is a form, then

$$\nabla\nabla(\omega \wedge s) = \nabla(d\omega \wedge s) + (-1)^p \nabla(\omega\nabla s) = \omega\nabla^2 s.$$

Therefore, ∇^2 commutes with multiplication by forms and thus it is determined by Ω.

□

7.2. **The first Chern class - the baby case of Chern-Weil theory.** Let us figure out what the curvature look like for a complex line bundle $L \to M$. Let s be a local section of L without zeroes, in other words a local basis. As ∇s is an L-valued 1-form, we can write it as $\nabla s = \omega \wedge s$, for a unique complex valued 1-form ω. But then

$$\nabla^2 s = \nabla(\omega \wedge s) = d\omega \wedge s - \omega \wedge \nabla s = (d\omega) \wedge s - \omega \wedge \omega s = (d\omega) \wedge s.$$

Here we used in an essential way that we talked about a *line* bundle. The relevant property is that L has *abelian* structure group, as we will see. We have shown that the curvature form is just $\Omega = d\omega$. That it is a scalar valued 2-form should not come as a surprise: the endomorphism bundle of a line bundle is trivial, in a canonical way (the identity endomorphism is a global section without zeroes). Let us make some remarks. We know that the form Ω is a *globally defined* 2-form, by Proposition 7.1.6! The formula $\Omega = d\omega$ seems to suggest that Ω is an exact form, but this is fallacious: the form ω *depends on the choice of s* and therefore, it does exist only *locally.* But the formula $\Omega = d\omega$ *does give* important information - namely that Ω is closed, which is by definition a local property of a form. So we get a cohomology class $[\Omega] \in H^2(M)$. If L is trivial, then there exists a global section s without zeroes, and thus the form ω has a global meaning, which tells us at once that Ω is exact and $[\Omega] = 0$.

On the other hand, the trivial bundle $M \times \mathbb{C}$ admits a connection whose curvature is zero, namely the exterior derivative! These observations might lead us to the suspicion that the cohomology class of the curvature form contains relevant information about the global structure of the bundle. We now prove that is indeed correct.

Definition 7.2.1. Let $L \to M$ be a complex line bundle and ∇ a connection on L with curvature form Ω. The *first Chern class* of L is the class

$$c_1(L) := -\frac{1}{2\pi i}[\Omega].$$

Lemma 7.2.2. *The cohomology class of Ω is independent of the choice of the connection.*

Proof. Indeed, the difference $\nabla_1 - \nabla_0$ of two connections on L is an operator of order 0 and hence given by a (complex-linear) vector bundle homomorphism $L \to T^*M \otimes_{\mathbb{R}} L$ or equivalently by a 1-form with values in the endomorphism bundle

$\mathrm{End}(L)$. As $\mathrm{End}(L)$ is trivial, this is just a 1-form $\alpha \in \mathcal{A}^1(M, \mathbb{C})$. Hence $\nabla_1 = \nabla_0 + \alpha$ for a globally defined 1-form α. With respect to a local section s, this shows that $\omega_1 = \omega_0 + \alpha$ (locally) and hence $\Omega_1 = \Omega_0 + d\alpha$ *globally.* □

One can (and we will) show that if $f : M \to N$ is a smooth map, then $f^* c_1(L) = c_1(f^* L)$, but this won't be simpler than the general argument given below.

Example 7.2.3. Consider the tautological line bundle $L \to \mathbb{CP}^1$. Recall that $L = \{(\ell, v) \in \mathbb{CP}^1 | v \in \ell\}$. Let $U \subset \mathbb{CP}^1$ be the set of all $[1 : z] \in \mathbb{CP}^1$; this is the complement of a point. We will now compute the projection connection ∇ and its curvature. There is a complex chart $\mathbb{C} \to U$, $z \mapsto [1 : z]$. Over U, we have the section $s : U \to L|_U$, which expressed in these coordinates is given by

$$s(z) := ([1 : z], (1, z)).$$

In other words, the section s is given by the vector valued function, also denoted $s(z) = \begin{pmatrix} 1 \\ z \end{pmatrix}$. The projection operator is

$$p(z) = \frac{1}{\|s(z)\|} \begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} 1 & \bar{z} \end{pmatrix} = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}.$$

With these formulae, we compute

$$\nabla s = p(ds) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} \begin{pmatrix} 0 \\ dz \end{pmatrix} = \frac{1}{1 + |z|^2} \begin{pmatrix} \bar{z} dz \\ \bar{z} z dz \end{pmatrix} = \frac{\bar{z} dz}{1 + |z|^2} \begin{pmatrix} 1 \\ z \end{pmatrix} = \omega s$$

with

$$\omega = \frac{\bar{z} dz}{1 + |z|^2}.$$

The curvature is the exterior derivative of this form, but we do not need to compute the derivative. Let us show that $[\Omega]$ and hence $c_1(L)$ is nontrivial, by computing $\int_{\mathbb{CP}^1} c_1(L)$. Since the complement of the coordinate patch U is a point, it has measure zero, and we compute

$$-\int_{\mathbb{CP}^1} c_1(L) = \frac{1}{2\pi i} \int_{\mathbb{C}} d\omega = \frac{1}{2\pi i} \lim_{r \to \infty} \int_{|z| \leq r} d\omega = \frac{1}{2\pi i} \lim_{r \to \infty} \int_{|z| = r} \omega$$

by Stokes theorem. But for $|z| = r$, we have $\bar{z} = r^2 z^{-1}$ and hence

$$\frac{1}{2\pi i} \lim_{r \to \infty} \int_{|z| = r} \omega = \frac{1}{2\pi i} \lim_{r \to \infty} \frac{r^2}{1 + r^2} \int_{|z| = r} z^{-1} dz = 1.$$

The minus sign has "historical" reasons, the $2\pi i$ factor emphasizes the integral structure of cohomology! The "historical" reason is to line up with the Euler class.

Theorem 7.2.4. *Let $L \to M$ be a complex line bundle. Then $c_1(L) = e(L)$.*

The proof is a prelude to the proof of the Gauß-Bonnet-Chern theorem, and is a good introduction to the techniques involved in the theory of characteristic classes. First, we use that any complex line bundle has a bundle map to $L \to \mathbb{CP}^n$, the tautological line bundle. The naturality of the Euler class, together with the not yet proven naturality of the first Chern class, shows that it is enough to consider the tautological line bundle.

Now the dual of a complex vector bundle is isomorphic to the complex-conjugate bundle. The effect on the Euler class is that orientation is changed by $(-1)^k$, k the rank. Therefore, the Euler class of L is -1, the same as $c_1(L)$.

The computation for the integral of the first Chern class on $\mathbb{CP}^1$ proves that it is equal to the Euler class. The case of higher dimensional projective spaces follows because $H^2(\mathbb{CP}^n) \to H^2(\mathbb{CP}^1)$ is an isomorphism!

However, not all vector bundles are line bundles, and we now study connections and curvature in more detail for higher rank bundles. It is the failure of commutativity of the Lie groups GL_n that makes this more difficult.

7.3. **The coordinate description of a connection.** In the same way as principal bundles gave us more flexibility when dealing with bundles, we will gain flexibility with connections by introducing a new concept a connection on a G-principal bundle when G is a Lie group. Just as principal bundles is a notion to keep track in a systematic way of all trivializations, the notion of a connection on a principal bundle arises from a systematic study of the way the connection is written when coordinates are chosen.

Let $V \to M$ be a vector bundle and ∇ be a connection on V. Let $U \subset M$ be open and s a local section of the frame bundle, defined over U. We consider s as a map $U \times \mathbb{R}^n \to V|_U$ (a bundle isomorphism). We obtain isomorphisms

$$\phi_s : \mathcal{A}^p(U; \mathbb{R}^n) \to \mathcal{A}^p(U; V).$$

These isomorphisms have the following explicit description. Let e_i be the ith unit vector in $\mathbb{R}^n$. Recall that for $x \in U$, $s(x)$ gives a vector space isomorphism $\mathbb{R}^n \to V_x$, and we let $s_i(x) \in V_x$ be the image of e_i. Of course, we obtain continuous sections s_i of $V|_U$, and this is another way of describing the local frame. For $p = 0$, the map ϕ_s sends a function $a = (a_1, \dots, a_n)$ to the section $\sum_i a_i s_i$. The same is true for p-forms, i.e. if a_i is a p-form. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{A}^p(U, \mathbb{R}^n) \arrow[r, "\phi_s"] \arrow[d, "?"] & \mathcal{A}^p(U; V) \arrow[d, "\nabla"] \\
\mathcal{A}^{p+1}(U; \mathbb{R}^n) \arrow[r, "\phi_s"] & \mathcal{A}^{p+1}(U; V);
\end{tikzcd} \tag{7.3.1}$$

the left vertical map is defined so that the diagram commutes (this is possible since the horizontal maps are isomorphisms). Let us describe the map "?". There exists uniquely determined forms

$$\theta_{ij} \in \mathcal{A}^1(U) \text{ such that } \nabla s_j = \sum_i \theta_{ij} s_i (\text{sic!}).$$

We summarize them in the matrix $\theta = \theta_s = (\theta_{ij})_{i,j=1,\dots n}$. Now we follow a tuple of p-forms $a = (a_1, \dots, a_n)$ in the diagram 7.3.1. By ϕ_s, it is sent to $\sum_i a_i s_i$ and

$$\nabla \sum_i a_i s_i = \sum_{i=1}^n da_i \otimes s_i + (-1)^p \sum_{j,i} a_j \theta_{ij} s_i$$

With these notations, we can write

$$\nabla \phi_s(a) = \sum_i da_i \otimes s_i + (-1)^p \sum_j \sum_i a_j \theta_{ij} s_i = \phi_s(da) + \phi_s(\theta a),$$

the $(-1)^p$ factor is absorbed because a_i is a scalar-valued p-form and the entries of θ are 1-forms and we interchanged the order of multiplication.

Thus the connection can be written in local coordinates s as $d+\theta$, for a 1-form $\theta \in \mathcal{A}^1(U;\mathfrak{gl}_n(\mathbb{R}))$. Here $\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$; which as a vector space is the same as $\mathrm{Mat}_{n,n}(\mathbb{R})$. Not yet, but soon it will become clear why we use the Lie algebra notation here. The curvature is easily computes in this formalism as

$$\Omega_s a = (d+\theta)(d+\theta)a = (d+\theta)(da+\theta a) = dda+d\theta a-\theta\wedge da+\theta\wedge d\theta+\theta\wedge\theta a = (d\theta+\theta\wedge\theta)a,$$

the notation Ω_s indicates that the matrix-valued 2-form $\Omega_s \in \mathcal{A}^2(U,\mathfrak{gl}_n(\mathbb{R}))$ depends on s. As a side remark, this gives another proof that the curvature is a tensor field. Finally, we remark that $\theta\wedge\theta$ is in general not zero, because the ring $\mathrm{Mat}_{n,n}(\mathbb{R})$ is not commutative.

The next step is to figure out how the forms θ_s and Ω_s change if the local frame s changes. Let $g : U \to \mathrm{GL}_n(\mathbb{R})$ be a smooth function and s a local frame. We obtain a new local frame sg, with components given by

$$(sg)_i = (\sum_j g_{j1}s_j,\ldots,\sum_j g_{j1}s_j).$$

The functions / forms a_i are changed to $g^{-1}a$ (matrix multiplication).

The map ϕ_{sg} is the composition $\phi_s\circ(g\ldots)$, where $(g\cdot)$ is the map given by matrix multiplication. To find out the change-of-frame formula, we look at the diagram

$$\begin{tikzcd}
C^\infty(U,\mathbb{R}^n) \arrow[r, "g\cdot"] \arrow[d, "d+\theta_{sg}"] & C^\infty(U,\mathbb{R}^n) \arrow[r, "\phi_s"] \arrow[d, "d+\theta_s"] & \mathcal{A}^0(U,V) \arrow[d, "\nabla"] \\
\mathcal{A}^1(U,\mathbb{R}^n) \arrow[r, "g\cdot"] & \mathcal{A}^1(U,\mathbb{R}^n) \arrow[r, "\phi_s"] & \mathcal{A}^1(U,V);
\end{tikzcd}$$

the horizontal compositions are the maps ϕ_{sg}. From that, one derives the formula

$$\theta_{sg} = g^{-1}dg + g^{-1}\theta_s g. \tag{7.3.2}$$

The curvature form transforms as

$$\begin{aligned}
\Omega_{sg} = d\theta_{sg} + \theta_{sg}\wedge\theta_{sg} = d(g^{-1}dg) + d(g^{-1}\theta_s g) + (g^{-1}dg + g^{-1}\theta_s g)\wedge(g^{-1}dg + g^{-1}\theta_s g) = \\
= d(g^{-1}dg) + d(g^{-1}\theta_s g) + g^{-1}dgg^{-1}dg + g^{-1}dgg^{-1}\theta g + g^{-1}\theta dg + g^{-1}\theta\wedge\theta g = \\
= d(g^{-1}\theta_s g) + g^{-1}dgg^{-1}\theta g + g^{-1}\theta dg + g^{-1}\theta\wedge\theta g = \\
= g^{-1}dgg^{-1}\theta_s g + g^{-1}d\theta g - g^{-1}\theta dg + g^{-1}dgg^{-1}\theta g + g^{-1}\theta dg + g^{-1}\theta\wedge\theta g = \\
= g^{-1}(d\theta+\theta\wedge\theta)g.
\end{aligned}$$

Let us summarize the calculations so far:

Proposition 7.3.3. *There is a bijection between connections on V and rules that assign a form $\theta_s \in \mathcal{A}^1(U,\mathrm{Mat}_{n,n}(\mathbb{R}^n))$ to a local frame s, such that $\theta_{sg} = g^{-1}dg + g^{-1}\theta_s g$, for each change-of-frame function $g : U \to \mathrm{GL}_n(\mathbb{R})$. The curvature is given by $\Omega_s = d\theta_s + \theta_s\wedge\theta_s$, and the change-of-frame formula is $\Omega_{sg} = g^{-1}\Omega_s g$. Any such a rule defines a connection; in a frame s it is $\nabla = d+\theta_s$.*

This description is not very practical yet; we want to package the information of the "rule" in a single 1-form $\theta \in \mathcal{A}^1(\mathrm{Fr}(V); \mathfrak{gl}_n(\mathbb{R}))$, such that $\theta_s := s^*\theta$. And we want to talk about other Lie groups than $\mathrm{GL}_n(\mathbb{R})$.

7.4. **The very basics of Lie theory.** We need to pause a little and introduce some basic constructions of Lie theory, beyond the mere definitions.

Definition 7.4.1. Let G be a Lie group. The *Lie algebra* of G is the vector space $T_1G = \mathfrak{g} = \mathrm{Lie}(G)$.

If V is a real vector space, then $\mathfrak{gl}(V) = \mathrm{End}(V)$.

Definition 7.4.2. A *Lie algebra* over a field $\mathbb{F}$ of characteristic $\neq 2$ is a $\mathbb{F}$-vector space $\mathfrak{g}$, together with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$ such that

(1) $[X, Y] = -[Y, X]$ and
(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$ (the *Jacobi identity*)

hold.

How is the structure of a Lie algebra on T_1G defined? Each $g \in G$ defines a smooth map $C_g : G \to G$, $h \mapsto ghg^{-1}$, and $C_g(1) = 1$.

Definition 7.4.3. Let G be a Lie group. The *adjoint representation* of G on $\mathfrak{g}$ is defined by $\mathrm{Ad}(g)X := D_1C_g(X)$.

It is easy to see that

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h)$$

and that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a smooth group homomorphism.

Definition 7.4.4. Let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the derivative of Ad at the identity. We define, for $X, Y \in \mathfrak{g}$: $[X, Y] := \mathrm{ad}(X)(Y)$.

It is not a complete triviality to prove:

Theorem 7.4.5. *(Lie's first theorem) $\mathfrak{g}$, equipped with $[,]$ is a Lie algebra. If $\phi : G \to H$ is a homomorphism of Lie groups, then $D_1\phi$ is a homomorphism of Lie algebras.*

This can be found in basic texts on Lie theory, e.g. [6], [9].

Example 7.4.6. Let $G = \mathrm{GL}_n(\mathbb{R})$. Then $\mathfrak{gl}_n(\mathbb{R})$ is the space of $n \times n$-matrices. The conjugation map $C_g(h) := ghg^{-1}$. To compute the adjoint representation, let $X \in \mathfrak{gl}_n(\mathbb{R})$ and $g \in \mathrm{GL}_n(\mathbb{R})$. Then $t \mapsto \exp(tX)$ is a curve through 1 with tangent vector X, and

$$\mathrm{Ad}(g)X = \frac{d}{dt}|_{t=0} g \exp(tX) g^{-1} = gXg^{-1}.$$

Moreover

$$\mathrm{ad}(Y)(X) = \frac{d}{dt}|_{t=0} \exp(tY) X \exp(-tY) = YX - XY.$$

Definition 7.4.7. A *representation* of a Lie group G on the vector space V is a smooth group homomorphism $\phi : G \to \mathrm{GL}(V)$. A *representation* of a Lie algebra is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

If a Lie group representation $\phi : G \to \mathrm{GL}(V)$ is given, we obtain a Lie algebra representation as the derivative of ϕ, at the identity. There are some ways to produce new representations of Lie groups/algebras out of old ones. We denote the action by $g \in G$ or $X \in \mathfrak{g}$.

Examples 7.4.8. (1) *The trivial representations:* $g \cdot v := v$, $X \cdot v = 0$.
(2) *Direct sums of representations are representations (in both cases).*
(3) *If V is a representation, then the dual space has the following representation:* $g \cdot \ell := \ell \circ g^{-1}$, $X \cdot \ell := -\ell \circ X$.
(4) *If V and W are representations, the tensor product $V \otimes W$ is a representation:* $g(v \otimes w) := gv \otimes gw$, $X(v \otimes w) := Xv \otimes w + v \otimes Xw$.
(5) $\mathrm{Hom}(V, W)$ *is a representation:* $g \cdot f := g \circ f \circ g^{-1}$, $X \cdot f := X \circ f - f \circ X$.

Lemma 7.4.9. *Let $\phi : G \to \mathrm{GL}(V)$ be a representation and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced Lie algebra representation. Then φ is G-equivariant, in other words for all $g \in G$ and $x \in \mathfrak{g}$:*

$$\phi(g)\varphi(X)\phi(g^{-1}) = \varphi(\mathrm{Ad}(g)X) \in \mathfrak{gl}(V).$$

Proof. Let $c_t : (-\epsilon, \epsilon) \to G$ be a curve with $c_0 = 1$ and $\dot{c}_0 = X$. Then

$$\phi(g)\varphi(X)\phi(g^{-1}) = \phi(g)\frac{d}{dt}|_{t=0}\phi(c_t)\phi(g^{-1}) = \frac{d}{dt}|_{t=0}\phi(gc_tg^{-1}) =$$
$$= \frac{d}{dt}|_{t=0}\phi(C_g(c_t)) = \varphi(\frac{d}{dt}|_{t=0}C_gc_t) = \varphi(\mathrm{Ad}(g)X).$$

□

If M is a manifold and $\mathfrak{g}$ a Lie algebra, we can talk about the space $\mathcal{A}^p(M; \mathfrak{g})$ of p-forms with values in $\mathfrak{g}$. One can combine the wedge product and the Lie bracket:

$$[;] : \mathcal{A}^p(M; \mathfrak{g}) \otimes \mathcal{A}^q(M; \mathfrak{g}) \xrightarrow{\wedge} \mathcal{A}^{p+q}(M; \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{[,]} \mathcal{A}^{p+q}(M, \mathfrak{g});$$

more concretely, if ω, η are real valued forms and $X, Y \in \mathfrak{g}$, then $[\omega \otimes X, \eta \otimes Y] := \omega \wedge \eta \otimes [X, Y]$.

Example 7.4.10. Let $\mathfrak{g} = \mathfrak{gl}(V)$ and $X, Y \in \mathfrak{gl}(V)$, $\omega \in \mathcal{A}^p(M)$, $\eta \in \mathcal{A}^q(M)$. Then

$$[\omega \otimes X, \eta \otimes Y] = \omega \wedge \eta \otimes XY - \omega \wedge \eta \otimes YX = (\omega \otimes X) \wedge (\eta \otimes Y) - (-1)^{pq}\eta \wedge \omega \otimes YX =$$
$$(\omega \otimes X) \wedge (\eta \otimes Y) + (-1)^{pq+1}(\eta \otimes Y) \wedge (\omega \otimes X).$$

In other words, if $\omega \in \mathcal{A}^p(M, \mathfrak{gl}(V))$ and $\eta \in \mathcal{A}^q(M, \mathfrak{gl}(V))$, we find that

$$[\omega, \eta] = \omega \wedge \eta - (-1)^{pq}\eta \wedge \omega, \tag{7.4.11}$$

in particular, for $\omega \in \mathcal{A}^1(M; \mathfrak{g})$:

$$\omega \wedge \omega = \frac{1}{2}[\omega, \omega].$$

In general, one can prove easily that $\mathcal{A}^p(M, \mathfrak{g})$ has the structure of a *differential graded Lie algebra*:

Proposition 7.4.12. *Assume that $\omega \in \mathcal{A}^p(M; \mathfrak{g})$, $\eta \in \mathcal{A}^q(M; \mathfrak{g})$, and $\zeta \in \mathcal{A}^r(M; \mathfrak{g})$. Then*

(1) $d[\omega,\eta] = [d\omega,\eta] + (-1)^p[\omega, d\eta]$,
(2) $[\omega,\eta] = (-1)^{pq+1}[\eta,\omega]$,
(3) $(-1)^{pr}[[\omega,\eta],\zeta] + (-1)^{qp}[[\eta,\zeta],\omega] + (-1)^{rq}[[\zeta,\omega],\eta] = 0$.

A Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ induces, in an obvious manner, a map $\varphi_* : \mathcal{A}^*(M,\mathfrak{g}) \to \mathcal{A}^*(M,\mathfrak{h})$. Smooth maps $g : M \to G$ act on $\mathfrak{g}$-valued differential forms by the adjoint representation. I.e., if $\omega \in \mathcal{A}^p(M,\mathfrak{g})$, then $\mathrm{Ad}(g)\omega$ is the form that takes tangent vectors $X_1,\dots,X_p \in T_xM$ to

$$\mathrm{Ad}(g(x))(\omega(X_1,\dots,X_p)).$$

The mother of all Lie algebra valued forms is a canonical 1-form that exists on every Lie group.

Definition 7.4.13. Let G be a Lie group and $\pi : TG \to G$ be the tangent bundle. By R_g, L_g, we denote the left and right translations by $g \in G$. The maps $TG \to G\times\mathfrak{g}$, $v \mapsto (\pi(v), L_{\pi(v)^{-1}*}v)$ and $G\times\mathfrak{g} \to TG$, $(g,x) \mapsto L_{g*}x$ are two mutually inverse bundle isomorphisms. The 1-form $\omega_G \in \mathcal{A}^1(G;\mathfrak{g})$, $v \mapsto L_{\pi(v)^{-1}*}v$ is called *Maurer-Cartan-form.*

Example 7.4.14. Let $G = \mathrm{GL}_n(\mathbb{R})$. Then there is the trivialization $TG = \mathrm{GL}_n(\mathbb{R})\times \mathrm{Mat}_{n,n}(\mathbb{R})$, since $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{Mat}_{n,n}(\mathbb{R})$ is an open subset. The map $v \mapsto L_{\pi(v)^{-1}*}v$ becomes in this trivialization

$$(g,X) \mapsto g^{-1}X,$$

because the left-translation map is "linear". So we can write the Maurer-Cartan form $\omega_{\mathrm{GL}_n(\mathbb{R})} = g^{-1}dg$.

The Maurer-Cartan form has some useful properties.

Proposition 7.4.15. *Let G and H be Lie groups and $\phi : G \to H$ be a homomorphism with derivative $\varphi : \mathfrak{g} \to \mathfrak{h}$. Let $\mu : G\times G \to G$ be the multiplication. Then the following statements are true:*

(1) *For all $g \in G$: $L_g^*\omega_G = \omega_G$; $R_g^*\omega_G = \mathrm{Ad}(g^{-1})\circ\omega_G$.*
(2) *$\phi^*\omega_H = \varphi_*\omega_G$.*
(3) *$\mu^*\omega_G = p_2^*\omega_G + \mathrm{Ad}(p_2^{-1})p_1^*\omega_G$, where $p_i : G\times G \to G$ are the two projections.*
(4) *(structural equation) $d\omega_G + \frac{1}{2}[\omega_G,\omega_G] = 0$.*
(5) *Let $\eta \in \mathcal{A}^p(M)$ and $g : M \to G$. Then*

$$d(\mathrm{Ad}(g^{-1})\eta) = -g^*\omega_G \wedge \mathrm{Ad}(g^{-1})(\eta) + \mathrm{Ad}(g^{-1})d\eta + (-1)^p\mathrm{Ad}(g^{-1})\eta g^*\omega_G.$$

We will only give the proof in the special case when G is a *linear* group, i.e. if $G \subset \mathrm{GL}_n(\mathbb{R})$. This is not a serious restriction for our purposes; we will only consider linear groups, but simplifies the proof, because the objects are easier to grasp.

Proof. Ad 1) and 2): these follow immediately from the definitions, and for arbitrary Lie groups.

Ad 3) This is the computation

$$(gh)^{-1}d(gh) = (gh)^{-1}gdh+(gh)^{-1}(dg)h = h^{-1}g^{-1}gdh+h^{-1}g^{-1}(dg)h = h^{-1}dh+h^{-1}(g^{-1}dg)h,$$

together with the formulae for the adjoint representation and the Maurer-Cartan form.

Ad 4) This is the computation

$$d(g^{-1}dg) + \frac{1}{2}[g^{-1}dg, g^{-1}dg] \overset{7.4.11}{=} -g^{-1}dgg^{-1} \wedge dg + g^{-1}dg \wedge g^{-1}dg = 0.$$

Ad 5)

$$d(\mathrm{Ad}(g^{-1})\eta) = d(g^{-1}\eta g) = -g^{-1}dgg^{-1}\eta g + \mathrm{Ad}(g^{-1})d\eta + (-1)^p g^{-1}\eta dg =$$

$$= -g^*\omega_G \wedge \mathrm{Ad}(g^{-1})(\eta) + \mathrm{Ad}(g^{-1})d\eta + (-1)^p \mathrm{Ad}(g^{-1})\eta g^*\omega_G.$$

□

7.5. **Back to connections.** Recall Proposition 7.3.3, and let us reformulate it in terms of the Lie algebraic data we found. Let $E \to M$ be a real vector bundle and $\mathrm{Fr}(E) = P$ be its frame bundle. Let $G = \mathrm{GL}_n(\mathbb{R})$ and $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{R})$.

Proposition 7.5.1. *There is a bijection between connections on E and rules that assign a form $\theta_s \in \mathcal{A}^1(U, \mathfrak{g})$ to a local section s of P, such that for each change-of frame function $g : U \to G$, we have*

$$\theta_{sg} = g^*\omega_G + \mathrm{Ad}(g^{-1})\theta_s.$$

The curvature is given by

$$\Omega_s = d\theta_s + \frac{1}{2}[\theta_s, \theta_s].$$

and the change-of-frame formula for the curvature is

$$\Omega_{sg} = \mathrm{Ad}(g^{-1})\Omega_s.$$

Note that the above structure can be expressed *entirely using the frame bundle and the Lie group/Lie algebra structure.* No reference, implicit or explicit, is made to the fact that $\mathrm{GL}_n(\mathbb{R})$ is a linear Lie group. We also want to see how the connection on E (the differential operator) can be reconstructed from these data, but we do this below and form an abstract version. We will have two versions of the same thing. One is an abstraction of the properties of Proposition 7.5.1. We will formulate it, for later use, in a slightly more general setting. The other is a single 1-form on the total space P.

Assume that $P \to M$ is a G-principal bundle. Let $(U_i, s_i)_{i\in I}$ be a bundle atlas, U_i open and s_i a local section over U_i. For $i, j \in I$, we denote $U_{ij} = U_i \cap U_j$. There is a unique smooth function $g_{ij} : U_{ij} \to G$ such that $s_i g_{ij} = s_j$. These functions satisfy $g_{ij}g_{jk} = g_{ik}$, the *cocycle identity.*

If $p \in P$ is a point, the *orbit map* is denoted $j_p : G \to P$, $j_p(g) := xg$ - this identifies the fibres of P with G. Note that j_p is G-equivariant when G carries the action by right-multiplication.

Definition 7.5.2. A *connection rule* is a family $\theta_i \in \mathcal{A}^1(U_i, \mathfrak{g})$, such that for each $i, j \in I$, we have $\theta_j = g_{ij}^*\omega_G + \mathrm{Ad}(g_{ij}^{-1})\theta_i$ on U_{ij}. A *connection 1-form* is an element $\theta \in \mathcal{A}^1(P, \mathfrak{g})$ such that $R_g^*\theta = \mathrm{Ad}(g^{-1})\theta$ for all g and for all $p \in P$: $j_p^* = \omega_G$.

Theorem 7.5.3. *Sending the connection 1-form θ to the family $(\theta_i)_{i\in I}$, $\theta_i := s_i^*\theta$ defines a bijection from the set of connection 1-forms to the set of connection rules.*

Proof. We begin by figuring out what a connection 1-form on the trivial bundle looks like. Let $p : M \times G \to M$ and $q : M \times G \to G$ be the projections and $s_0 : M \to M \times G$ be the section $x \mapsto (x, 1)$. We claim that a general connection 1-form can be written - uniquely - as

$$\theta = q^*\omega_G + \mathrm{Ad}(q^{-1})p^*\eta \tag{7.5.4}$$

for a form $\eta \in \mathcal{A}^1(M, \mathfrak{g})$. It is easily verified that the above formula indeed defines a connection 1-form, and the uniqueness of η is clear, since $s_0^*\theta = \eta$. To prove the existence of the above formula, write $\eta := s_0^*\theta$, put $\theta' := q^*\omega_G + \mathrm{Ad}(q^{-1})p^*\eta$ and we have to show that $\theta = \theta'$. This is a purely local problem, and we write

$$\theta = \sum_i a_i(x, g)q^*\omega_i + \sum_j b_j(x, g)p^*dx_j$$

for some scalar valued 1-forms ω_i on G and dx_i on M and $\mathfrak{g}$-valued functions. The orbit map $\iota : G \to M \times G$, $g \mapsto (x, g)$ satisfies $\iota^*p^*dx_j = 0$ and $q \circ \iota = \mathrm{id}_G$, and so $\iota^*\theta = \omega_G$ implies already that $\sum_i a_i(x, g)\omega_i = \omega_G$ for all $x \in G$, or that

$$\theta = q^*\theta_G + \sum_j b_j(x, g)p^*\eta_j$$

The condition $R_h^*\theta = \mathrm{Ad}(h^{-1})\theta$ for $h \in G$ enforces $b_j(x, g) = \mathrm{Ad}(g^{-1})b_j(x, 1)$. But $\eta = s_0^*\theta = \sum_j b_j(x, 1)\eta_j$ and therefore θ has to be of the form 7.5.4.

To see that $\theta_i = s_i^*\theta$ is a connection rule when θ is a connection 1-form, we have to prove the transformation property. Since it refers to a subset of M over which the bundle has a cross-section, we may assume that the bundle is trivial and the connection 1-form is given by 7.5.4. A function $g : M \to G$ determines a section $s_g(x) := (x, g(x))$, and

$$s_g^*\theta = g^*\omega_G + \mathrm{Ad}(g^{-1})\eta.$$

Thus we can compare, using Proposition 7.4.15,

$$s_{gh}^*\theta = (gh)^*\omega_G + \mathrm{Ad}((gh)^{-1})\eta = h^*\omega_G + \mathrm{Ad}(h^{-1})g^*\omega_G + \mathrm{Ad}(h^{-1})\mathrm{Ad}(g^{-1})\eta = h^*\omega_G + \mathrm{Ad}(h^{-1})s_g^*\theta$$

as claimed.

The above computations already show that a connection 1-form is uniquely determined - over $U \subset M$ - by $s^*\theta$ for a section s. This proves injectivity. To show surjectivity, we need to see that any connection rule comes from a unique connection 1-form. Recall that any local section s_i defines a bundle isomorphism $U_i \times G \cong P|_{U_i}$. We define a 1-form $\rho_i \in \mathcal{A}^1(P|_{U_i}, \mathfrak{g})$ by $\rho_i = q^*\omega_G + \mathrm{Ad}(q^{-1})p^*\theta_i$ on U_i and transplant it to P_{U_i}. This is a connection form (as shown above) and $s_i^*\rho_i = \theta_i$ (since the section s_i corresponds to the unit section in these coordinates). We have to prove that $\rho_i = \rho_j$ on the intersection $P|_{U_{ij}}$; and this proves that the ρ_i glue together to a global form. But we have seen that a connection 1-form is determined by its pullback along one section, and thus it is enough to prove that $s_j^*\rho_i = \theta_j$. In these coordinates, the section s_j is given by $x \mapsto (x, g_{ij}(x))$ and thus $(g = g_{ij})$

$$s_j^*\rho_i = g^*\omega_G + \mathrm{Ad}(g^{-1})\theta_i = \theta_j$$

by the definition of a connection rule. ∎

The theorem has some useful consequences.

Corollary 7.5.5. *Let $f : M \to N$ be smooth, $P \to N$ and $Q \to M$ be G-principal bundles and $\hat{f} : Q \to P$ be a bundle map. If $\theta \in \mathcal{A}^1(P, \mathfrak{g})$ is a connection 1-form, then $\hat{f}^*\theta$ is a connection 1-form on Q.*

Corollary 7.5.6. *Let $P \to M$ be a G-principal bundle and $\phi : G \to H$ be a Lie group homomorphism with derivative φ. Form the H-principal bundle $Q = P \times_G H$ and let (s_i) be a bundle atlas for P. Then (t_i) is a bundle atlas for Q, with $t_i(x) = [s_i(x), 1]$. Let $\theta_i := s_i^*\theta$ be the connection rule. Then $\sigma_i := \varphi_*\theta_i$ is a connection rule for the H-principal bundle Q. Thus connections can be prolonged along Lie group homomorphisms.*

Corollary 7.5.7. *Each principal bundle admits a connection 1-form.*

Proof. Let λ_i be a partition of unity, subordinate to the open cover of a bundle atlas. Glue together local connections... □

7.6. **Linear connections induced by a principal connection.** Now let $\phi : G \to \mathrm{GL}(V)$ be a representation, $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be its derivative. A local frame s for P defines an isomorphism

$$\psi_s : \mathcal{A}^p(U, V) \to \mathcal{A}^p(U, P \times_G V).$$

We *define* a covariant derivative on $P \times_G V$ by setting

$$\psi_s^{-1} \nabla \psi_s f := df + \varphi(\theta_s) f. \tag{7.6.1}$$

It is clear that the Leibniz rule holds, but that this definition is independent of the frame needs to be verified. We have to prove that for each function $g : U \to G$, the diagram

$$\begin{array}{ccc} \mathcal{A}^p(U,V) & \xrightarrow{\phi(g)\cdot} & \mathcal{A}^p(U,V) \\ \big\downarrow{\scriptstyle d+\varphi(\theta_{sg})} & & \big\downarrow{\scriptstyle d+\varphi(\theta_s)} \\ \mathcal{A}^{p+1}(U,V) & \xrightarrow{\phi(g)\cdot} & \mathcal{A}^{p+1}(U,V) \end{array}$$

commutes and follow a form f in the left upper corner. If we map it to the lower left corner along the three other maps, then it becomes

$$\phi(g)^{-1} d(\phi(g) f) + \phi(g)^{-1} \varphi(\theta_s) \phi(g) f = \phi(g)^{-1} d(\phi(g)) f + df + \varphi(\mathrm{Ad}(g^{-1})\theta_s) f$$

by Lemma 7.4.9. But $\phi(g)^{-1} d(\phi(g)) = g^*\phi^*\omega_{\mathrm{GL}(V)}$ by the computation of the Maurer-Cartan form on $\mathrm{GL}(V)$. Moreover, $g^*\phi^*\omega_{\mathrm{GL}(V)} = g^*\varphi(\omega_G) = \varphi(g^*\omega_G)$ by Proposition 7.4.15. Therefore

$$\phi(g)^{-1} d(\phi(g)) f + df + \varphi(\mathrm{Ad}(g^{-1})\theta_s) f = df + \varphi(g^*\omega_G + \mathrm{Ad}(g^{-1})\theta_s) f$$

as claimed and 7.6.1 gives indeed a well-defined covariant derivative.

Example 7.6.2. Let $V \to M$ be a Riemannian vector bundle and ∇ a metric connection. Prove that if an orthogonal frame is chosen, then the 1-form θ_s takes values in $\mathfrak{o}(n)$.

If $P \to M$ is a G-principal bundle, θ a connection on P and $\phi : G \to \mathrm{GL}(V)$ a representation, we denote the connection induced on $P \times_G V$ by $\nabla^{\theta,V}$. The following Lemma is easy, but fundamental.

Lemma 7.6.3. *Let V, W be two G-representations. Then*

(1) *For $\omega \in \mathcal{A}^p(M; P \times_V)$ and $\eta \in \mathcal{A}^q(M; P \times_G W)$, we have $\nabla^{\theta,V\otimes W}(\omega \wedge \eta) = \nabla^{\theta,V}\omega \wedge \eta + (-1)^p \omega \wedge \nabla^{\theta,W}\eta$.*
(2) *If $f : V \to W$ is an equivariant map, which induces a bundle homomorphism $f : P \times_G V \to P \times_G W$, then f is parallel, i.e. $f\nabla^{\theta,V}\omega = \nabla^{\theta,W} f\omega$.*

The curvature of the induced connection $\nabla^{\theta,V}$ is called $\Omega^{\theta,V}$. It is a 2-form with values in $\mathrm{End}(P \times_G V) = P \times_G \mathrm{End}(V)$. By the defining formula for $\nabla^{\theta,V}$ 7.6.1, we have in a local frame:

$$\Omega_s^{\theta,V} = d\varphi(\theta_s) + \frac{1}{2}[\varphi(\theta_s), \varphi(\theta_s)] = \varphi(d\theta_s + \frac{1}{2}[\theta_s, \theta_s]).$$

This can be interpreted as follows: the curvature forms $\Omega_s = d\theta_s + \frac{1}{2}[\theta_s, \theta_s]$ define a 2-form Ω^θ with values in the adjoint bundle $P \times_G \mathfrak{g}$, and since φ is G-equivariant, it defines a bundle homomorphism $\varphi : P \times_G \mathfrak{g} \to P \times_G \mathrm{End}(V)$. The curvature $\Omega^{\theta,V}$ is obtained by

$$\Omega^{\theta,V} = \varphi\Omega^\theta. \tag{7.6.4}$$

We can now easily prove the following fundamental result:

Theorem 7.6.5. *(The Bianchi identity) The curvature $\Omega^{\theta,V}$ is parallel.*

Proof. By 7.6.4 and Lemma 7.6.3, we have

$$\nabla^{\theta,\mathrm{End}(V)}\Omega^{\theta,V} = \nabla^{\theta,\mathrm{End}(V)}\varphi\Omega^\theta = \varphi\nabla^{\theta,\mathfrak{g}}\Omega^\theta.$$

Therefore it is enough to prove that $\nabla^{\theta,\mathfrak{g}}\Omega^\theta = 0$, and we do this in a local frame. The curvature is given by $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$, and the connection applied to it is

$$d\Omega + [\theta, \Omega]$$

and what we have to show is thus that

$$d(d\theta + \frac{1}{2}[\theta, \theta]) + [\theta, d\theta] + \frac{1}{2}[\theta, [\theta, \theta]] = 0.$$

This identity holds for arbitrary $\mathfrak{g}$-valued 1-forms: by the graded Jacobi identity, $[\theta, [\theta, \theta]] = 0$ and

$$\frac{1}{2}d[\theta, \theta] + [\theta, d\theta] = \frac{1}{2}[d\theta, \theta] - \frac{1}{2}[\theta, d\theta] + [\theta, d\theta] = \frac{1}{2}[d\theta, \theta] + \frac{1}{2}[\theta, d\theta].$$

For degree reasons, $[\theta, d\theta] = -[d\theta, \theta]$, which concludes the proof. □

7.7. **The Chern-Weil construction.** Now we can give the general construction of characteristic classes for general G-principal bundles. Here are the ingredients:

(1) $P \to M$ is a G-principal bundle,
(2) $\theta \in \mathcal{A}^1(P, \mathfrak{g})$ a connection with curvature form $\Omega \in \mathcal{A}^2(M; P \times_G \mathfrak{g})$.
(3) $p \in (\mathfrak{g}^\vee)^{\otimes k}$ is a G-invariant tensor, viewed as an equivariant map $\mathfrak{g}^{\otimes k} \to \mathbb{R}$ (or $\mathbb{C}$). Later, we see that we can restrict to *symmetric tensors*.

We know by the Bianchi identity (Theorem 7.6.5) that $\nabla^{\theta,\mathfrak{g}}\Omega = 0$. Next one considers

$$\Omega^{\otimes k} \in \mathcal{A}^{2k}(P \times_G \mathfrak{g}^{\otimes k}).$$

This is parallel, by the Bianchi identity and by the Leibniz rule 7.6.3. As $p : \mathfrak{g}^{\otimes k} \to \mathbb{C}$ is equivariant, it induces a bundle map $P \times_G \mathfrak{g}^{\otimes k} \to P \times_G \mathbb{C} = M \times \mathbb{C}$. By Lemma 7.6.3, the $2k$-form $\mathbf{CW}(\theta, p) := p(\Omega^{\otimes k}) \in \mathcal{A}^{2k}(M)$ is parallel. But the trivial line bundle is induced by the trivial representation, and the connection induced on the trivial line bundle is the exterior derivative. Hence we conclude

$$dp(\Omega^{\otimes k}) = 0. \tag{7.7.1}$$

Let us inspect the multilinear algebra involved a bit closer. The tensor algebra $(\mathfrak{g}^*)^{\otimes *}$ has a product, namely two tensors p and q of degrees k and l are multiplied by the rule

$$p \otimes q(v_1, \ldots, v_{k+l} := p(v_1, \ldots, v_k)q(v_{k+1}, \ldots, v_{k+l}).$$

Here we used implicitly the identification of tensors with multilinear forms. In other words, the diagram

$$\begin{tikzcd} V^{\otimes k} \otimes V^{\otimes l} \arrow[r] \arrow[d, "p\otimes q"] & V^{\otimes k+l} \arrow[d, "p\otimes q"] \\ \mathbb{K} \otimes \mathbb{K} \arrow[r] & \mathbb{K} \end{tikzcd}$$

commutes, where the two vertical arrows have the same name, but different meanings.

To understand the construction a little better, let us pick a local frame of P in which the curvature is given by a form $\Omega \in \mathcal{A}^2(U, \mathfrak{g})$. It follows that $(p \otimes q)(\Omega^{\otimes(k+l)}) = p(\Omega^{\otimes k}) \wedge q(\Omega^{\otimes l})$. Before we analyze the situation closer, we summarize the basic properties of this construction.

Theorem 7.7.2. *Let $P \to M$ be a G-principal bundle, θ a connection on P and $p \in ((\mathfrak{g}^*)^{\otimes k})^G$ be an invariant symmetric tensor. Then*

1. *The form $\mathbf{CW}(\theta, p) := p(\Omega^{\otimes k})$ is closed.*
2. *If $f : N \to M$ is a smooth map and $f^*\theta$ the pullback-connection, then $f^*\mathbf{CW}(\theta, p) = \mathbf{CW}(f^*\theta, p)$.*
3. *The cohomology class of $\mathbf{CW}(\theta, p)$ is independent of θ.*
4. *The map $p \mapsto \mathbf{CW}(\theta, p)$ defines a homomorphism of algebras $(V^*)^{\otimes} \to \mathcal{A}^{ev}(M)$.*
5. *Let $\phi : G \to H$ is a Lie group homomorphism and $\varphi : \mathfrak{g} \to \mathfrak{h}$ its derivative. Let $\varphi_*\theta \in \mathcal{A}^1(P \times_G H; \mathfrak{h})$ be the prolonged connection. Let $p \in ((\mathfrak{h}^*)^{\otimes k})^G$ and $\varphi^* p$ be the pulled back tensor. Then $\mathbf{CW}(\theta, \varphi^* p) = \mathbf{CW}(\varphi_*\theta; p)$.*

Proof. We have already proven parts (1) and (4). Part (2) is trivial (why?). We won't prove part (5). Part (3) is an easy consequence of (1) and (2): Let θ_0, θ_1 be two connections. Let $\pi : M \times [0,1] \to M$ be the projection and $t : M \times [0,1] \to \mathbb{R}$ be the other projection. Then $\theta = (1-t)\pi^*\theta_0 + t\pi^*\theta_1$ is a connection and $j_i^*\theta = \theta_i$, where $j_i : M \to M \times [0,1]$ are the inclusions. By part (1), $\mathbf{CW}(\theta, p) \in \mathcal{A}^{2k}(M \times [0,1])$

is closed, and by part (2), $\mathbf{CW}(\theta_i;p) = j_i^* \mathbf{CW}(\theta,p)$. The result follows from the homotopy invariance of the de Rham cohomology. $\square$

The tensor algebra is not very practical and we will now replace it by something simpler. Let $\Omega \in \mathcal{A}^2(U,\mathfrak{g})$ be the curvature. To understand the algebra, let us assume for a second that $p = \ell_1 \otimes \ldots \otimes \ell_k$ is a tensor product of linear forms. Then $p(\Omega^{\otimes k})$ is given by

$$(7.7.3)\quad \ell_1(\Omega)\wedge\ldots\wedge\ell_k(\Omega) \overset{!}{=} \frac{1}{k!}\sum_{\sigma\in\Sigma_k} \ell_{\sigma(1)}(\Omega)\wedge\ldots\wedge\ell_{\sigma(k)}(\Omega) = S(\ell_1\otimes\ldots\otimes\ell_k)(\Omega^{\otimes k}),$$

(since Ω is a 2-form and 2 is even!) where S is the *symmetrization operator* on $(\mathfrak{g}^*)^k$ defined by

$$S(p)(v_1,\ldots,v_k) = \frac{1}{k!}\sum_{\sigma\in\Sigma_k} p(v_{\sigma(1)},\ldots,v_{\sigma(k)}).$$

Let $\mathrm{Sym}^l(\mathfrak{g}^*) \subset (\mathfrak{g}^*)^{\otimes k}$ be the subspace of symmetric tensors, i.e. the image of the idempotent S. We have seen that $\mathbf{CW}(\theta,p)$ only depends on $S(p)$ (and on θ, but that is not the point right now). Furthermore, 7.7.3 shows that we are only interested in evaluating a symmetric tensor on *equal* arguments.

Let $\mathrm{Pol}^k(V)$ be the space of homogeneous polynomial functions $p: V \to \mathbb{K}$ of degree k. There is a map

$$T : \mathrm{Sym}^k(V^*) \to \mathrm{Pol}^k(V);\ p \mapsto Tp;\ Tp(v) := a_k p(v,\ldots,v)$$

which is an isomorphism; the inverse is given by a polarization procedure. The composition TS is an algebra homomorphism:

$$(TS(p\otimes q))(v) := S(p\otimes q)(v,\ldots,v) = \frac{1}{(k+l)!}\sum_{\sigma\in\Sigma_{k+l}} (p\otimes q)(v,\ldots,v) = (p\otimes q)(v,\ldots,v)$$

and

$$(TSp)(v)(TSq)(v) = Sp(v,\ldots,v)Sq(v,\ldots,v) = p(v,\ldots,v)q(v,\ldots,v) = (p\otimes q)(v,\ldots,v).$$

The polished form of the Chern-Weil construction takes invariant polynomials as an input.

Definition 7.7.4. Let G be a Lie group and $k \in \mathbb{N}$. By $I(G)$, we denote the vector space of degree k homogeneous polynomial functions $\mathfrak{g} \to \mathbb{C}$ which are invariant under the adjoint representation.

It is not recommended to try explicit computations of $p(\Omega)$ in terms of forms, local coordinates etc. We will do all relevant computations on the Lie algebra level, the only computation on a manifold was the case of the tautological line bundle on $\mathbb{CP}^1$, and this is already done.

Nevertheless, it is conceptually helpful to have a concrete *interpretation*, even if it does not give a handy recipe for computations. As usual, Ricci calculus is the supreme language here. So let $P \to M$ be a G-principal bundle and θ a connection. Pick a local frame, so that θ is given by a form $\theta \in \mathcal{A}^1(U;\mathfrak{g})$. The curvature is given by $d\theta + [\theta,\theta]$. The Lie bracket can be computed by means of a basis. Fix a basis $X_1,\ldots,X_n$ of $\mathfrak{g}$, the *structure constants* are defined by

$$[X_i, X_j] := c_{ij}^l X_k$$

(Einstein convention). Write $\theta = \theta^i X_i$; then

$$[\theta, \theta] := c_{ij}^l \theta^i \wedge \theta^j X_l$$

or

$$[\theta, \theta]^l := c_{ij}^l \theta^i \wedge \theta^j$$

(yes, if you follow the rules for the Ricci calculus, it thinks for you). The curvature tensor is written as $\Omega = \Omega^i X_i$ with

$$\Omega^i = d\theta^i + c_{jl}^i \theta^j \wedge \theta^l.$$

The tensor p is given, in terms of the dual base X^i of $\mathfrak{g}$, by

$$p = p_{i_1,\ldots,i_k} X^{i_1} \otimes \ldots \otimes X^{i_k},$$

the symmetry condition is expressed by the invariance of the numbers $p_{i_1,\ldots,i_k}$ under permutations of the indices. The G-invariance takes care of itself (!). The final formula is that the $2k$-form is given by

$$p(\Omega) = \frac{1}{k!} p_{i_1,\ldots,i_k} \Omega^{i_1} \wedge \Omega^{i_k}.$$

If you wish to know how to insert vector fields into this form, it becomes more complicated; likewise, picking coordinates on M given a new layer of indices.

7.8. **Chern classes and the proof of the Riemann-Roch theorem.** For us, the most important Lie groups are $\mathrm{GL}_n(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $U(n)$, $O(n)$ and $SO(n)$. We will eventually determine the algebras of invariant polynomials in all of these cases, but we first consider complex bundles and prove the Riemann-Roch theorem. In the next section, we give a formula for the Euler class of even-dimensional real oriented vector bundles and prove the Gauß-Bonnet-Chern theorem. A more detailed study of the characteristic classes of real vector bundles will be undertaken in the next semester.

Recall that $I(G)$ is the graded algebra of G-invariant polynomials on the Lie algebra $\mathfrak{g}$. A homomorphism $G \to H$ induces a map $I(H) \to I(G)$, which is compatible with the Chern-Weil construction. We will only consider *complex-valued* polynomials. Let us begin with $G = \mathrm{GL}_n(\mathbb{C})$.

Definition 7.8.1. Let $c_k \in I(\mathrm{GL}_n(\mathbb{C}))$ be defined by the formula

$$c_k(A) := (\frac{-1}{2\pi i})^k \operatorname{Tr}(\Lambda^k A).$$

Here $\Lambda^k A$ is the endomorphism of $\Lambda^k \mathbb{C}^n$ induced by A.

Clearly, the polynomial c_k has degree k. Up to factors of $2\pi i$, c_k is the k-th elementary symmmetric polynomial in the eigenvalues of A. If $V \to M$ is a vector bundle of rank n and θ a connection, we call $\mathbf{CW}(\theta, c_k) =: c_k(V)$ the kth *Chern class* of V. It is easy to see that $c_0 = 1$ and that for $n = 1$, c_1 agrees with the previously defined first Chern class.

Theorem 7.8.2. *The homomorphism* $\mathbb{C}[c_1,\ldots,c_k] \to I(\mathrm{GL}_n(\mathbb{C}))$ *is an isomorphism.*

Proof. An element $p \in I(\mathrm{GL}_n(\mathbb{C}))$ is determined by its values on the diagonal matrices: the set of diagonalizable matrices is Zariski dense in the vector space $\mathfrak{gl}_n(\mathbb{C})$, and the adjoint action is given by conjugation, and so by invariance the claim follows.

So let $\mathfrak{d}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ be the subspace of diagonal matrices. Any permutation of the entries can be realized by a conjugation (embed Σ_n as the permutation matrices). So the restriction map $I(\mathrm{GL}_n(\mathbb{C})) \to \mathbb{C}[x_1,\ldots,x_n]^{\Sigma_n}$ is injective. The theorem follows by applying the main theorem on symmetric functions. □

For a vector bundle $V \to M$, we put $c_i(V) = 0$ for $i > \mathrm{rank}(V)$ and $c(V) := \sum_{k\geq 0} c_k(V) \in H^*(M)$.

Theorem 7.8.3. *The Chern classes have the following properties:*

(1) *Naturality.*
(2) $c(V\oplus W) = c(V)c(W)$*. More precisely* $c_k(V\oplus W) = \sum_{p+q=k} c_p(V)c_q(W)$*.*
(3) *The tautological line bundle* $L \to \mathbb{CP}^1$ *has Chern class* $c(L) = 1 - x$*, where* $x \in H^2(\mathbb{CP}^1)$ *is the unique element with* $\int_{\mathbb{CP}^1} x = 1$*.*
(4) *Let* $L_i \to M$*,* $i = 0,1$*, be two line bundles. Then* $c_1(L_0 \otimes L_1) = c_1(L_0) + c_1(L_1)$*.*

Proof. Naturality is part of Theorem 7.7.2 and we did the computation for part (3) in example 7.2.3. For part (2), we consider the homomorphism $\phi : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{m+n}(\mathbb{C})$ and let φ be its derivative. Let $\Pi_n : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be the projection and π_n be its derivative. Π_m and π_m are defined in a similar fashion.

We use the functorial isomorphism $\Lambda^k(V\oplus W) \cong \bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W$ and the relation $\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$ and compute

$$\varphi^* c_k(A,B) = c_k(A\oplus B) = (\frac{-1}{2\pi i})^k \sum_{p+q=k} \mathrm{Tr}(\Lambda^p(A)\otimes\Lambda^q(B)) =$$

$$= \sum_{p+q=k} c_p(A)c_q(B) = \sum_{p+q=k} \pi_n^* c_p(A,B)\pi_m^* c_q(A,B).$$

In short

$$\varphi^* c_k = \sum_{p+q=k} \pi_n^* c_p \pi_m^* c_q.$$

Now let $V = P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n$ and $W = Q \times_{\mathrm{GL}_m(\mathbb{C})} \mathbb{C}^m$ and let $R := \Delta^*(P\times Q)$ be the product $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$-principal bundle. Let θ_n be a connection on P and θ_m a connection on Q. Both connections together define a connection θ on R such that

$$(\Pi_n)_*\theta = \theta_n;\ (\Pi_m)_*\theta = \theta_m.$$

The connection $\phi_*\theta$ is a connection on the frame bundle of $V\oplus W$. Therefore

$$c_k(V\oplus W) = \mathbf{CW}(\phi_*\theta, c_k) \overset{7.7.2(5)}{=} \mathbf{CW}(\theta, \varphi^* c_k) = \sum_{p+q=k} \mathbf{CW}(\theta, \pi_n^* c_p \pi_m^* c_q) \overset{7.7.2(4)}{=}$$

$$= \sum_{p+q=k} \mathbf{CW}(\theta, \pi_n^* c_p)\mathbf{CW}(\theta, \pi_m^* c_q) = \sum_{p+q=k} \mathbf{CW}((\pi_n)_*\theta, c_p)\mathbf{CW}((\pi_m)_*\theta, c_q) =$$

$$= \sum_{p+q=k} \mathbf{CW}(\theta_n, c_p)\mathbf{CW}(\theta_m, c_q) = \sum_{p+q=k} c_p(V) c_q(W).$$

For part (4), we proceed by a similar philosophy. Consider $\phi : \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$, $(A, B) \mapsto AB$, and the derivative $\varphi(A, B) = A + B$, which gives the result. □

Now we turn to the Riemann-Roch theorem.

Theorem 7.8.4. *Let M be a connected Riemann surface of genus g and $V \to M$ be a holomorphic vector bundle. Then*

$$\operatorname{ind}(\bar\partial_V) = \operatorname{rank}(V)(1-g) + \int_M c_1(V).$$

Recall that we have already done quite a bit of work: Proposition 4.6.10 states that there is a unique homomorphism $I : K^0(M) \to \mathbb{Z}$ such that for each holomorphic vector bundle $V \to M$, the identity $I(V) = \operatorname{ind}(\bar\partial_V)$ holds. Moreover, we have shown by Hodge theory (Theorem 4.6.4) that

$$I(\underline{\mathbb{C}}) = 1 - g;\ I(\Lambda^{1,0}T^*M) = g - 1.$$

So by inspection Theorem 7.8.4 is true for the bundle $\underline{\mathbb{C}}$. The bundle $\Lambda^{1,0}T^*M$ is the dual of TM, whence

$$c_1(\Lambda^{1,0}T^*M) = -c_1(TM) = -e(TM)$$

by Theorem 7.8.3 (4), Theorem 7.2.4. By the topological Gauß-Bonnet theorem 6.4.1, we obtain

$$\int_M c_1(\Lambda^{1,0}T^*M) = -\chi(M) = 2g - 2$$

and therefore

$$\operatorname{rank}(\Lambda^{1,0}T^*M)(1-g) + \int_M c_1(\Lambda^{1,0}T^*M) = g - 1 = I(\Lambda^{1,0}T^*M).$$

Thus the Riemann-Roch theorem holds for the two vector bundles $\underline{\mathbb{C}}$ and $\Lambda^{1,0}T^*M$.

The right-hand side of the Riemann-Roch formula can also be interpreted in terms of K^0: if V and W are two vector bundles on M, we get

$$c_1(V \oplus W) = c_1(V) + c_1(W)$$

by the product formula for Chern classes; therefore $V \mapsto (\operatorname{rank}(V), \int_M c_1(V))$ defines a homomorphism

$$J : K^0(M) \to \mathbb{Z}^2;\ V \mapsto (\operatorname{rank}(V), \int_M c_1(V)).$$

We have not yet proven that $\int_M c_1(V)$ is an integer, but that is part of the next theorem. We will now prove:

Theorem 7.8.5. *The homomorphism J takes values in $\mathbb{Z}^2$ and is an isomorphism, for any connected Riemann surface.*

Proof of Riemann-Roch, assuming Theorem 7.8.5. Since $K^0(M)$ has rank 2, a homomorphism $K^0(M) \to \mathbb{Z}$ is uniquely determined by its values on these elements. We need to distinguish the two cases $g \neq 1$ and $g = 1$. If $g \neq 1$, the elements $\underline{\mathbb{C}}$ and $\Lambda^{1,0}T^*M$ are linearly independent (over $\mathbb{Q}$) in $K^0(M)$. If $g = 1$, both bundles have the same image in K^0, and so this is not enough (in fact both bundles are isomorphic), and we need a bundle with nonzero Chern number for which the Riemann-Roch formula is true. The bundle $L_{(x)}$ discussed in Theorem 4.7.4 has Chern number 1, by the Poincaré-Hopf theorem. The index was computed in Theorem 4.7.4 and is 1, as desired. □

Now we delve into the proof of Theorem 7.8.5

Proposition 7.8.6. *Let $V \to M$ be a vector bundle of rank $r > 1$. Then there exists a line bundle $L \to M$ and an isomorphism $L \oplus \mathbb{C}^{r-1} \cong V$.*

Proof. Take a section $s : M \to V$ which is transverse to the zero section. If $\operatorname{rank}(V) > 1$, then s does not have a zero, and so V splits as $V' \oplus \mathbb{C} \cong V$. The result follows by induction. □

Corollary 7.8.7. *For each vector bundle $V \to M$, the number $\int_M c_1(V)$ is an integer.*

Proof. The above Proposition and the sum formula reduce the statement to line bundles. But for line bundles, the first Chern class is the Euler class, which is integral by the Poincaré-Hopf theorem. □

Proof of Theorem 7.8.5. Since $J(\underline{\mathbb{C}}) = (1,0)$, it is, for the surjectivity, enough to produce a bundle L with $J(L) = (1,1)$. Take a map $f : M \to S^2 = \mathbb{CP}^1$ of degree 1. The bundle f^*H has Chern number +1.

The injectivity is more difficult (and more important for us). A general element $\xi \in K^0(M)$ can be written as $\sum_i a_i[V_i]$, where V_i are vector bundles and $a_i \in \mathbb{Z}$. Using the relation $[V] + [W] = [V \oplus W]$ in $K^0(M)$, we can write $\xi = [V] - [W]$. So the injectivity of J follows from the next claim:

(1) Let $V, W \to M$ be vector bundles with $\operatorname{rank}(V) = \operatorname{rank}(W)$ and $c_1(V) = c_1(W)$. Then $V \cong W$.

Because of Proposition 7.8.6 and the sum formula, it is enough to assume that V and W are line bundles. Moreover, as $0 = c_1(V) - c_1(W) = c_1(W^* \otimes V) = c_1(\operatorname{Hom}(W,V))$, it is enough to prove that a line bundle on M with trivial Chern class is zero. Let L be such a line bundle. By the classification theorem, there exists a smooth map $f : M \to \mathbb{CP}^n = \operatorname{Gr}_1(\mathbb{C}^{n+1})$ with $f^*H = L$. Next, we show that we can assume $n = 1$. If $n > 1$, pick a regular value of f. For dimension reasons, this must be a point $\ell \in \mathbb{CP}^n \smallsetminus f(M)$. Without loss of generality, we can assume that $\ell = \ell_0 = 0 \times \mathbb{C} \subset \mathbb{C}^{n+1}$. This is because the connected group $U(n+1)$ acts transitively on $\mathbb{CP}^n$. But $\mathbb{CP}^n \smallsetminus \ell$ is diffeomorphic to H_{n-1}, the total space of the dual tautological line bundle on $\mathbb{CP}^{n-1}$. A diffeomorphism is given by $(\ell, h) \mapsto \Gamma_h$: a linear form on some $\ell \subset \mathbb{C}^n$ is sent to the graph $\Gamma_h \subset \ell \times \mathbb{C} \subset \mathbb{C}^{n+1}$. Vice versa, each line in $\mathbb{C}^{n+1}$, with the exception of ℓ_0, is the graph of some linear form.

But H_n deformation restracts onto its zero section, namely $\mathbb{CP}^{n-1}$. So we have proven that $f : M \to \mathbb{CP}^n$ is homotopic to a map $M \to \mathbb{CP}^{n-1}$, if $n > 1$; so altogether, we may assume that there is a map

$$f : M \to \mathbb{CP}^1;\ f^* H \cong L.$$

But

$$\int_M c_1(L) = \int_M f^* c_1(H) = \int_M f^* x$$

with $\int_{\mathbb{CP}^1} x = 1$ and this is the mapping degree $\deg(f)$. So we have to show that a map $f : M \to S^2$ with degree 0 is nullhomotopic. This is a general fact: if M^n is a closed oriented connected manifold, then $\deg : [M; S^n] \to \mathbb{Z}$ is a bijection. This is a classical theorem by Hopf, the proof can be found in [19], p. 50f. The idea is very similar to the proof that $\pi_n(S^n) \cong \mathbb{Z}$. □

7.9. **Proof of the Gauß-Bonnet-Chern theorem.** We now arrive at the capstone of the first part of this course: the *Gauss-Bonnet-Chern theorem*. Let M^{2n} be a closed oriented Riemann manifold. Recall that there is the operator $D = d + d^* : \mathcal{A}^{ev}(M) \to \mathcal{A}^{odd}(M)$. We computed its index, and the result was, by Hodge theory, the Euler characteristic of M

$$\operatorname{ind}(D) = \chi(M).$$

We found two different descriptions of the Euler number. If X is a tangential vector field on M which is transverse to the zero section, then the Poincaré-Hopf theorem states that

$$\chi(M) = \sum_{X(x)=0} I_x X,$$

the sum of local indices of the vector field X. On the other hand, there is the Euler class $e_{top}(TM) := e(TM) \in H^{2n}(M)$, and the topological Gauss-Bonnet theorem is the formula

$$\chi(M) = \int_M e_{top}(TM).$$

The Gauss-Bonnet-Chern theorem states that there is a construction of the Euler class in terms of the Chern-Weil construction. Above, we wrote e_{top} for the Euler class that was constructed using the Thom class, which came from Poincaré duality. We will now construct the class e_{geo}, the *geometric Euler class*, using the Chern-Weil theorem. After the construction is done, we will prove that $e_{geo} = e_{top}$.

We have proved that the cohomology class of the Chern-Weil forms do not depend on the chosen connection. In the case of the tangent bundle of a Riemann manifold, there is a special connection, the Levi-Civita connection, and one can express the Euler form in terms of the curvature of the metric, i.e. by a geometric quantity. In the case of $\dim(M) = 2$, we obtain the classical Gauß-Bonnet theorem.

Now we embark on the construction. Let $V \to M$ be an oriented Riemann vector bundle of rank $2n$, equipped with a metric connection. Let $\mathrm{Fr}^O(V) \to M$ be the oriented orthonormal frame bundle, an $SO(2n)$-principal bundle. We wish to find an invariant polynomial P_n on the Lie algebra $\mathfrak{so}(2n)$ so that $P(\Omega)$ is the Euler class. The Lie algebra $\mathfrak{so}(2n)$ is the space of all skew-symmetric $n \times n$-matrices.

We have several constraints on P_n.

(1) P_n needs to have degree n.

(2) If $A \in \mathfrak{so}(2n)$ and $B \in \mathfrak{so}(2m)$, then $P_{n+m}(A \oplus B) = P_n(A)P_n(B)$. This expresses the multiplicative property of the Euler class.

(3) A 2-dimensional oriented Riemann vector bundle is the same as a hermitian line bundle; this is the bundle theoretic expression of the isomorphism $SO(2) \cong U(1)$. There are two such isomorphisms, let us fix one, namely

$$\phi : \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto a + ib$$

with derivative $\varphi : \mathfrak{so}(2) \to \mathfrak{u}(1)$. Since the Euler class of a complex line bundle is the same as the first Chern class, the polynomial P_1 should be the polynomial defining the first Chern class.

We will now prove that there are unique invariant polynomials Pf_n on $\mathfrak{so}(2n)$ satisfying these properties. We first describe Pf_1. Both Lie algebras are 1-dimensional, and the isomorphism φ is given by

$$\phi : \begin{pmatrix} & -a \\ a & \end{pmatrix} \mapsto ia.$$

The first Chern form is given by the linear form $ia \mapsto \frac{-1}{2\pi i} ia = \frac{-1}{2\pi} a$, and so

$$\mathrm{Pf}_1(\begin{pmatrix} & -a \\ a & \end{pmatrix}) = \frac{-1}{2\pi} a$$

is the right definition. Now write $R_a := \begin{pmatrix} & -a \\ a & \end{pmatrix}$, and for $a_1, \ldots, a_n \in \mathbb{R}$ let

$$A(a_1, \ldots, a_n) := \begin{pmatrix} R_{a_1} & & & \\ & R_{a_2} & & \\ & & \ldots & \\ & & & R_{a_n} \end{pmatrix}.$$

Multiplicativity says that we need to have

$$\mathrm{Pf}(A(a_1, \ldots, a_n)) = \frac{(-1)^n}{2^n \pi^n} \prod_{i=1}^n a_i. \tag{7.9.1}$$

Note, by the way, the identity

$$\mathrm{Pf}(A)^2 = \frac{1}{(2\pi)^{2n}} \det(A).$$

Now recall from Linear Algebra II that each skew-symmetric matrix is conjugate to one of the same form as A. Therefore, an invariant polynomial is uniquely determined by the three properties. It remains to construct the polynomial Pf.

Lemma 7.9.2. *Let V be a euclidean m-dimensional vector space. Let*

$$\Phi : \Lambda^2 V^* \to \mathfrak{so}(V);\ v_1 \wedge v_2 \mapsto \langle v_1, \_\rangle v_2 - \langle v_2, \_\rangle v_1.$$

and

$$\Psi : \mathfrak{so}(V) \to \Lambda^2 V^*;\ A \mapsto ((v_1, v_2) \mapsto \langle A v_1, v_2 \rangle).$$

Then Φ and Ψ are mutually inverse equivariant isomorphisms.

Proof. Equivariance is clear; and it is easy to calculate that $\Psi\Phi = \mathrm{id}$. Both spaces have the same dimension, namely $\frac{1}{2}m(m-1)$, which completes the proof. □

One calculates that

$$\Phi(\sum_{i=1}^{n} a_i e^{2i-1} \wedge e^{2i}) = A(a_1, \ldots, a_n).$$

Now we define

$$\mathrm{Pf}_n(A)\mathrm{vol} := \frac{(-1)^n}{n!(2\pi)^n} \Phi^{-1}(A)^{\wedge n} \tag{7.9.3}$$

It is clear that Pf_n is an $SO(2n)$-invariant polynomial on $\mathfrak{so}(2n)$ of degree n. Note that Pf_n is not invariant under the group $O(2n)$ with the same Lie algebra; this is because in the definition, we used the orientation, more specifically the volume form. The identity 7.9.1 follows from

$$\Phi^{-1}(A(a_1, \ldots, a_n))^{\wedge n} = (\sum_{i=1}^{n} a_i e^{2i-1} \wedge e^{2i})^{\wedge n} = a_1 \cdots a_n n! \mathrm{vol},$$

which implies the normalization and multiplicativity.

Definition 7.9.4. Let $V \to M$ be an oriented $2n$-dimensional Riemann vector bundle, equipped with a metric connection ∇. The *geometric Euler class* is represented by the closed $2n$-form $\mathbf{CW}(\nabla, \mathrm{Pf}_n) \in \mathcal{A}^{2n}(M)$.

Theorem 7.9.5. *Let $V \to M$ be an oriented vector bundle of rank $2n$. Then $e_{geo}(V) = e_{top}$.*

We already proved Theorem 7.9.5 in the case $n = 1$, see Theorem 7.2.4. The proof of Theorem 7.9.5 will be by a localization procedure. By the classification of oriented vector bundles and because the oriented Grassmann manifold is compact, it is enough to prove Theorem 7.9.5 when the base manifold M is compact.

The localization will be by means of a section. Assume that s is a section of V, and that $Z := s^{-1}(0)$ is compact. Let $U \supset Z$ be a relatively compact neighborhood of Z. We say that a metric connection ∇ on V is *adapted* if ∇ preserves the orthogonal decomposition

$$V|_{M-Z} = \mathrm{span}\{s\} \oplus \mathrm{span}\{s\}^{\perp}$$

on some open neighborhood of $M \setminus U$. Adapted connections exist: pick a connection on each of the two bundles $\mathrm{span}\{s\}$ and $\mathrm{span}\{s\}^{\perp}$ and take the direct sum. Then pick any connection on $V|_U$ and glue the connections together by means of a partition of unity.

Lemma 7.9.6. *Let ∇ be an adapted connection. Then the form $\mathbf{CW}(\nabla, \mathrm{Pf}_n)$ has support in U.*

Proof. The condition on the section s and the connection means that (outside U) the bundle V has a reduction of the structure group to $SO(2n-1)$ and the connection is induced from an $SO(2n-1)$-connection. Thus it will be enough to show that the polynomial Pf_n vanishes when restricted to $\mathfrak{so}(2n-1)$. Recall that $\mathrm{Pf}_n(A)^2 = c\det(A)$ for a nonzero constant. But if $A \in \mathfrak{so}(2n-1) \subset \mathfrak{so}(2n)$, then $\det(A) = 0$, as desired. □

Definition 7.9.7. The *relative Euler class* $e_{geo}(V,s)$ is the cohomology class (in $H_c^{2n}(M)$) of the form $\mathbf{CW}(\nabla, \mathrm{Pf}_n)$ of an adapted connection.

The relative Euler class is defined for each section s whose zero set is compact. It is clear that under $H_c^{2n}(M) \to H^{2n}(M)$, the relative Euler class maps to the (absoute) geometric Euler class. The key for the proof of Theorem 7.9.5 is

Proposition 7.9.8. *Let $\pi : V \to M$ be an oriented $2n$-dimensional vector bundle and let $s(v) := (v,v)$ be the tautological section of $\pi^* V \to V$. Then $e_{geo}(\pi^* V, s) \in \mathcal{A}_c^{2n}(V)$ is a Thom class.*

Proof of Theorem 7.9.5, assuming Proposition 7.9.8. Let $t : M \to V$ be the zero section. Then

$$e_{top}(V) = t^* \tau_V = t^* e_{geo}(\pi^* V, s) = t^* e_{geo} \pi^* V = (\pi \circ t)^* e_{geo} V = e_{geo}(V).$$

□

Proof. By the characterization of the Thom class, it is enough to prove that $\int_{V_x} e(\pi^* V, s) = 1$. But by the naturality of the Euler class, this means that it is enough to show the relative Euler class of the trivial vector bundle $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, relative to the identity section, has integral 1.

For this computation, we view $\mathbb{R}^{2n}$ as $\mathbb{C}^n$.

The proof is completed by embedding this trivial vector bundle into a bundle over a closed manifold, whose Euler class we can compute geometrically. Consider the dual $H \to \mathbb{CP}^n$ of the tautological line bundle. For $i = 1, \ldots, n$, we get a section s_i, by taking the linear form e^i on $\mathbb{C}^{n+1}$ defining $s_i(\ell) := e^i|_\ell$.

Consider $h : \mathbb{C}^n \to \mathbb{CP}^n$, $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n]$. A bundle chart k for H over $h(\mathbb{C}^n)$ is given by

$$\ell^* \ni \alpha \mapsto \alpha(1, z_1, \ldots, z_n).$$

Now compute

$$k(s_i(h(z_1, \ldots, z_n))) = k s_i([1 : z_1 : \ldots : z_n]) = k(e^i|_{[1:z_1:\ldots:z_n]}) = e^i(1, z_1, \ldots, z_n) = z_i.$$

These computations prove the following. Consider the section $s = (s_1, \ldots, s_n)$ of $H^n \to \mathbb{CP}^n$. Then in the bundle chart $k \oplus \ldots k$, this section becomes the identity section of $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$. The whole point is now that the identity section extends to a continuous section over a closed manifold. We are now ready for the final argument. The image of $e_{geo}(\underline{\mathbb{C}^n}, \mathrm{id})$ under the map $h_! : H_c^{2n}(\mathbb{C}^n) \to H^{2n}(\mathbb{CP}^n)$ is the same as $e_{geo}(H^n)$. Therefore

$$\int_{\mathbb{C}^n} e_{geo}(\underline{\mathbb{C}^n}, \mathrm{id}) = \int_{\mathbb{CP}^n} e_{geo}(H^n) = \int_{\mathbb{CP}^n} e_{geo}(H)^n = \int_{\mathbb{CP}^n} c_1(H)^n = 1,$$

by the multiplicativity of the Euler class, Theorem 7.2.4 and Theorem 6.3.12. □

7.10. **Remarks.** There are many sources for the Chern-Weil construction. Basically, there are two approaches: one possibility is to stay in the realm of vector bundles. You find expositions in [20], [14] and in many other places. The other approach is to use only principal bundles, see [16] and the superb monograph [7]. One disadvantage of [7] is that he does not make a close connection to vector bundles. The above exposition mixes both approaches, which in my opinion is clearer.

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