# Exercises for Index theory I

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The purpose of this exercise sheet is to prove the Atiyah-Singer index theorem for the manifold $S^1$ by bare hands: each elliptic differential operator on $S^1$ of order 1 has index zero.

Via the usual map $\mathbb{R}/\mathbb{Z} \to S^1$, $t \mapsto e^{2\pi i t}$, we can identify (vector-valued) functions on $S^1$ with 1-periodic functions $C^\infty(\mathbb{R};\mathbb{C}^n)_1$. Now let $A : \mathbb{R} \to \mathrm{Mat}_{n,n}(\mathbb{C})$ be a smooth, 1-periodic, matrix valued function. We consider the linear differential operator

$$D : C^\infty(\mathbb{R};\mathbb{C}^n)_1 \to C^\infty(\mathbb{R};\mathbb{C}^n)_1;\ f \mapsto f' + Af. \tag{1}$$

This is in fact an elliptic differential operator on $S^1$. Recall from Analysis II the solution theory of linear ODEs of order 1, forgetting for the moment that $A$ is assumed to be periodic. There exists a (unique) function $W : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ such that $W(0) = 1$ and $W' = -AW$. If $v \in \mathbb{C}^n$, then $f(t) = W(t)v$ is the unique solution to the ODE $Df = 0$ with initial value $f(0) = v$, which is why we call $W$ the *fundamental solution.* We also need to talk about *inhomogeneous* solutions, namely solution $f$ of the ODE

$$Df = u. \tag{2}$$

Let us try to solve the equation 2, first with the intial value $f(0) = 0$. To find the solution, we make the ansatz $f(t) = W(t)c(t)$ for a yet to be determined function $c : \mathbb{R} \to \mathbb{C}^n$ (with $c(0) = 0$). Applying the equation 2, we find that

$$c' = W^{-1}u \text{ or } c(t) = \int_0^t W(s)^{-1}u(s)ds.$$

The general solution to the initial value problem $Df = u$, $f(0) = v$ is then given by

$$f(t) = W(t)v + W(t)\int_0^t W(s)^{-1}u(s)ds. \tag{3}$$

We have proven so far that $D : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is surjective and has $n$-dimensional kernel. But we want to talk about *periodic solutions.*

**Exercise 1.** *Assume that $A$ is 1-periodic and let $W(t)$ be the fundamental solution. Prove that*

$$W(t+1) = W(t)W(1)$$

*and that the linear map* $v \mapsto W(t)v$, $\mathbb{C}^n \to C^\infty(\mathbb{R};\mathbb{C}^n)$ *induces an isomorphism from the eigenspace* $\ker(W(1)-1)$ *to the kernel of the operator 1.*

Now turn to the determination of the cokernel of the operator 1.

**Exercise 2.** *Let* $u$ *be a periodic function. Prove that there exists a* periodic *solution to* $Df = u$ *if and only if the vector*

$$\int_0^1 W(s)^{-1}u(s)ds \in \mathrm{Im}(W(1)-1).$$

*Derive that* $D : C^\infty(\mathbb{R};\mathbb{C}^n)_1 \to C^\infty(\mathbb{R};\mathbb{C}^n)_1$ *has index zero. Hint: use the solution formula 3.*

We go one step further. The vector space $C^\infty(\mathbb{R},\mathbb{C}^n)_1$ has an inner product $\langle f; g\rangle := \int_0^1 (f(t); g(t))\,dt$, using the integral and the inner product on $\mathbb{C}^n$. Now we consider the *adjoint operator* to $D$:

$$D^* f(t) := -f'(t) + A(t)^* f(t).$$

Let $V : \mathbb{R} \to \mathrm{Mat}_{n,n}(\mathbb{C})$ be the fundamental solution for $D^*$, i.e. $V(0) = 1$ and $V' = A^*V$.

**Exercise 3.** *Prove:*

*a)* $D^*$ *is indeed the adjoint of* $D$ *in the sense that* $\langle D^*f; g\rangle = \langle f; Dg\rangle$ *holds for all functions* $f, g$ *(partial integration).*

*b)* $V^*W = 1$ *(differentiate!).*

*c)* $\mathrm{Im}(W(1)-1) = (\ker(V(1)-1))^\perp$.

*d) Conclude that* $u \in \mathrm{Im}(D)$ *if and only for all* $w \in \ker(V(1)-1)$*, the equation* $\int_0^1 (V(s)w, u(s))\,ds = 0$ *holds.*

*e) Prove that there is an orthogonal sum decomposition* $C^\infty(\mathbb{R};\mathbb{C}^n)_1 = \mathrm{Im}(D)\oplus\ker(D^*)$.

In two cases, there are explicit formulae for the solution operator. If $n = 1$, then $W(t) = \exp(-\int_0^t A(s)ds)$. The other easy case is when $A(s) \equiv A$ is constant, in which case the fundamental solution is $\exp(At)$.

**Exercise 4.** *Assume that* $n = 1$*. Prove that* $\dim\ker(D) = 1$ *if and only if* $\int_0^1 a(s)ds \in 2\pi i$ *(in the other case, the kernel is trivial). Assume that* $n \geq 1$ *and* $A$ *is constant. Show that* $\dim(\ker(D)) = \sum_{k\in\mathbb{Z}} \mathrm{Eig}(A, 2\pi k)$.