Exercises for Index theory I

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Exercise 1. *Let V, W be separable Hilbert spaces (i.e, there exist* countable *orthonormal bases). Prove that the space of operators of finite rank is dense in the space of compact operators. Hint: pick an orthonormal basis and let P_n be the projection onto the span of the first n basis vectors. For each $F \in \mathrm{Kom}(V, W)$, $P_n F$ has finite rank, and $P_n F \to F$ pointwise. Prove that the convergence is uniform (Arzela-Ascoli theorem!)*

Exercise 2. *Let $T : V \to W$ be a Fredholm operator between Hilbert spaces. Show:*

- *If $\mathrm{ind}(T) \geq 0$, then for each $\epsilon > 0$, there exists a surjective Fredholm operator S with $\|S - T\| < \epsilon$.*
- *If $\mathrm{ind}(T) \leq 0$, then for each $\epsilon > 0$, there exists an injective Fredholm operator S with $\|S - T\| < \epsilon$.*
- *If $\mathrm{ind}(T) = 0$, then for each $\epsilon > 0$, there exists a bijective operator S with $\|S - T\| < \epsilon$.*

Exercise 3. *We can extend the Toeplitz operators to matrix-valued functions. Let $H^n \subset L^2(S^1; \mathbb{C}^n)$ be the subspace spanned by the function $f(z) = z^k v$, for $k \geq 0$ and $v \in \mathbb{C}^n$. Let $P_n : L^2(S^1; \mathbb{C}^n) \to H^n$ be the orthogonal projection. For a continuous function $f : S^1 \to \mathbb{C}(n) := \mathrm{Mat}_{n,n}(\mathbb{C})$, we define $T_f := P_n M_f$ in the same way as for scalar-valued functions. Prove that*

- *If $f : S^1 \to \mathrm{GL}_n(\mathbb{C})$, then T_f is Fredholm. Hint: it is useful if you write T_f as an $n \times n$-matrix of operators in some way.*
- *$f \mapsto \mathrm{ind}(T_f)$ is a well-defined map $J_n : [S^1; \mathrm{GL}_n(\mathbb{C})] \to \mathbb{Z}$ and a group homomorphism (the multiplication in $[S^1; \mathrm{GL}_n(\mathbb{C})]$ is induced by multiplication in $\mathrm{GL}_n(\mathbb{C})$).*
- *Let $m \geq n$ and let $s_{n,m} : GL_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ denote the inclusion $A \mapsto A \oplus 1_{m-n}$. Make precise and prove: $J_m \circ s_{n,m} = J_n$.*