# Exercises for Index theory I

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**Exercise 1.** Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function with $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. Write $\varphi_t(x) := \frac{1}{t^n}\varphi(\frac{x}{t})$, for $t > 0$. Determine all $s \in \mathbb{R}$ such that the limit $\lim_{t\to 0} \varphi_t$ exists as an element of the Sobolev space $W^s$ (Hint: you can recycle an estimate from the proof of the Sobolev embedding theorem). If the limit $\delta \in W^s$ exists, describe the functional on $W^{-s}$ determined by $\delta$. Of course, the name $\delta$ is not an accident.

**Exercise 2.** Let $\varphi$ be as in the previous exercise, with the additional assumption that $\varphi(-x) = \varphi(x)$. For $t > 0$, we define the *Friedrich mollifier* by $F_t(u) := \varphi_t * u$. Prove that:

a) If $u \in L^2(\mathbb{R}^n)$, then $F_t u \in \mathcal{S}(\mathbb{R}^n)$.

b) $F_t$ extends to a bounded operator $W^s \to W^s$, with $\|F_t\| \leq 1$.

c) $F_t$ commutes with any differential operator with constant coefficients.

d) $F_t : W^0 \to W^0$ is self-adjoint.

e) The space $C_0^\infty(\mathbb{R}^n)$ lies dense in $\mathcal{S}(\mathbb{R}^n)$, with respect to each $W^s$-norm. Hint: it is enough to consider $s \in \mathbb{N}$!

f) For each $u \in W^s$, we have $F_t u \to u$ in the $W^s$-norm. Hint: use the density of $C_0^\infty$ in $W^s$.

**Exercise 3.** Let $A \in \mathrm{GL}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Let $L_{A,b} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax + b$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Give a formula for $\widehat{f \circ L_{A,b}}$.