Exercises for Index theory I

Sheet 6 J. Ebert / W. Gollinger

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The purpose of this exercise sheet is to prove the spectral decomposition of a formally self-adjoint elliptic operator on a closed manifold. Here is the statement.

Theorem A. *Let M be a closed Riemann manifold, $E \to M$ a hermitian vector bundle and $D : \Gamma(M;E) \to \Gamma(M;E)$ a formally selfadjoint elliptic operator, of order $k > 0$. Let $V_\lambda := \ker(D - \lambda) \subset \Gamma(M;E)$. Then:*

a) For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the eigenspace $V_\lambda = 0$.

b) If $\lambda \neq \mu$, then $V_\lambda \perp V_\mu$.

c) For each $\Lambda \geq 0$, the sum of eigenspaces $U_\Lambda := \bigoplus_{|\lambda| \leq \Lambda} V_\lambda$ is finite-dimensional.

d) The direct sum $\bigoplus_{\lambda \in \mathbb{R}} V_\lambda$ is dense in $\Gamma(M;E)$.

Exercise 1. Prove the first three parts of Theorem A. Hint: the first two are easy. For the third part, prove that there is a constant $C \geq 0$ such that for all Λ and $u \in U_\Lambda$, one has $\|u\|_1^2 \leq C(1 + \Lambda^2)\|u\|_0^2$ (Gardings inequality). Then apply Rellichs theorem to show that the closed unit ball in U_Λ is compact.

The last part of the spectral theorem is more involved, and we prove it by reduction to the spectral theorem for compact operators. We study the operator $L := 1 + D^2$, which is again elliptic.

Exercise 2. Prove:

a) for each $l \geq 0$, the operators $L : W^{l+2k} \to W^l$ and $L : \Gamma(M;E) \to \Gamma(M;E)$ are invertible.

b) The inverse $S : \Gamma(M;E) \to \Gamma(M;E)$ extends to a bounded operator $L^2 \to W^{2k}$. We denote by T the composition $L^2 \stackrel{S}{\to} W^2 \subset L^2$.

c) T is compact.

d) T is self-adjoint and positive in the sense that $\langle Tu, u \rangle \geq 0$ for all u.

Hints:

a) a straightforward application of the analytical results from the lecture.

b) open mapping theorem

c) Rellich's theorem

d) because T is continuous (!), it is enough to check $\langle Tu, v\rangle = \langle u, Tv\rangle$ for smooth sections u and v. Use that each smooth section can be written as Lx.

Exercise 3. Take the next best functional analysis textbook, look for "spectral theorem for compact selfadjoint operators", read and understand the proof.

Exercise 4. Derive Theorem A. There are the following steps:

a) If $x \in L^2$, $u \in \Gamma(M; E)$, then $\langle Tx; Lu\rangle = \langle x, u\rangle$.

b) 0 is not an eigenvalue of T.

c) If $Tx = \mu x$, then x is orthogonal to the image of $1 - \mu L$.

d) $1 - \mu L$ is elliptic and thus all eigenfunctions of T are smooth.

e) Derive Theorem A for L and then for D.

f) Locate the place in the overall argument where the assumption $k > 0$ was needed.

Exercise 5. Let M be a closed Riemann manifold and let $[x] \in H^p_{dR}(M)$ be given. The Hodge theorem states that there is a unique harmonic form $\omega \in [x]$. Prove that ω can also be characterized as the unique minimum of the function $[x] \to \mathbb{R}$, $\eta \mapsto \|\eta\|^0_{L^2}$.