SEMINAR ON COBORDISM CATEGORIES AND THE DWYER-WEISS-WILLIAMS THEOREM, SUMMER 2020

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Cobordism categories and Thom spectra.

Talk 1 (The Pontrjagin-Thom construction and the Becker-Gottlieb transfer, Johannes Ebert)**.** The classical Pontrjagin-Thom construction, parametrized Pontrjagin-Thom construction, Thom spectra. Becker-Gottlieb transfer and relation to the Euler number. References: [2] for the transfer, [18] for the classical Pontrjagin-Thom construction, [12], [9] or many other places.

Talk 2 (Cobordism categories, Johannes Ebert)**.** This talk should introduce the cobordism categories and the Madsen-Tillmann spectra, and state the main theorem of [9]. Introduce the fibre sequence of spectra explained in [9], §3. Explain the computation of $\pi_0(\mathrm{MTSO}(d))$. [9] is formulated in a sheaf-theoretic language, which was developped in [13]. Do not take this language too seriously; it is merely a device to properly write down a homotopy type. Do not forget to talk about classifying spaces of categories. Tangential structures are important for all applications of the result, so they should also be covered.

Talk 3 (Proof of the GMTW-Theorem I, Julian Poetke)**.** Discuss the proof of the main result of [9], which is a nice blend of the classical Pontrjagin-Thom theory with the h-principle.

Talk 4 (Proof of the GMTW-Theorem II)**.** This talk gives an alternative proof of the GMTW-theorem [7]. We give both proofs in the seminar, as both are beautiful and very illuminating.

Talk 5 (Cobordism categories of manifolds with boundaries, Johannes Ebert)**.** Genauer [10] introduced the cobordism categories of manifolds with boundary and proved an analog of the GMTW-theorem. Instead of working through [10], one can either adapt the arguments from [7], or use an abstract argument that was given by Steimle [17].

Madsen-Weiss type results.

Talk 6 (Barratt-Priddy-Quillen-Segal theorem, Konrad Bals)**.** The 0-dimensional case of the GMTW theorem can be used to prove an important basic result in homotopy theory: the group completion of the topological monoid $\coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n$ is homotopy equivalent to the infinite loop space $Q(X_+)$ of the suspension spectrum of X. To my knowledge, this is only written down in [11], but also check [5]. On the way, you need to introduce the group completion theorem [14], in particular, the technical heart of it, which is [14, Proposition 4], see also §6 of [6] for a detailed and fairly elementary proof.

Talk 7 (Madsen-Weiss type theorems, Christoph Schrade)**.** This is a side-track of the main part of the seminar. The Madsen–Weiss theorem [13] was the main motivation for the development of the theory of cobordism categories. Outline the proof of the Madsen-Weiss theorem [13], following [9], §7. It is possible to sketch the proof of the surgery step in [9], §6.

The high dimensional version by Galatius–Randal-Williams [8] should also be discussed. The proof cannot sensibly be sketched, as it is very long and would keep us busy for the rest of the term, but the precise formulation should be given. The result of [3] also fits in here.

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Algebraic K-theory of spaces.

Talk 8 (K-Theory of Waldhausen categories, Lukas Stöveken, Thomas Spelten, Konrad Bals)**.** This talk and the next two should give an overview of the construction of Waldhausen's K-theory spectrum $A(X)$ of a space X, and the most important properties. The standard reference is still the foundational paper [19]. The first of those talks is on a very abstract level: introduce Waldhausen categories and the $S_\bullet$-construction, as in [19], §1.1–1.3. Then introduce the additivity theorem [19], §1.4 (the proof is too long to be discussed here). The additivity theorem is used in [19], p. 342, to define the K-theory *spectrum* of a Waldhausen category. The approximation theorem [19], Theorem 1.6.7 is also important for us. See also [21], §V.1 for the additivity theorem and §V.2 for the approximation theorem.

Talk 9 (Waldhausen's $A(X)$, Lukas Stöveken, Thomas Spelten, Konrad Bals)**.** Introduce the category $\mathcal{R}^f(X)$ of finite retractive spaces over X as in [19], and define $A(X)$ as its K-theory spectrum (the definition of $A(X)$ in terms of equivariant homotopy theory in loc.cit. is needed for the proofs of the structural properties, not for the applications that we discuss in the seminar). Discuss its most important properties: functoriality and homotopy invariance [19], Prop. 2.1.7 (which is a consequence of the approximation theorem). Give the computation of $\pi_0(A(X))$.

Talk 10 (Waldhausen's $A(*)$ and spaces of matrices, Lukas Stöveken, Thomas Spelten, Konrad Bals)**.** In this talk, the description of $A(X)$ in terms of spaces of matrices should be discussed, following [19], §2.2. Concentrate on the case $X = *$. Using this result, one can introduce the linearization map $L : A(X) \to K(R[\pi_1(X)])$. In the case $X = *$, $R = \mathbb{Z}$, L is a rational homotopy equivalence; mention Borel's computation of $\pi_*(K(\mathbb{Z})) \otimes \mathbb{Q}$. Can you compute $\pi_1(A(X))$?

The Dwyer-Weiss-Williams index theorem.

Talk 11 (The linearization map to K-theory of rings, Jannes Bantje)**.** This talk should introduce the Dwyer–Weiss–Williams index theorem as in the introduction of [4]. We will only see the proof of a weaker version of that result, which is also stated in the introduction of [15]. Depending on the previous talk, you should introduce the linearization map. Applying it to the Dwyer–Weiss–Williams theorem yields the statement which is described in [4], p.2. In loc.cit., this statement is also connected to the Atiyah–Singer index theorem for the Euler characteristic operator.

Talk 12 (Bivariant A-theory, Jens Reinhold?)**.** Bivariant A-theory was developed by Williams [23], and is the tool to construct the parametrizes Euler characteristic $\chi(p)$ of a fibrations p with homotopy finite fibres. Follow the exposition of bivariant A-theory in §3.1–3.3 of [15], and finish with the definition of $\chi(p)$, given at the beginning of [15], §4.

Talk 13 (Assembly, Coassembly and the unit map, Achim Krause)**.** Follow the exposition of general assembly maps given in [22], §1; consult also [4, §8]. Then discuss the construction of the unit map $X_+ \to A(X)$ as in [15], §3.4. Then move to coassembly, as described in [15, §4.1]. The idea of viewing a homotopy limit as a space of sections should be carefully discussed; it is explained in [4, §1].

Talk 14 (Proof of the index theorem, Jens Reinhold?)**.** In this talk, all the strands are put together to give the proof of the index theorem, as in [15], §5. §4.3 of that paper is also relevant. To be more precise, we show a simpler version, which can be explained with the following diagram, taken from [15], §4.2: if $p : E \to B$ is a bundle of smooth compact d-manifolds, there is a diagram:

$$\begin{array}{ccccccc} & & (Q_+)_B(E) & \longrightarrow & Q(E_+) & \longrightarrow & Q(BO(d)_+) \\ & \nearrow & \downarrow{\scriptstyle \eta_p} & & \downarrow{\scriptstyle \eta_E} & & \downarrow{\scriptstyle \eta_{BO(d)}} \\ B & \longrightarrow & A_B(E) & \longrightarrow & A(E) & \longrightarrow & A(BO(d)). \end{array}$$

The two squares commute for formal reasons; and [4] shows that the triangle on the left commutes up to homotopy. This statement is also shown in [16], which is the follow up to [15]. [15] only shows that the diagram with the maps η_p and η_E removed commutes up to homotopy.

Talk 15 (Concluding talk)**.** We could end the seminar with an outlook. Either, we can discuss higher torsion invariants, following [1], or the stable parametrized h-cobordism theorem of [20] and the "converse Riemann–Roch theorem" [4], part III.

References

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