
A lecture course on Cobordism Theory

Johannes Ebert

Mathematisches Institut Universität Münster Einsteinstr. 62

48149 Münster, Germany

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Contents

1	Differential Topology	2
1.1	Transversality	4
1.2	Orientations	5
1.3	The cobordism relation	5
2	Orientation and (Co)homology	7
2.1	Orientation and Duality	7
2.2	Homological Interpretation of the Mapping Degree	9
2.3	The Leray - Hirsch theorem	9
2.4	Thom-Isomorphism	12
2.5	Fiber Bundles	17
2.6	Homotopy Invariance of Fiber Bundles	18
2.7	Classification of G -Bundles	20
3	Characteristic Classes	26
3.1	Definition and Basic Properties	26
3.2	Universal Classes	30
3.3	Cohomology of classifying spaces	32
3.4	Multiplicative Sequences	35
3.5	Bordism vs. Homotopy: The Pontrjagin-Thom Construction	37
3.6	Pontrjagin-Thom Construction and Homology	38
4	Spectra and the Bordism Ring	39
4.1	Spectra	39
4.2	Calculation of the Oriented Bordism Ring	41
4.3	The Signature	41

1 Differential Topology

Definition 1.0.1 Regular and Singular Value Let M^m, N^n , be two smooth manifolds and $f : M \rightarrow N$ be a smooth map. An element $x \in N$ is a regular value if for all $y \in M$ with $f(y) = x$, the differential $T_y f : T_y M \rightarrow T_x N$ is surjective. A singular value is an $x \in N$ which is not a regular value.

Remark 1.0.2 If $m < n$, then the condition that $x \in N$ is a regular value means that x does not lie in the image of f .

Definition 1.0.3 Let $M^m \subset N^n$ be a submanifold. The codimension of M is the number $n - m$.

Proposition 1.0.4 If $x \in N$ is a regular value of the map $f : M^m \rightarrow N^n$, then $f^{-1}(x) \subset M$ is a submanifold of codimension n .

We now come to the most important fundamental result of differential topology.

Definition 1.0.5 Let M^m be a smooth manifold and $C \subset M$ be a subset. We say that C has measure zero if for each chart $M \supset U \xrightarrow{\phi} \mathbb{R}^n$, the set $\phi(C \cap U) \subset \mathbb{R}^n$ has Lebesgue measure zero.

This notion is well defined because of two facts: a C^1 -diffeomorphism of open subsets of \mathbb{R}^n maps sets of measure zero to measure zero; and any manifold has a countable atlas. Note that the union of countably many sets of measure zero has again measure zero and that the complement of a set of measure zero is never empty (unless M itself is empty). The following theorem is fundamental for all of differential topology.

Theorem 1.0.6 Sard's Theorem Let $f : M^m \rightarrow N^n$ be a smooth map. Then the set of singular values of f $\text{Crit}(f) \subset N$ has measure zero.

For the proof, see [1], p. 58 ff. The most immediate application, or rather, the simplest special case, of Sard's theorem is:

Corollary 1.0.7 If $n > m$ and $f : M^m \rightarrow N^n$ is smooth, then $f(M) \subset N$ has measure zero, hence $N \setminus f(M) \neq \emptyset$.

Theorem 1.0.8 The Whitney Embedding Theorem Let M^m, N^n be manifolds and $A \subset M$ be closed. Assume that $n \geq 2m + 1$. Let N carry a complete metric. Let $f : M \rightarrow N$ be smooth and assume that there exists a neighborhood $A \subset U$ such that $f|_U$ is an injective immersion. Let $\epsilon : M \rightarrow (0, \infty)$ be a continuous function. Then there exists an injective immersion $g : M \rightarrow N$ such that $g|_A = f|_A$ and such that $d(f(x), g(x)) \leq \epsilon(x)$.

We discussed the proof that is given in [2], section 15.7, but only in the case $N = \mathbb{R}^n$. Recall that a proper injective immersion is an embedding. In particular, if we started with a proper map f and a bounded ϵ , the resulting g will be an embedding. Each manifold admits a proper map to \mathbb{R} : pick a countable, locally finite cover by relatively compact open sets U_i of M , let $f_i : M \rightarrow \mathbb{R}_{\geq 0}$ be a function with compact support and $f_i|_{V_i} = 1$, where $V_i \subset U_i$ are smaller subsets that cover X . Then the function $f = \sum_i i f_i$ is proper. So any manifold has a proper map to \mathbb{R}^n for each n . This shows that we can realize any manifold as a closed submanifold of \mathbb{R}^n for $n \geq 2m + 1$. In particular, any manifold admits a complete metric.

Definition 1.0.9 Let $f : M \rightarrow N$ be an immersion. The normal bundle of f is the vector bundle $\nu f := f^*TN/TM$ on M . If f is the inclusion of a submanifold, we write ν_M^N or simply ν_M for the normal bundle.

Remark 1.0.10 Suppose that N has a Riemann metric. Then the normal bundle is isomorphic to the orthogonal complement of TM in the Riemannian vector bundle f^*TN .

Before we define tubular neighborhoods, we add a small remark on vector bundles. Let $\pi : E \rightarrow M$ be a smooth vector bundle. There is a canonical bundle monomorphism $\eta : E \rightarrow TE|_M$, sending $e \in E$ to the equivalence class of the curve $\mathbb{R} \rightarrow E; t \mapsto te$. Moreover, the differential of π yields a vector bundle epimorphism $T\pi : TE|_M \rightarrow TM$. These maps yield an exact sequence

$$0 \rightarrow E \rightarrow TE|_M \rightarrow TM \rightarrow 0, \tag{1}$$

which admits a canonical splitting, namely the differential of the zero section $\iota : M \rightarrow E$.

Definition 1.0.11 Tubular Map *Let $f : M \rightarrow N$ be an embedding. A tubular map for f is a smooth map $t : \nu f \rightarrow N$ with the following properties:*

1. *The restriction of t to the zero section is equal to f .*
2. *There is an open neighborhood $U \subset \nu f$ of the zero section such that $t|_U$ is an embedding.*
3. *The composition $\nu f \xrightarrow{\eta} T\nu f \xrightarrow{Tt} f^*TN \rightarrow \nu f$ (the first map was described before the definition, the third map is the quotient map) is the identity map.*

Theorem 1.0.12 *Each embedding has a tubular map.*

We followed the proof given in [2] and we used a lemma from [1]. To gain some flexibility in dealing with tubular maps, we need two lemmata.

Lemma 1.0.13 *There exists a smooth map $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(r, \epsilon, t, x) \mapsto \sigma_t^{r, \epsilon}(x)$ such that:*

1. *for each fixed r, ϵ , the map $(t, x) \mapsto \sigma_t^{r, \epsilon}(x)$ is an isotopy.*
2. $\sigma_0^{r, \epsilon} = \text{id}$.
3. $\sigma_t^{r, \epsilon}|_{B_r(0)} = \text{id}$.
4. $\sigma_1^{r, \epsilon}(\mathbb{R}^n) = B_{r+\epsilon}(0)$.

Using this lemma, we show: if $t : \nu f \rightarrow N$ is a tubular map, then there exists a tubular map $\tilde{t} : \nu f \rightarrow N$ that coincides with f on a neighborhood of the zero section and that embeds the whole of νf into N . The image of such a tubular map is called a *tubular neighborhood*. The following theorem explains in which sense a tubular neighborhood is uniquely determined by f .

Theorem 1.0.14 *Let $f : M \rightarrow N$ be an embedding and $t_0, t_1 : \nu f \rightarrow N$ be tubular maps. Then there exists an isotopy from t_0 to t_1 .*

An application of the Whitney embedding theorem and tubular maps is the following theorem.

Theorem 1.0.15 Smooth approximation theorem *Let M, N be manifolds, N with a complete Riemann metric. Let $f : M \rightarrow N$ be a continuous map.*

1. *There exists a function $\epsilon : M \rightarrow (0, \infty)$, such that any function $g : M \rightarrow N$ which satisfies $d(f(x), g(x)) < \epsilon(x)$ is homotopic to f .*
2. *For any continuous function $\epsilon : M \rightarrow (0, \infty)$, there is a smooth map $g : M \rightarrow N$ with $d(f(x), g(x)) < \epsilon(x)$. If there is a closed $A \subset M$ and f is already smooth on a neighborhood of A , we can pick g to coincide with f on A .*

1.1 Transversality

The most important notion in differential topology is transversality.

Definition 1.1.1 *Let $M; X; Y$ be smooth manifolds, $f : X \rightarrow M, g : Y \rightarrow N$ be smooth maps. Then f and g are called transverse if for all x, y with $f(x) = g(y) = z$, we have that*

$$(T_x f)(T_x X) + (T_y g)(T_y Y) = T_z Z.$$

We write $f \pitchfork g$ when f and g are transverse. If g is the inclusion of a submanifold $Y \subset M$, we write $f \pitchfork Y$.

One of the important properties of transversality is:

Proposition 1.1.2 *Let $M; X; Y$ be smooth manifolds, $f : X \rightarrow M$ be a smooth map and $Y \subset N$. Assume that $f \pitchfork Y$. Then $f^{-1}(Y) \subset X$ is a submanifold and the normal bundles are related by $\nu_{f^{-1}(Y)}^X \cong f^* \nu_Y^M$.*

The proof is an exercise. The following result is one of the key results of differential topology:

Theorem 1.1.3 The Transversality Theorem *Let $f : M \rightarrow N$ and $g : L \rightarrow N$ be smooth maps. Let N be endowed with a Riemann metric and $\epsilon : M \rightarrow (0, \infty)$ be a function. Then there exists a map $h : M \rightarrow N$ with $d(f(x), h(x)) < \epsilon(x)$ such that $h \pitchfork g$. If $A \subset M$ is closed and if f is transverse to g at all points of A , then we can choose h to coincide with f on A . Moreover (due to Theorem 1.0.15), if ϵ is suitably small, then h will be homotopic to f relative to A .*

The proof presented in the lecture was taken from [1], §14. The first application of transversality is to computations of some homotopy groups (see any book on algebraic topology, e.g. [2], for the definition of homotopy groups).

Theorem 1.1.4 *Let M^m and N^n be smooth manifolds, $g : N^n \rightarrow M^m$ be a smooth map. Let $x \in M \setminus g(N)$ and $j : M \setminus g(N) \rightarrow M$ be the inclusion. The map $\pi_i(M \setminus g(N), x) \rightarrow \pi_i(M, x)$ induced by j is an epimorphism if $j \leq m - n - 1$ and an isomorphism if $j < m - n - 1$.*

By Theorem 1.0.15, any homotopy class has a smooth representative, which can be chosen to be transverse to g . A dimension count and a similar argument, applied to homotopies, gives the proof.

Example 1.1.5 *$M = S^m, N = *$. Then $M \setminus N \cong \mathbb{R}^m$ and hence contractible. We obtain that $\pi_i(S^m) = 0$ for $i \leq m - 1$.*

Example 1.1.6 *Let $\text{Mon}(\mathbb{R}^n, \mathbb{R}^m) \subset \text{Mat}_{m,n}(\mathbb{R})$ be the subspace of injective linear maps. It is an open subset; and the complement $\text{Mat}_{m,n} \setminus \text{Mon}(\mathbb{R}^n, \mathbb{R}^m)$ is the union $\bigcup_{r=0}^{n-1} M_r$, where M_r is the subset of all matrices of rank r . M_r is a submanifold of dimension $(m - r)r + rn$ and therefore $\pi_i(\text{Mon}(\mathbb{R}^n, \mathbb{R}^m)) = 0$ for $i < m - n$.*

Example 1.1.7 *Let M be an m -dimensional manifold. The ordered configuration space of M is the manifold $F^k(M) := \{(x_1, \dots, x_k) \in M^k \mid i \neq j \Rightarrow x_i \neq x_j\}$. Let $\Delta_{i,j} := \{(x_1, \dots, x_k) \mid x_i = x_j\}$, a submanifold of M^k of codimension m . Then $F^k(M) = M^k \setminus \bigcup_{i \neq j} \Delta_{i,j}$. Therefore the inclusion $F^k(M) \rightarrow M^k$ is highly connected. In particular, for $M = \mathbb{R}^m$, we obtain $\pi_i(F^k(M)) = 0$ as long as $i \leq m - 2$ (this does not depend on k).*

1.2 Orientations

For many purposes in differential topology, orientations are quite important. We begin by fixing definitions and conventions. Let V be a finite-dimensional real vector space. An *orientation* of V is an equivalence of bases of V , where two bases are equivalent if their transformation matrix has positive determinant. Another way (and in some sense better) to define orientations is by means of the exterior algebra.

Definition 1.2.1 *Let V be an n -dimensional real vector space. Let $\Lambda^n V$ be the top exterior power; a one-dimensional vector space. An orientation of V is one of the two components of $\Lambda^n V \setminus \{0\}$. The standard orientation of \mathbb{R}^n is the component containing the element $e_1 \wedge \dots \wedge e_n$. If V and W are two oriented vector spaces, we orient their sum $V \oplus W$ so that, if (v_1, \dots, v_n) is an oriented basis of V and (w_1, \dots, w_m) is an oriented basis of W , then $(v_1, \dots, v_n, w_1, \dots, w_m)$ is an oriented basis of $V \oplus W$.*

The problem with this definition is that the vector spaces $V \oplus W$ and $W \oplus V$ are canonically isomorphic, but that the isomorphism is not orientation preserving (its determinant is $(-1)^{mn}$). It can be shown that it is impossible to remove this problem: direct sum of oriented vector spaces is not a symmetric operation, or in more fancy terms: oriented vector spaces do not form a symmetric monoidal category.

Definition 1.2.2 *Let $V \rightarrow X$ be a real vector bundle. An orientation of V is a family $\{x\}$, $x \in X$, where x is an orientation of the fibre V_x , such that for each $y \in X$, there exists a neighborhood U of y and a bundle map $h : V|_U \rightarrow U \times \mathbb{R}^n$, such that for each $x \in U$, the linear isomorphism $h_x : V_x \rightarrow \mathbb{R}^n$ is orientation-preserving, where \mathbb{R}^n is endowed with the standard orientation.*

Again, a more elegant definition can be given using the exterior algebra. Let $\Lambda^n V \rightarrow X$ be the bundle of top exterior powers. It is a real line bundle. We form $\text{Or}(V) := (\Lambda^n V \setminus 0) / \mathbb{R}_{>0} \rightarrow X$; which is a twofold covering. We define an orientation to be a cross-section of $\text{Or}(V) \rightarrow X$. The above orientation conventions prompt an orientation convention for submanifolds. If M is an oriented manifold and $N \subset M$ a submanifold, we say that orientations of N and the normal bundle ν_N^M are compatible if the (almost) canonical isomorphism $TM|_N \cong TN \oplus \nu_N^M$ is orientation-preserving.

1.3 The cobordism relation

Definition 1.3.1 (Co)bordism *Let M_i , $i = 0, 1$ be closed manifolds. A bordism (or cobordism) from M_0 to M_1 is a triple (W, ϕ_0, ϕ_1) , where W is a compact manifold with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$, which is decomposed into open a disjoint union of closed and open subsets. The maps $\phi_i : \partial_i W \rightarrow M_i$ are diffeomorphisms.*

Let M be a manifold with boundary. If $x \in \partial M$, the tangent space $T_x \partial M \subset T_x M$ is a codimension 1 subspace. A vector $v \in T_x M \setminus T_x \partial M$ is an inward vector if there is a smooth curve $c : [0, 1) \rightarrow M$, $c(t) \in \partial M \Rightarrow t = 0$, $c(0) = x$ and $\frac{d}{dt} c(0) = v$. A vector v is an outward vector if $-v$ is an inward one. The subset of all inward vectors in an open half-space in the tangent space.

Definition 1.3.2 Normal Vector Field *Let (W, ϕ_0, ϕ_1) be a bordism from M_0 to M_1 . A normal vector field is a section $v : \partial M \rightarrow TM|_{\partial M}$ such that for all $x \in \partial_0 W$, the vector $v(x)$ is an inward vector and for all $x \in \partial_1 W$, $v(x)$ is outward.*

One can show that there is always a normal vector field and the space of normal vector fields is convex (a partition of unity argument). Any normal vector field gives rise to a vector bundle isomorphism

$$\mathbb{R} \oplus T\partial W \cong TW|_{\partial W}; \quad (u, t) \mapsto u + tv. \quad (2)$$

Using this isomorphism, we define the notion of an oriented bordism.

Definition 1.3.3 Oriented Bordism Let $M_i, i = 0, 1$ be oriented closed manifolds. An oriented bordism from M_0 to M_1 is a bordism (W, ϕ_0, ϕ_1) , where W is in addition oriented, the above vector bundle isomorphism and ϕ_i are orientation-preserving.

Definition 1.3.4 Bordism Group The unoriented bordism group of n -dimensional manifolds is the set

$$\mathcal{N}_n := \{M^n \mid M \text{ closed smooth } n\text{-manifold}\} / \sim$$

with

$$M_0 \sim M_1 \quad :\Leftrightarrow \quad \text{there is a bordism from } M_0 \text{ to } M_1.$$

The oriented bordism group is the set

$$\Omega_n := \{M^n \mid M \text{ closed oriented smooth } n\text{-manifold}\} / \sim$$

where the equivalence relation is now taken to be only by oriented bordisms.

Theorem 1.3.5 Bordism and oriented bordism are equivalence relations. The assignment $[M_0] + [M_1] \mapsto [M_0 \amalg M_1]$ turns both \mathcal{N}_n and Ω_n into abelian groups. Moreover, defining $[M_0][M_1] := [M_0 \times M_1]$, turns $\mathcal{N}_* := \bigoplus_{n \geq 0} \mathcal{N}_n$ into a commutative ring and $\Omega_* := \bigoplus_{n \geq 0} \Omega_n$ into a graded-commutative ring.

Proof: We only prove that being bordant is an equivalence relation for Ω_n (\mathcal{N}_n is easier).

- Reflexivity: $[0, 1] \times M$ is a bordism from M to M .
- Symmetry: If $W : M_0 \rightarrow M_1$ is a bordism, then so is $-W : M_1 \rightarrow M_0$.
- Transitivity: If $W_{01} : M_0 \rightarrow M_1$ and $W_{12} : M_1 \rightarrow M_2$ are bordisms, then by gluing along M_1 we get a bordism $W_{01} \cup_{M_1} W_{12} : M_0 \rightarrow M_2$.
- Neutral element: $\emptyset \amalg M = M \amalg \emptyset = M$
- Inverse: We have $[M] + [-M] = \emptyset$, which can be seen, by considering the cylinder over M and bending one end over the other.

□

Due to the classification of 1-dimensional manifolds, the zero dimensional bordism groups are easy to calculate. The result is that $\Omega_0 \cong \mathbb{Z}$ and $\mathcal{N}_0 \cong \mathbb{Z}/2$. This calculation already gives rise to interesting results. Consider the following situation. Let M^m be an oriented manifold, $N^n \subset M^m$ be a submanifold. Let furthermore K^k be a third oriented and closed manifold and $f : K \rightarrow M$ be a smooth map. Assume that $f \pitchfork N$. Then $f^{-1}(N) \subset K$ is a submanifold of K ; it has dimension $k - m + n$ and the normal bundle is $\nu_{f^{-1}(N)}^K \cong f^* \nu_N^M$. Since M and N are oriented, ν_N^M has an orientation by the above convention and we orient $f^{-1}(N)$ by this orientation convention. Taking the bordism class of $f^{-1}(N)$, we obtain an element

$$\sharp(f; N) := [f^{-1}(N)] \in \Omega_{k-m+n}.$$

That this definition makes sense is the content of the next result. The proof gives a first insight of the connection between homotopies and bordism and is an important motivation for the Pontrjagin-Thom construction.

Proposition 1.3.6 If $f_i : K \rightarrow M, i = 0, 1$, are both transverse to N and if $f_0 \sim f_1$, then $[f_0^{-1}(N)] = [f_1^{-1}(N)] \in \Omega_{k-m+n}$.

The simplest special case of the invariant $\sharp(f; N)$ is when $k - m + n = 0$; in this case we get an element in $\Omega_0 = \mathbb{Z}$. We discuss two special cases in more detail. The first is the mapping degree. Let $f : M^n \rightarrow N^n$ be a smooth map between closed oriented manifolds of the same dimension. Let $z \in N$ be a regular value. Let $1 \times f : M \rightarrow M \times N$ be the map $x \mapsto (x, f(x))$ (the graph embedding).

Definition 1.3.7 Mapping Degree *The mapping degree at a regular value $z \in N$ of a smooth map $f : M \rightarrow N$ is the element $\deg_z(f) := \sharp(1 \times f; M \times z) \in \Omega_0 = \mathbb{Z}$.*

Proposition 1.3.8 *The mapping degree has the following properties:*

1. If N is connected, then $\deg_z(f)$ does not depend on the choice of the regular value z .
2. If $f_0 \sim f_1$, then $\deg(f_0) = \deg(f_1)$.
3. $\deg fg = \deg(f) \deg(g)$.

The first two properties are not entirely trivial consequences of the homotopy invariance. For the first, we consider the submanifold $\text{graph}(f) = \{(x, f(x)) \in M \times N\}$ and the map $\iota_z : M \rightarrow M \times N$, $x \mapsto (x, z)$ and note that $\sharp(1 \times f; M \times z) = (-1)^{n^2} \sharp(\iota_z, \text{graph}(f))$. Now we can give a purely differential-topological proof of one of the pivotal results of homotopy theory. Let M^n be a closed oriented connected n -manifold. Let $[M; S^n]$ be the set of homotopy classes of continuous maps. Given $f : M \rightarrow S^n$, we know that f is homotopic to a smooth map. The homotopy invariance of the mapping degree shows that $[f] \mapsto \deg(f)$ is a well-defined map $[M; S^n] \rightarrow \mathbb{Z}$.

Theorem 1.3.9 Hopf *The map $\deg : [M; S^n] \rightarrow \mathbb{Z}$ is a bijection for each closed oriented connected n -manifold M .*

Corollary 1.3.10 *There is an isomorphism of groups $\pi_n(S^n) \cong \mathbb{Z}$.*

Another important special case of the intersection index is the Euler number of a vector bundle. Let $V \rightarrow M$ be a rank k oriented vector bundle over a compact oriented manifold of dimension n . Then the *Euler number* $\text{Eul}(V)$ is defined to be the element $\sharp(s; M) \in \Omega_{n-k}$, where $M \subset V$ is the zero section and $s : M \rightarrow V$ is any section transverse to the zero section. Surely $\text{Eul}(V)$ does not depend on the choice of s and if V has a section without zero, then $\text{Eul}(V) = 0$. Computation of some examples (spheres, complex-projective spaces, oriented surfaces) give credibility to the following theorem.

Theorem 1.3.11 *If M is a closed oriented manifold, then*

$$\text{Eul}(TM) = \chi(M) = \sum_{k=0}^n (-1)^k \dim H_k(M; \mathbb{Q}).$$

The proof can be given either using Morse theory [6] or Poincaré duality (see Theorem 2.4.15 below).

2 Orientation and (Co)homology

2.1 Orientation and Duality

Definition 2.1.1 Orientation Conventions *The singular 1-simplex*

$$\begin{aligned} \Delta^1 &\longrightarrow \mathbb{R}, \\ (1-t)e_0 + te_1 &\longmapsto 2t - 1 \end{aligned}$$

gives a cycle in $H_1(\mathbb{R}, \mathbb{R} \setminus 0; \mathbb{Z})$ whose equivalence class is a generator, denoted μ_1 . We set

$$\mu_1 \in H_1(\mathbb{R}, \mathbb{R} \setminus 0; \mathbb{Z}) \cong \mathbb{Z}, \quad \mu_n := \mu_1 \times \cdots \times \mu_1 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; \mathbb{Z}) \cong \mathbb{Z}$$

and define $\mu^n \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; \mathbb{Z})$ by the requirement $\langle \mu^n; \mu_n \rangle = 1$. Then we have

$$\mu_m \times \mu_n = \mu_{m+n}, \quad \mu^n \times \mu^m = (-1)^{mn} \mu^{m+n}, \quad \langle \mu^n, \mu_n \rangle = 1.$$

Definition and Lemma 2.1.2 Fundamental Class *Let M^n be a topological manifold and $x \in K \subset M^n$ with K compact. Then there is exactly one class $[M]_K \in H_n(M, M - K)$, such that for all $x \in K$ and every oriented chart $x \in U \xrightarrow{\varphi} \mathbb{R}^n$ with $\varphi(x) = 0$, we have that*

$$H_n(M, M - K) \longrightarrow H_n(M, M - x) \xrightarrow{\cong} H_n(U, U - x) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n - 0).$$

$$[M]_K \longrightarrow \mu_n$$

Theorem 2.1.3 Poincare – Lefschetz – Alexander Duality *Let M^n be an oriented topological manifold with $L \subseteq K \subseteq M$ and K, L compact and nice, then the following is an isomorphism*

$$\boxed{(-) \cap [M]_K : H^p(K, L) \xrightarrow{\cong} H_{n-p}(M - L, M - K)}$$

Remark 2.1.4 *Neighborhood retracts (e.g. submanifolds) are "nice". If K, L are not nice, then the above isomorphism still holds, if one replaces $H^p(K, L)$ by the Cech-cohomology*

$$\check{H} := \operatorname{colim}_{U, V} H^p(U, V),$$

where $L \subset U \subset V \supset K$ and U, V are open in M .

Example 2.1.5

1. Let M^m, N^n be oriented topological manifolds and $K \subseteq M, L \subseteq N$ compact, then

$$[M \times N]_{K \times L} = [M]_K \times [N]_L \in H_{m+n}((M \times N), (M \times N) - (K \times L))$$

due to the uniqueness of the fundamental class.

2. Let M_0, M_1 be oriented, compact n -manifolds and $(W, \operatorname{id}_0, \operatorname{id}_1)$ an oriented bordism. We glue necks onto W :

$$\widehat{W} := W \cup_{M_0} ((-\infty, 0] \times M_0) \cup_{M_1} ([0, \infty) \times M_1),$$

consider the isomorphism $\varphi : H_{n+1}(\widehat{W}, \widehat{W} - W) \xrightarrow{\cong} H_{n+1}(W, \partial W)$ and define

$$[W] := \phi_* [\widehat{W}]_W \in H^{n+1}(W, \partial W).$$

Theorem 2.1.6 *Let W be a bordism with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$. Consider the homology sequence of the pair $(W, \partial W)$*

$$\cdots \longrightarrow H_n(W, \partial W) \xrightarrow{\partial} H_n(\partial W) \xrightarrow{j} H_n(W) \longrightarrow \cdots$$

It holds that

$$\partial[W] = -[\partial_0 W] + [\partial_1 W].$$

Proof: See [7], Lemma VI.9.1. □

Corollary 2.1.7 *Let W be a bordism with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$ and $x \in H^n(W)$. Consider the inclusions $j_i : \partial_i W \hookrightarrow W$, then*

$$\langle j_0^* x, [\partial_0 W] \rangle = \langle j_1^* x, [\partial_1 W] \rangle.$$

Proof: Let $j = j_0 \sqcup j_1$, then

$$\langle j_0^* x, [\partial_0 W] \rangle - \langle j_1^* x, [\partial_1 W] \rangle = \langle j^* x, \partial[W] \rangle = \langle x, (j_* \partial)[W] \rangle = 0,$$

where in the first step we have used the previous theorem and in the last step we used that $(j_* \partial) = 0$, due to the exactness of the homology sequence. □

2.2 Homological Interpretation of the Mapping Degree

Theorem 2.2.1 *Let M^n, N^n be closed connected oriented topological manifolds and $f : M \rightarrow N$ a continuous mapping, then due to $H_n(M) \cong \mathbb{Z} \cong H_n(N)$, we have that*

$$\exists! d \in \mathbb{Z} : f_*[M] = d[N], \quad \text{and} \quad \deg(f) = d.$$

Proof: W.l.o.g. let f be smooth, $x \in N$ be a regular value. Let further $x \in U \subseteq N$ with U open and all $u \in N$ be regular values. Then

$$f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$$

is a covering, i.e. $f^{-1}(U) = \sqcup_{i=1}^r V_i$ and $f|_{V_i} \rightarrow U$ is a diffeomorphism. Let further $\varphi : U \rightarrow \mathbb{R}^n$ be a map with $\varphi(x) = 0$ and let $f^{-1}(x) = \{y_1, \dots, y_r\}$, then consider the following commutative diagram

$$\begin{array}{ccc} H_n(M) & \xrightarrow{f_*} & H_n(N) \\ \downarrow & & \downarrow \cong \\ H_n(M, M - \{y_1, \dots, y_r\}) & \xrightarrow{f_*} & H_n(N, N - x) \\ \downarrow & & \downarrow \varphi_* \\ H_n(\sqcup_{i=1}^r V_i, \sqcup_{i=1}^r (V_i - \{y_i\})) & & H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \\ \downarrow \cong & \nearrow (a_1, \dots, a_r) \rightarrow \sum_i a_i & \\ \bigoplus_{i=1}^r H_n(\mathbb{R}^n, \mathbb{R}^n - 0) & & \end{array}$$

where in the left column $H_n(M) \ni [M] \mapsto (\varepsilon_1 \mu_n, \dots, \varepsilon_r \mu_n) \in \bigoplus_{i=1}^r H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ with $\varepsilon_i := \text{sgn}(\det d_{y_i} f)$ and in the right column $H_n(N) \ni [N] \mapsto \mu_n \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$.

Due to $\bigoplus_{i=1}^r H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \ni (\varepsilon_1 \mu_n, \dots, \varepsilon_r \mu_n) \mapsto \sum_{i=1}^r \varepsilon_i \mu_n = \deg(f) \mu_n \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$, and the commutativity of the diagram, we indeed get $f_*[M] = \deg(f)[N]$.

□

2.3 The Leray - Hirsch theorem

Remark 2.3.1 Thom – Isomorphism Special Case *Let M^m be an oriented closed manifold and $V \rightarrow M$ an oriented n -vector bundle. Consider the disc bundle $W := \{v \in V \mid |v| \leq 1\}$ which is a compact manifold and gives rise to the following commutative diagram*

$$\begin{array}{ccc} H^k(M) & \longrightarrow & H^{n+k}(W, \partial W) \\ \downarrow \cong & & \downarrow \cong \\ H_{m-k}(M) & \xrightarrow{\cong} & H_{m-k}(W) \end{array}$$

which lets the above map be an isomorphism called *Thom-isomorphism*. The image of $1 \in H^0(M)$ under this map is referred to as the *Thom class*.

For the general situation, we consider a principal ideal domain R , a topological space X and a fiber bundle $\pi : E \rightarrow X$ with fiber $F^x := \pi^{-1}(x)$ and a subbundle $E_0 \subset E$. We make the following definitions and remarks:

- $H^*(E, E_0) := H^*(E, E_0; R)$ is a $H^*(X)$ -module via $x \cdot y := \pi^*(x) \cup y$, where $x \in H^*(X)$, $y \in H^*(E, E_0)$.
- Let $F_0^x := F^x \cap E_0$ and $j_x : (F^x, F_0^x) \hookrightarrow (E, E_0)$

The Leray-Hirsch theorem makes a statement on the structure of $H^*(E; E_0)$ as a module over $H^*(X)$ under the following assumption.

Assumption 2.3.2 *There is a set $\mathcal{B} \subseteq H^*(E, E_0)$ of homogeneous elements, such that:*

- For every $n \in \mathbb{N}$ there are only finitely many $b \in \mathcal{B}$ of degree n .
- For all $x \in X$ the set $\{j_x^* b \mid b \in \mathcal{B}\}$ is a R -basis of $H^*(F^x, F_0^x)$.

Theorem 2.3.3 Leray – Hirsch Theorem *Let everything be as above and assume 2.3.2, then*

$$\begin{aligned} \Phi : H^*(X) \otimes_R R\mathcal{B} &\xrightarrow{\cong} H^*(E, E_0; R) \\ x \otimes b &\longmapsto \pi^* x \cdot b \end{aligned}$$

is an isomorphism of $H^(X)$ -modules (but not a ring isomorphism!).*

Proof:

1.) Formalities

a) Let $f : Y \rightarrow X$ be a continuous function. Consider

$$\begin{array}{ccc} (f^* E, F^* E_0) & \xrightarrow{\pi|_f} & Y \\ \downarrow \tilde{f} & & \downarrow f \\ (E, E_0) & \xrightarrow{\pi} & X \end{array}$$

and we set $f^* \mathcal{B} := \{\tilde{f}^* b \mid b \in \mathcal{B}\} \subseteq H^*(f^* E, f^* E_0; R)$, then $(f^* E, F^* E_0) \rightarrow Y$ and $f^* \mathcal{B}$ fulfill the requirements of the theorem.

b) Φ is a natural transformation of functors

$$\mathbf{Top}_X \longrightarrow R\text{-Mod}$$

where \mathbf{Top}_X has as objects (f, Y) where $f : Y \rightarrow X$ is a continuous function and the morphisms are commutative triangles

$$\begin{array}{ccc} Y_1 & \xrightarrow{f_1} & X \\ \downarrow & \nearrow f_2 & \\ Y_2 & & \end{array}$$

Φ is given by

$$\begin{aligned} \Phi : H^*(Y) \otimes_R R\mathcal{B} &\longrightarrow H^*(f^* E, f^* E_0; R) \\ y \otimes b &\longmapsto \pi|_f^* y \cdot \tilde{f}^* b \end{aligned}$$

2.) Let now X be a zero dimensional CW complex, then

$$H^*(X; R) \otimes_R R\mathcal{B} \cong \left(\prod_{x \in X} R \right) \otimes_R R\mathcal{B}, \quad H^*(E, E_0; R) \cong \prod_{x \in X} H^*(F^x, F_0^x; R),$$

where the first isomorphism is given by $r \otimes b \mapsto (r_x)_{x \in X} \otimes b$ and the second by $b \mapsto (b_x)_{x \in X}$. Under these identifications, Φ becomes

$$\begin{aligned} \Phi : \left(\prod_{x \in X} R \right) \otimes_R R\mathcal{B} &\longrightarrow \prod_{x \in X} H^*(F^x, F_0^x; R), \\ (r_x)_{x \in X} \otimes b &\longmapsto (r_x b_x)_{x \in X} \end{aligned}$$

which is an isomorphism.

- 3.) We check the Mayer-Vietoris property. Take $A, B \subseteq X$ open, and assume the theorem holds for $A, B, A \cap B$, then we shall prove that the theorem also holds for $A \cup B$. We introduce the notation

$$L^n(A) := (H^*(A) \otimes_R RB)^n \text{ (the piece of degree } n\text{); } K^n(A) := H^n(E|_A, E_0|_A; R);$$

Φ becomes a natural transformation of functors $L^n \rightarrow K^n$ from spaces over X to R -modules. Now consider the following diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} L^{n-1}(A) \oplus L^{n-1}(B) & \longrightarrow & L^{n-1}(A \cap B) & \xrightarrow{\partial} & L^n(A \cup B) & \longrightarrow & L^n(A) \oplus L^n(B) & \longrightarrow & L^n(A \cap B) \\ \cong \downarrow \Phi & & \cong \downarrow \Phi & & \cong \downarrow \Phi & & \cong \downarrow \Phi & & \cong \downarrow \Phi \\ K^{n-1}(A) \oplus K^{n-1}(B) & \longrightarrow & K^{n-1}(A \cap B) & \xrightarrow{\partial} & K^n(A \cup B) & \longrightarrow & K^n(A) \oplus K^n(B) & \longrightarrow & K^n(A \cap B) \end{array}$$

All the above squares commute, only commutativity of the square containing the boundary maps ∂ needs to be checked. Commutativity here amounts to

$$\pi^*(\partial x) \cup b = \partial(\pi^* x \cup b).$$

This holds, since considering open subsets $U_1, U_2 \subseteq Y$ such that $U_1 \cup U_2 = Y$, we have a commutative diagram of cochain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{\mathcal{U}}^*(Y) & \longrightarrow & C^*(U_1) \oplus C^*(U_2) & \longrightarrow & C^*(U_1 \cap U_2) & \longrightarrow & 0 \\ & & \downarrow \times z & & \downarrow \times z & & \downarrow \times z & & \\ 0 & \longrightarrow & C_{\mathcal{U} \times Z}^*(Y \times Z) & \longrightarrow & C^*(U_1 \times Z) \oplus C^*(U_2 \times Z) & \longrightarrow & C^*(U_1 \cap U_2 \times Z) & \longrightarrow & 0 \end{array}$$

with $z \in C^*(Z)$ and $\partial z = 0$, where the subscript denotes the complex of small simplices with respect to the covers $\mathcal{U} = \{U_1, U_2\}$ and $\mathcal{U} \times Z = \{U_1 \times Z, U_2 \times Z\}$. These complexes are chain equivalent to the full singular cochain complex by the small simplex theorem. Thus the boundary map ∂ from the Mayer-Vietoris sequence commutes with the cohomology cross product \times . Thus we indeed have the above commutativity.

- 4.) Let $f : Y \rightarrow X$ be a weak homotopy equivalence (i.e. $f_n : \pi_n(Y) \xrightarrow{\cong} \pi_n(X) \forall n$). We assume that the theorem holds for Y and show that this also lets the theorem hold for X .

The map f induces a map between the long exact homotopy sequences:

$$\begin{array}{ccccccccc} \pi_*(F^x) & \longrightarrow & \pi_*(f^*E) & \longrightarrow & \pi_*(Y) & \longrightarrow & \pi_{*-1}(F^x) & \longrightarrow & \dots \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\ \pi_*(F^x) & \longrightarrow & \pi_*(E) & \longrightarrow & \pi_*(X) & \longrightarrow & \pi_{*-1}(F^x) & \longrightarrow & \dots \end{array}$$

Thus by the 5-Lemma, $\tilde{f} : f^*E \rightarrow E$ is a weak homotopy equivalence and the same argument shows that $\tilde{f} : f^*E_0 \rightarrow E_0$ is weak homotopy equivalence. By the Hurewicz theorem and the 5-Lemma again,

$$\tilde{f}_* : H_*(f^*E; f^*E_0) \longrightarrow H_*(E; E_0)$$

is an isomorphism and the universal coefficient theorem proves that

$$\begin{array}{ccc} H^*(E, E_0; R) & \xrightarrow{\cong} & H^*(f^*E, f^*E_0; R) \\ H^*(X) & \xrightarrow{\cong} & H^*(Y) \end{array}$$

is an isomorphism.

- 5.) The theorem holds for all finite dimensional CW complexes X . This is proved by induction on the dimension n of X . We have already seen the case $n = 0$. So now let $\dim(X) = n > 0$ and let $D \subset X$ contain exactly one point out of every open n -cell. The theorem holds for D . Let

$$A := \bigcup \{\text{open } n\text{-cells}\} \simeq D, \quad B := X \setminus D \simeq X^{n-1};$$

this gives a cover $A \cup B = X$ and $A \cap B = \cup (\{\text{open } n\text{-cells}\} \setminus \{*\}) \simeq \cup S^{n-1}$. By induction hypothesis and step (4), the theorem holds for A, B and $A \cap B$ and so by step (3), it follows for X .

- 6.) We can now prove the theorem for an arbitrary CW complex X . We set $E^{(n)} := \pi^{-1}(X^{(n)})$ and have a commutative diagram

$$\begin{array}{ccc} H^*(X) \otimes_R RB & \xrightarrow{\Phi} & H^*(E, E_0; R) \\ \cong \downarrow (i) & & \downarrow (ii) \\ (\varprojlim H^*(X^{(n)})) \otimes_R RB & & \\ \downarrow (iii) & \xrightarrow{\varprojlim \Phi^{(n)}} & \varprojlim H^*(E^{(n)}, E_0^{(n)}) \\ \varprojlim (H^n(X^{(n)}) \otimes_R RB) & & \end{array}$$

where (i) is an isomorphism by the CW-homology theorem. The map (ii) is an isomorphism since $E^{(n)} \rightarrow E$ is a π_* -isomorphism for $* < n$ and thus a homology isomorphism in small degrees. (iii) is an isomorphism because the inverse system is eventually constant.

- 7.) For a general space X we now chose a CW-approximation $f : Y \rightarrow X$ with Y a CW-complex and f a weak homotopy equivalence

□

2.4 Thom-Isomorphism

Definition 2.4.1 Thom – Class Let $\pi : V \rightarrow X$ be a real vector bundle of rank n and set $V_0 := V \setminus 0$ where 0 is understood as the image of the zero section. A Thom-class is an element

$$\tau \in H^n(V, V_0; R) : \quad j_x^* \tau \in H^n(V^x, V_0^x; R) \cong R \text{ is a generator } \forall x \in X.$$

Corollary 2.4.2 of Theorem 2.3.3 Let τ be a Thom-class, then

$$\begin{array}{ccc} \text{th} : H^*(X; R) & \xrightarrow{\cong} & H^{*+n}(V, V_0; R) \\ x & \longmapsto & \pi^* x \cup \tau \end{array}$$

is an isomorphism.

Remark 2.4.3

- If τ is a Thom-class for $V \rightarrow Y$ and $f : X \rightarrow Y$, then

$$\tilde{f}^* \tau \in H^n(f^* V, f^* V_0)$$

is a Thom-class also.

- Consider two vector bundles $V^n \rightarrow X$ and $W^m \rightarrow Y$ with Thom-classes τ_V, τ_W , then

$$\tau_V \times \tau_W \in H^{n+m}(V \times W, (V \times W)_0)$$

is a Thom-class also.

Definition 2.4.4 Thom – Space Let $V \rightarrow X$ be a vector bundle with metric, then consider the associated disc and sphere bundles

$$DV := \{v \in V \mid |v| \leq 1\}, \quad SV := \{v \in V \mid |v| = 1\}.$$

The Thom-space of V is defined as

$$\text{Th}(V) := DV/SV \cong \mathbb{P}(V \oplus \mathbb{R})/\mathbb{P}V,$$

where $\mathbb{P}V$ is the projective bundle of V .

Remark 2.4.5

- If X is compact, then the Thom-space coincides with the one-point-compactification:
 $\text{Th}(V) = V^+$.
- We have

$$\tilde{H}^*(\text{Th}(V)) \cong H^*(V, V_0), \quad \text{Th}(V \times W) \cong \text{Th}(V) \wedge \text{Th}(W).$$

Definition 2.4.6 Compatible Thom – Class Let $V \rightarrow X$ be an oriented vector bundle. A Thom-class τ is called compatible with the orientation, iff for all $x \in X$ and every orientation preserving isomorphism $\varphi: \mathbb{R}^n \rightarrow V^x$, we have that

$$H^n(V, V_0; R) \xrightarrow{j_x^*} H^n(V^x, V_0^x; R) \xrightarrow{\varphi^*} H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; R), \quad \tau \mapsto \mu^n.$$

Remark 2.4.7 If $V^n \rightarrow X$ and $W^m \rightarrow Y$ are oriented vector bundles with compatible Thom-classes τ_V, τ_W , then $(-1)^{nm} \tau_V \times \tau_W$ is a compatible Thom-class for $V \times W$.

Theorem 2.4.8 Let $V \rightarrow X$ be a vector bundle of rank n , then set $\mathcal{H}_V := \coprod_{x \in X} H^n(V^x, V_0^x; R)$ (which has a unique topology that turns it into a bundle (of discrete abelian groups) over X).

1.) The following is an isomorphism of abelian groups:

$$\begin{aligned} \psi: H^n(V, V_0; R) &\xrightarrow{\cong} \Gamma(X; \mathcal{H}_V). \\ \alpha &\mapsto (x \mapsto j_x^* \alpha) \end{aligned}$$

2.) We have $H^k(V, V_0; R) = 0$ for all $k < n$.

Proof:

The proof follows a similar pattern as the proof of the Leray-Hirsch theorem.

1.) ψ is natural w.r.t. bundle maps. We have $\mathcal{H}_{f^*X} = f^* \mathcal{H}_V$ and the following commutes for $f: Y \rightarrow X$

$$\begin{array}{ccc} H^n(V, V_0; R) & \xrightarrow{\psi} & \Gamma(X; \mathcal{H}_V) \\ \downarrow & & \downarrow \\ H^n(f^*V, f^*V_0; R) & \xrightarrow{\psi} & \Gamma(Y; \mathcal{H}_{f^*V}) \end{array}$$

2.) Let $V = X \times \mathbb{R}^n$, then $\mathcal{H}_V \cong X \times H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ and the following commutes

$$\begin{array}{ccc} H^n(X \times (\mathbb{R}^n, \mathbb{R}^n - 0)) & \xrightarrow{=} & H^0(X) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \\ \downarrow \psi & & \downarrow = \\ \Gamma(\mathcal{H}_V) = H^0(X, H^n(\mathbb{R}^n, \mathbb{R}^n - 0)) & \xrightarrow{=} & H^0(X) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \end{array}$$

where we have used the Künneth theorem and $\Gamma(X; \mathcal{H}_V) = C^0(X, H^n(\mathbb{R}^n, \mathbb{R}^n - 0))$. So the theorem holds for trivial bundles.

- 3.) We now run a Mayer-Vietoris argument. Let $X = A \cup B$ with A, B open. We assume that the theorem holds for $A, B, A \cap B$. We get the commutative diagram

$$\begin{array}{ccccccc} \cdots 0 & \longrightarrow & H^n(V, V_0) & \longrightarrow & H^n(V|_A, V_0|_A) \oplus H^n(V|_B, V_0|_B) & \longrightarrow & H^n(V|_{A \cap B}, V_0|_{A \cap B}) \\ \downarrow = & & \downarrow (i) & & \downarrow \cong & & \downarrow \cong \\ \cdots 0 & \longrightarrow & \Gamma(\mathcal{H}_V) & \longrightarrow & \Gamma(A, \mathcal{H}_V) \oplus \Gamma(B, \mathcal{H}_V) & \longrightarrow & \Gamma(A \cap B, \mathcal{H}_V). \end{array}$$

The upper row is the Mayer-Vietoris sequence, and we have used that $H^{n-1}(V|_{A \cap B}, V_0|_{A \cap B}) = 0$ by assumption. The exactness of the bottom sequence is clear. With the 5-lemma we get that (i) is also an isomorphism. As in the proof of Theorem 2.3.3, we conclude that the statement is true for bundles over CW complexes.

□

Theorem 2.4.9 *Let $V \rightarrow X$ be a vector bundle of rank n , then*

- 1.) *if V is oriented, then there is exactly one compatible Thom-class $\tau_V \in H^*(V, V_0; \mathbb{Z})$,*
- 2.) *if V is arbitrary, then there is exactly one compatible Thom-class $\tau_V \in H^*(V, V_0; \mathbb{Z}_2)$.*

Proof:

- 2.) For $R = \mathbb{Z}/2$ we have $H^n(V^x, V_0^x, \mathbb{Z}/2) = \mathbb{Z}/2$, then $H_X(V, V_0, \mathbb{Z}/2) \rightarrow X$ is a $\mathbb{Z}/2$ group-bundle (not a principal bundle!) and thus trivial. Considering the isomorphism in (2.4.8):

$$\psi : H^n(V, V_0; R) \xrightarrow{\cong} \Gamma(\mathcal{H}_V; R), \quad \alpha \mapsto (x \mapsto j_x^* \alpha)$$

it follows that α is a Thom class, iff $\psi(\alpha)(x)$ is a generator for all $x \in X$, but in a $\mathbb{Z}/2$ there is only one such generator.

- 1.) In the bundle $H^n(V, V_0; R)$ we have the subbundle \mathcal{U}_V of all generators, which is a $\mathbb{Z}/2$ principal bundle and thus not necessary trivial. Considering the orientation bundle $\text{Or}(V)$ we have the isomorphism

$$\begin{array}{ccc} \text{Or}(V) & \xrightarrow{\cong} & \mathcal{U}_V, \\ o_X & \mapsto & \varphi_x^* u_n \end{array}$$

where $\varphi_x : V^x \rightarrow \mathbb{R}^n$ is orientation preserving.

□

Definition 2.4.10 Euler – Class *Let $\pi : V \rightarrow X$ be a vector bundle of rank n and $s : (X, \emptyset) \rightarrow (V, V - 0)$ be the zero section. We define*

- 1.) $e(V) := s^* \tau_V \in H^n(X, \mathbb{Z}/2)$ *the mod 2 - Euler-class.*
- 2.) *If V is oriented, we define the Euler-class $e(V) := s^* \tau_V \in H^n(X, \mathbb{Z})$.*

Theorem 2.4.11 Properties of the Euler – Class *Let V, V_1, V_2 be vector bundles over X of rank n, n_1 and n_2 respectively. Let $f : Y \rightarrow X$ be a continuous function, then the following hold*

- 1.) $e(f^* V) = f^*(e(V)),$
- 2.) $e(V_1 \oplus V_2) = (-1)^{n_1 n_2} e(V_1) \cup e(V_2),$
- 3.) *if there is a nowhere vanishing section $z : X \rightarrow V$, then $e(V) = 0$,*

4.) if $n = \text{rank}(V)$ is odd, then $2e(V) = 0$ and thus $e(V_1 \oplus V_2) = e(V_1) \cup e(V_2)$ is always true.

Proof:

- 1.) Follows from naturality of the Thom-class τ_V .
- 2.) Follows from (2.4.7).
- 3.) Since two sections are always homotopic, we have that z is homotopic to the zero section s , so

$$e(V) = s^* \tau_V = z^* \tau_V.$$

Since z factors over (X, X) :

$$\begin{array}{ccc} (X, \emptyset) & \xrightarrow{z} & (V, V_0) \\ \downarrow & \nearrow & \\ (X, X) & & \end{array}$$

and $H^n(X, X) = 0$, we have $e(V) = 0$.

- 4.) We denote V with opposite orientation by V^- and consider the orientation reversing map $f(v) = -v$. With the commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \downarrow & \nearrow & \\ X & & \end{array}$$

we have that $V \cong V^-$ as oriented bundles and thus $e(V) = e(V^-) = -e(V)$.

□

Lemma 2.4.12 Thom – Classes of Submanifolds *Let M^m be an oriented manifold and $N^n \subseteq M^m$ an oriented submanifold. Let $K \subseteq N$ be compact and $j : N \hookrightarrow M$ the inclusion. We consider a tubular neighborhood $U \supseteq N$ with the projection $\pi : U \rightarrow N$ and set $V := \pi^{-1}(N \setminus K)$. With the normal bundle ν_N^M , due to excision, it holds that*

$$\begin{array}{ccc} H^{m-n}(U, U \setminus N) & \xrightarrow{\cong} & H^{m-n}(M, M \setminus N), \\ \tau_{\nu_N^M} & \mapsto & \tau_N^M \end{array}$$

with $[M]_K \in H_m(M, M \setminus K)$ and $j_* : H_n(N, N \setminus K) \rightarrow H_n(M, V)$, we have

$$\tau_N^M \cap [M]_K = (-1)^{n(m-n)} j_* [N]_K \in H_n(M, V)$$

If both M and N are compact and $N = K$, then

$$\tau_N^M \cap [M]_K = (-1)^{m(m-n)} j_* [N].$$

We can express this loosely by saying that the fundamental class of N and the Thom-class of ν_N^M are Poincaré dual.

Proof: We can restrict to the following case

- $M = U$ due to excision,
- $K = \{*\}$ because of the definition of the fundamental class,

- $N = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^m$ and $K = \{0\}$.

This gives us

$$\tau_N = 1 \times u^{m-n} \in H^{m-n}(\mathbb{R}^n \times (\mathbb{R}^{m-n}, \mathbb{R}_0^{m-n}))$$

and we have $[M] = u_1 \times \cdots \times u_1 \in H_m(\mathbb{R}^m, \mathbb{R}_0^m)$, $j_*[N] = u_1 \times \cdots \times u_1 \times 1 \in H_n((\mathbb{R}^n, \mathbb{R}_0^n) \times \mathbb{R}^{m-n})$, which gives us

$$\begin{aligned} \tau_N^M \cap [M] &= (1 \times u^{m-n}) \cap (u_1 \times \cdots \times u_1) \\ &= (-1)^{n(m-n)} (1 \cap \underbrace{(u_1 \times \cdots \times u_1)}_n) \times (u^{m-n} \cap \underbrace{(u_1 \times \cdots \times u_1)}_{m-n}) \\ &= (-1)^{n(m-n)} \underbrace{(u_1 \times \cdots \times u_1)}_n \times 1 = (-1)^{n(m-n)} j_*[N] \end{aligned}$$

□

Corollary 2.4.13 *Let M^m be oriented and compact, $\pi : V \rightarrow M$ a vector bundle of rank k and $s : M \rightarrow V$ a section that is transversal to the zero section: $s \pitchfork 0$. Then $K^{m-k} := s^{-1}(0) \subseteq M$ is a submanifold with $\nu_Z^M = V|_Z$. It follows that*

$$j_*[Z] = (-1)^{(m-k)k} \tau_Z^M \cap [M] = (-1)^{(m-k)k} e(Z) \cap [M].$$

Corollary 2.4.14 Poincaré – Hopf *With the same conditions as in the previous corollary, but $k = m$ and thus $[Z] \in \Omega_0 \cong \mathbb{Z}$, it holds that*

$$\langle e(V), [M] \rangle = e(V) \cap [M] = j_*[Z] = \text{Eul}(V)$$

Theorem 2.4.15 Topological Gauss – Bonnet *Let M^n be a closed oriented manifold, then*

$$\chi(M) = \langle e(TM), [M] \rangle = \text{Eul}(TM)$$

Proof:

- If n is odd, then $2e(TM) = 0$ and thus $\langle e(TM), [M] \rangle = 0$, but also $\chi(M) = 0$, due to Poincaré duality and the universal coefficient theorem, i.e.

$$\dim H_k(M; \mathbb{Q}) = \dim H^{n-k}(M; \mathbb{Q}) = \dim H_{n-k}(M; \mathbb{Q}).$$

- We thus consider n even and chose \mathcal{B} a basis of $H_*(M; \mathbb{Q})$, where all $\alpha \in \mathcal{B}$ are homogeneous and $\{\alpha^\# \mid \alpha \in \mathcal{B}\}$ is a basis for $H^*(M; \mathbb{Q})$ with

$$\langle \alpha^\# \cup \beta, [M] \rangle = \delta_{\alpha, \beta} \quad \forall \alpha, \beta \in \mathcal{B}.$$

Note that such elements $\alpha^\# \in H^*(M; \mathbb{Q})$ exist, due to Poincaré duality. Now we consider the diagonal

$$\Delta := \{(x, x) \in M \times M\} \subset M \times M, \quad T\Delta \{(v, -v) \in TM \times TM\} \cong TM,$$

with the Thom-class $\tau = \tau_\Delta^{M \times M} \in H^n(M \times M; \mathbb{Q})$ of $\nu_\Delta^{M \times M}$. By the Künneth theorem there exists unique $c_{\gamma\delta} \in \mathbb{Q}$ such that

$$\tau = \sum_{\gamma, \delta} c_{\gamma\delta} (\gamma^\# \times \delta), \quad \gamma, \delta \in \mathcal{B}.$$

We now see that

$$\begin{aligned} \langle (\alpha^\# \times \beta) \cup \tau, [M \times M] \rangle &= \langle \alpha^\# \times \beta, \tau \cap [M \times M] \rangle = (-1)^n \langle \alpha^\# \times \beta, j_*[M] \rangle \\ &= \langle j^*(\alpha^\# \times \beta), [M] \rangle = \langle \alpha^\# \cup \beta, [M] \rangle = \delta_{\alpha, \beta}, \end{aligned}$$

and analogously get

$$\langle (\alpha^\# \times \beta) \cup \tau, [M \times M] \rangle = \sum_{\gamma, \delta} c_{\gamma\delta} \langle (\alpha^\# \times \beta) \cup (\gamma^\# \times \delta), [M \times M] \rangle = c_{\beta\alpha} (-1)^{|\beta|},$$

which gives $c_{\alpha\beta} = (-1)^{|\alpha|} \delta_{\alpha\beta}$ and thus $\tau = \sum_{\alpha} (-1)^{|\alpha|} (\alpha^\# \times \alpha)$, which in turn lets us calculate

$$\begin{aligned} \langle e(TM), [M] \rangle &= \langle j^* \tau, [M] \rangle = \sum_{\alpha} (-1)^{|\alpha|} \langle j^* (\alpha^\# \times \alpha); [M] \rangle \\ &= \sum_{\alpha} (-1)^{|\alpha|} \langle \alpha^\# \cup \alpha; [M] \rangle = \sum_{\alpha} (-1)^{|\alpha|} = \chi(M) \end{aligned}$$

□

2.5 Fiber Bundles

In the following, let G be a topological group, X, F be topological spaces and F be a left G -space.

Definition 2.5.1 Fiber Bundle with Structure Group *A fiber bundle over X with structure group G and fiber F is a triple*

$$(E \xrightarrow{\pi} X, P \xrightarrow{q} X, \varphi)$$

- $\pi : E \longrightarrow X$ is a fiber bundle,
- $q : P \longrightarrow X$ is a G -principal fiber bundle,
- $\varphi : P \times_G F \xrightarrow{\cong} E$ is an isomorphism of fiber bundles.

Remark 2.5.2 Alternative Definition *In the literature one usually finds the above definition in terms of an open, trivializing cover $\{U_i\}_{i \in I}$ of X with trivializations*

$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times F,$$

giving rise to trivialization changes

$$\begin{aligned} \varphi_{ij} := h_i \circ h_j^{-1} : (U_i \cap U_j) \times F &\longrightarrow (U_i \cap U_j) \times F \\ (x, f) &\longmapsto (x, g_{ij}(x)f) \end{aligned}$$

with the functions $g_{ij} : (U_i \cap U_j) \rightarrow G$ fulfilling the cocycle condition $g_{ij}g_{jk} = g_{ik}$. Definition 2.5.1 agrees with that provided $G \subset \text{Homeo}(F)$, i.e. G acts faithfully on F .

Example 2.5.3

- $G = Gl_n(\mathbb{R})$, $F = \mathbb{R}^n$ gives a vector bundle.
- $G = O(n)$, $F = \mathbb{R}^n$ gives a riemannian vector bundle.
- G and F discrete gives a covering.
- $G = \text{Homeo}(F)$ with KO -topology (here we need X to be compact, so that G is a topological group), F compact and metrizable gives a fiber bundle with fiber F .
- $G = \text{Diff}(X)$ with C^∞ -topology and X a compact manifold gives a smooth bundle with fibre X .

Theorem 2.5.4 Ehresmann's Fibration Theorem *Let E, X be manifolds, $f : E \rightarrow X$ be a smooth, proper submersion, then $f : E \rightarrow X$ is a fiber bundle with structure group $\text{Diff}(F)$ and fiber F .*

Remark 2.5.5 Classifying Space *We define*

$$\text{Prin}_G(X) := \{\pi : P \rightarrow X \mid \pi \text{ principal } G \text{ bundle}\}/\text{Iso}$$

*which gives the contravariant functor $\text{Prin}_G(-) : \text{Top} \rightarrow \text{Set}$. Since $f_0 \cong f_1$ gives $f_0^*P \cong f_1^*P$, the functor $\text{Prin}_G(-)$ in fact factorizes over HoTop :*

$$\text{Prin}_G(-) : \text{HoTop} \rightarrow \text{Set}.$$

Now the question is, if there is a G -bundle $\pi : E \rightarrow B$, such that

$$[-, B] \rightarrow \text{Prin}_G(-), \quad f \mapsto f^*E$$

is a bijection. The answer to this question is affirmative, if we restrict ourselves to base spaces X that are CW-complexes, and is given by the so called classifying space.

2.6 Homotopy Invariance of Fiber Bundles

Theorem 2.6.1 *Let X be paracompact and $P \rightarrow X \times [0, 1]$ a G -principal bundle. Consider $r : X \times [0, 1] \rightarrow X \times [0, 1]$, $r(x, t) := (x, 1)$, then there is a bundle map*

$$\begin{array}{ccc} P & \xrightarrow{R} & P \\ \downarrow & & \downarrow \\ X \times [0, 1] & \xrightarrow{r} & X \times [0, 1] \end{array}$$

with $R|_{P|_{X \times 1}} = \text{id}$.

Before we come to the proof of the theorem, let us state a few corollaries.

Corollary 2.6.2 *Let X, P, G be as in the theorem and $P_t := P|_{X \times \{t\}}$. Then there is a bundle isomorphism*

$$\begin{array}{ccc} P_1 \times [0, 1] & \xrightarrow{\varphi} & P \\ \downarrow & & \downarrow \\ X \times [0, 1] & \xrightarrow{\text{id}} & X \times [0, 1] \end{array}$$

with $\varphi|_{P_1 \times [0, 1]} = \text{id}$.

Proof: $P_1 \times [0, 1] = r^*P$

□

Corollary 2.6.3 *With X, P, G as in the theorem, it holds that $P_0 \cong P_1$.*

Proof: $P_0 \subseteq P \xrightarrow{R} P$ is a bundle map over $x \mapsto (x, 1)$ and thus $P_0 \cong P_1$.

□

Corollary 2.6.4 *Let $f_0, f_1 : X \rightarrow Y$ be homotopic and $P \rightarrow Y$ a G -principal bundle, then*

$$f_0^*P \cong f_1^*P.$$

Remark 2.6.5 *The proof of Theorem 2.6.1 is not very intuitive, and so we give a few examples to get a feeling of what is involved.*

1.) Let $\pi : E \rightarrow X \times [0, 1]$ a Galois cover, then

$$\begin{array}{ccc} E_0 \times 0 & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \pi \\ E_0 \times [0, 1] & \xrightarrow{\pi_0 \times \text{id}} & X \times [0, 1] \end{array}$$

where the diagonal map exists and is unique because of the homotopy lifting theorem.

2.) Vector bundles on compact spaces. Let $V \rightarrow X \times [0, 1]$ be a vector bundle and X a compact Hausdorff space. Let further $\text{pr} : X \times [0, 1] \rightarrow X$ be the projection onto the first component and $\iota_t : X \rightarrow X \times [0, 1]$ the embedding $\iota_t(x) := (x, t)$ for which we define $V_t := \iota_t^* V$, then it holds that

$$V|_{X \times \{t\}} \cong (\text{pr}^* V_t)|_{X \times \{t\}}$$

By the Tietze extension theorem, there is a neighborhood $X \times \{t\} \subseteq U \subseteq X \times I$ such that

$$V|_U \cong (\text{pr}^* V_t)|_U.$$

Since X is compact there is a $\varepsilon > 0$, such that $X \times (t - \varepsilon, t + \varepsilon) \subseteq U$ and thus

$$V_t \cong V_u, \quad \text{for } |u - t| < \varepsilon.$$

Since I is compact and connected, we get $V_0 \cong V_1$. (Lebesgue Lemma).

3.) Smooth fiber bundles with compact fibers. Let M be a manifold and $f : X \rightarrow M \times I$ a smooth fiber bundle with compact fiber. ∂_t is a vector field on $M \times I$. There is a vector field V on X with

$$(T_x f)(V_x) = \partial_t|_{f(x)}.$$

In the following the fact that f is proper is essential. We take the flow φ_t of V and get the maps

$$\begin{array}{ccc} X & \xrightarrow{\varphi_t} & X \\ \downarrow & & \downarrow \\ M \times \mathbb{R} & \xrightarrow{m} & M \times \mathbb{R} \end{array} \qquad \begin{array}{ccc} X_0 \times [0, 1] & \xrightarrow{\phi} & X \\ \downarrow f & & \downarrow f \\ M \times [0, 1] & \xrightarrow{\text{id}} & M \times [0, 1] \end{array}$$

with $m(x, s) = (x, s + t)$ and $\phi(x, t) := \varphi_t(x)$.

4.) Smooth G -principal bundle for a Lie group G . A connection ω on P is a G -equivariant splitting $\omega : p^* TM \rightarrow TP$ of

$$0 \rightarrow P \times \mathfrak{g} \rightarrow TP \xrightarrow{dp} p^* TM \rightarrow 0$$

such that $\omega(p^* TM) \subset TP$ is a subbundle. A connection defines a unique lift of any vector field on M to a G -equivariant vector field on P . As in the previous example, the flow of this lifted vector field gives the desired isomorphism.

Proof: Of the theorem

- Let $(x, t) \in X \times [0, 1]$, then there is a cover $U_{x,t} \times I_{x,t}$ with $x \in U_{x,t}$ and $t \in I_{x,t}$, such that $P|_{U_{x,t} \times I_{x,t}}$ is trivial. Since $[0, 1]$ is compact, we find an open $U \subseteq X$ and a partition $0 = t_0 \leq \dots \leq t_r = 1$, such that $P|_{U \times [t_i, t_{i+1}]}$ is trivial. The number r can be reduced by the following property. Let

$$g : P|_{U \times [a,b]} \rightarrow U \times [a,b] \times G, \quad h : P|_{U \times [b,c]} \rightarrow U \times [b,c] \times G$$

be trivializations, then there is a function $f : U \rightarrow G$, such that for $x \in P|_{U \times [a,b]}$, we have

$$g(h^{-1}(u, b, xf(b))) = (u, b, x).$$

Now we can change the trivialization h using f to obtain a trivialization of P over $U \times [a, c]$.

- There is an open cover $(U_j)_{j \in \mathbb{N}}$ of X such that $P|_{U_j \times [0,1]}$ is trivial. Let λ_j be a partition of unity subordinate to $(U_j)_{j \in \mathbb{N}}$, and set

$$\lambda(x) := \max_{j \in \mathbb{N}} \lambda_j(x), \quad \mu_j(x) := \lambda_j(x)/\lambda(x), \quad \text{supp}(\mu_j) \subseteq U_j, \quad \max_{j \in \mathbb{N}} \mu_j(x) = 1.$$

We now define

$$\begin{aligned} r_j : X \times [0, 1] &\longrightarrow X \times [0, 1]. \\ (x, t) &\longmapsto (x, \max(\mu_j(x), t)) \end{aligned}$$

There is a bundle map R_j over r_j :

$$\begin{array}{ccc} P|_{U_j \times [0,1]} & \xrightarrow{R_j} & P|_{U_j \times [0,1]} \\ \downarrow \cong & & \downarrow \cong \\ U_j \times [0, 1] \times G & \xrightarrow{(r_j, \text{id}_G)} & U_j \times [0, 1] \times G \end{array}$$

and, outside of $\text{supp}(\lambda_j)$, we have that R_j is the identity. We can thus extend R_j to all of P . Now, due to the local finiteness of the partition of unity, the following expressions are well defined and give the maps stated in the theorem

$$\begin{aligned} R &:= \cdots \circ R_j \circ R_{j-1} \circ \cdots \circ R_2 \circ R_1, \\ r &:= \cdots \circ r_j \circ r_{j-1} \circ \cdots \circ r_2 \circ r_1. \end{aligned}$$

□

2.7 Classification of G -Bundles

In the following let G be a topological group and let $P \rightarrow X$, $E \rightarrow B$ be G -principal bundles. Let E be weakly contractible (i.e. $\pi_i(E) = 0 \forall i$). In this section we want to show that

$$\text{Prin}_G(X) \cong [X; BG],$$

where we restrict ourself to the case where X is a CW-complex. The proof of the general case can be found in [2], chapter 14.

Theorem 2.7.1 *Let X be a CW-complex, $A \subseteq X$ a subcomplex, $P \rightarrow X$ and $E \rightarrow B$ be G -principal bundles with $E \cong *$, then for each bundle map*

$$\begin{array}{ccc} P|_A & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

there is an extension to a bundle map $g : P \rightarrow E$, such that $g|_{P|_A} = f$.

Proof: The following set is partially ordered by inclusion:

$$\{(Y, g) \mid A \subseteq Y \subseteq X, Y \text{ complex } g : P|_Y \rightarrow E \text{ bundle map } g|_{P|_A} = f\},$$

and with Zorn's lemma there exists a maximal element. It now suffices to study the case where we just add one cell, i.e. X is the pushout in

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\psi} & X \end{array}$$

Now given the bundle map $f : P_Y \rightarrow E$, we recall the homotopy theorem and get

$$\pi^* P \cong D^n \times G.$$

Under this isomorphism, the map f becomes

$$S^{n-1} \times G \xrightarrow{\tilde{f}} E,$$

which is G -equivariant and thus determined by $\tilde{f}|_{S^{n-1} \times 1} \rightarrow E$. Now finally due to $\pi_{n-1}(E) = 0$, we see that we can extend \tilde{f} and thus also f . □

Corollary 2.7.2 *Let X be a CW-complex, $P \rightarrow X$ and $E \rightarrow B$ be G -principal bundles where E is weakly contractible. Then $\text{map}_G(P; E)$ is weakly contractible.*

Proof: For a space Z to be weakly contractible, it is equivalent, that for all $k \geq 0$ and every map $S^{k-1} \rightarrow Z$ factors over D^k :

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & Z \\ \downarrow & \nearrow & \\ D^k & & \end{array}$$

For $Z = \text{map}_G(P; E)$, this amounts to

$$\begin{array}{ccc} P \times S^{k-1} & \longrightarrow & E \\ \downarrow & \nearrow & \\ P \times D^k & & \end{array}$$

which is solved by the last theorem. □

Corollary 2.7.3 *Let X be a CW-complex, $P \rightarrow X$ and $E \rightarrow B$ be G -principal bundles where E is weakly contractible. Then*

$$\exists f : X \longrightarrow B : f^* B \cong P,$$

and f is unique up to homotopy.

Proof: The equivariance of the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f/G} & B \end{array}$$

gives $f^* E \cong P$. It only remains to show uniqueness. Let f_0, f_1 be two such maps, then define $f = (f_0, f_1) : X \times \{0, 1\} \rightarrow E$ and we have the isomorphism

$$\varphi : P \times \{0, 1\} \longrightarrow f^* E \longrightarrow E.$$

So we have a bundle $P \times [0, 1] \rightarrow X \times [0, 1]$ and the G -map φ , thus the extension of f to $X \times [0, 1]$ exists and we have $f_0 \cong f_1$.

□

Definition 2.7.4 Universal G – Bundle A G -principal bundle $E \rightarrow B$ with B a CW-complex and $E \simeq *$ is called universal G -bundle. For such a bundle we write

$$EG := E, \quad BG := B.$$

Example 2.7.5

1.) Let $G = \Sigma_n$ be the symmetric group, and

$$F^n(\mathbb{R}^m) := \{(x_1, \dots, x_n) \in (\mathbb{R}^m)^n \mid x_i \neq x_j, \text{ for } i \neq j\}$$

called the ordered configuration space, which has the action

$$\Sigma_n \times F^n(\mathbb{R}^m) \longrightarrow F^n(\mathbb{R}^m), \quad (\sigma, (x_1, \dots, x_n)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

and thus gives rise to the unordered configuration space $C^n(\mathbb{R}^m) := F^n(\mathbb{R}^m)/\Sigma_n$. Together they form the Σ_n -principal bundle

$$F^n(\mathbb{R}^m) \longrightarrow C^n(\mathbb{R}^m).$$

The configuration space has vanishing lower homotopy groups: $\pi_i(F^n(\mathbb{R}^m)) = 0$ for $i < m - 1$ by Theorem 1.1.4, thus the direct limit

$$F^n(\mathbb{R}^\infty) := \varinjlim_m F^n(\mathbb{R}^m)$$

is weakly contractible: $\pi_i(F^n(\mathbb{R}^\infty)) = 0 \forall i$ and we have

$$E\Sigma_n \cong F^n(\mathbb{R}^\infty), \quad B\Sigma_n \cong C^n(\mathbb{R}^\infty) = F^n(\mathbb{R}^\infty)/\Sigma_n.$$

Let now $\pi : Y \rightarrow X$ be any n -fold covering (i.e. a fiber bundle with structure group Σ_n and fiber $\underline{n} = \{1, \dots, n\}$), then considering

$$F^n(\mathbb{R}^\infty) = \{f : \underline{n} \rightarrow \mathbb{R}^\infty \mid f \text{ injective}\}, \quad C^n(\mathbb{R}^\infty) = \{S \subseteq \mathbb{R}^\infty \mid \#S = n\},$$

and an embedding j :

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \times \mathbb{R}^\infty \\ \downarrow \pi & \nearrow \text{pr}_1 & \\ X & & \end{array}$$

we can define $f_j : X \rightarrow C^n(\mathbb{R}^\infty)$, $f_j(z) := j(\pi^{-1}(z))$, which gives

$$Y \cong (f_j^* F^n(\mathbb{R}^\infty)) \times_{\Sigma_n} \underline{n}.$$

2.) Let $G = Gl_n(\mathbb{R})$. Recall the definition of the Stiefel manifolds

$$St_n^m := \{f \in \text{Mat}_{m,n} \mid f \text{ injective}\},$$

for the Stiefel manifolds it also holds that $\pi_i(St_n^m) = 0$, $i < m - n$ by an application of Theorem 1.1.4 and there is a $Gl_n(\mathbb{R})$ -action

$$Gl_n(\mathbb{R}) \times St_n^m \longrightarrow St_n^m, \quad (g, f) \mapsto f \circ g,$$

by which we can define the Grassmann manifolds $Gr_n^m := St_n^m/Gl_n(\mathbb{R})$. Again we work with their direct limits

$$St_n^\infty := \varinjlim_m St_n^m, \quad Gr_n^\infty = St_n^\infty/Gl_n(\mathbb{R}) = \varinjlim_m Gr_n^m,$$

which form the $Gl_n(\mathbb{R})$ -principal bundle $St_n^\infty \longrightarrow Gr_n^\infty$. We also get vector bundles

$$V_{n,m} := St_n^m \times_{Gl_n(\mathbb{R})} \mathbb{R}^n \longrightarrow Gr_n^m$$

whose total space can also be written as $V_{n,m} = \{(V, v) \mid V \in Gr_n^m, v \in V\}$.

For a vector bundle $\pi : V \rightarrow X$ of rank n over a paracompact base X , we take the local trivializations

$$\varphi_i : U_i \times \mathbb{R}^n \xrightarrow{\cong} \pi^{-1}(U_i),$$

with the open cover $(U_i)_{i \in \mathbb{N}}$, for which we find a subordinate partition of unity $(\lambda_i)_{i \in \mathbb{N}}$, which induces a bundle map

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & Gr_n^\infty \times \mathbb{R}^\infty \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Gr_n^\infty \end{array}$$

with $\varphi(v) := (\pi(v), (\lambda_1(\pi(v))\varphi_1^{-1}(v), \lambda_2(\pi(v))\varphi_2^{-1}(v), \dots))$ and $f(x) := \varphi(\pi^{-1}x)$. By the Gauss map we now have $f^*V_{n,\infty} \cong V$ and thus conclude

$$St_n^\infty \cong E(Gl_n(\mathbb{R})), \quad Gr_n^\infty \cong B(Gl_n(\mathbb{R})).$$

3.) We take any closed subgroup $G \subset Gl_n(\mathbb{R})$. Then G is a Lie group and

$$Gl_n(\mathbb{R}) \longrightarrow Gl_n(\mathbb{R})/G$$

is a smooth G -principal bundle. Analogous to the previous case, we have

$$\boxed{BG \cong St_n^\infty/G = (St_n^\infty \times_{Gl_n(\mathbb{R})} (GL_n(\mathbb{R})/G))}$$

4.) The previous examples are folklore; this one I found in the book [3]. We consider the diffeomorphism group $G = \text{Diff}(M)$ with C^∞ -topology of a compact manifold M . We consider the Fréchet submanifold of embeddings $\text{Emb}(M, \mathbb{R}^m) \subseteq C^\infty(M, \mathbb{R}^m)$, and the action

$$\text{Diff}(M) \times \text{Emb}(M, \mathbb{R}^m) \longrightarrow \text{Emb}(M, \mathbb{R}^m), \quad (f, j) \mapsto j \circ f.$$

We will check that this gives the $\text{Diff}(M)$ -principal bundle

$$\text{Emb}(M, \mathbb{R}^m) \longrightarrow \text{Emb}(M, \mathbb{R}^m)/\text{Diff}(M).$$

Fix a $j_0 : M \hookrightarrow \mathbb{R}^m$ and w.l.o.g. let this be the inclusion. We consider a tubular neighborhood $U \xrightarrow{r} M \hookrightarrow \mathbb{R}^m$ and set

$$\mathcal{U} := \{j : M \rightarrow \mathbb{R}^m \mid j(M) \subseteq U, r \circ j : M \rightarrow M \text{ diffeomorphism}\} \subseteq \text{Emb}(M, \mathbb{R}^m).$$

$\text{Diff}(M) \subseteq C^\infty(M, M)$ is open (note that for the homeomorphisms $\text{Homeo}(M) \subseteq C^0(M, M)$ is not open, since being a homeomorphism is not an open condition). The mapping

$$\mathcal{U} \longrightarrow \mathcal{U}, \quad j \mapsto j \circ (rj)^{-1}$$

is $\text{Diff}(M)$ -equivariant, since for $h \in \text{Diff}(M)$ also $j \circ h \mapsto j \circ (rj)^{-1}$, and thus induces

$$\psi : \mathcal{U}/\text{Diff}(M) \longrightarrow \mathcal{U}.$$

Its inverse is $\psi^{-1}(j) = \text{Im}(j)$, and we indeed get local trivializations

$$\begin{aligned} \mathcal{U}/\text{Diff}(M) \times \text{Diff}(M) &\longrightarrow \mathcal{U} \\ ([j], h) &\longmapsto \psi([j]) \circ h \end{aligned}$$

So just as in the finite dimensional cases, we get

$$\boxed{B(\text{Diff}(M)) \cong \text{Emb}(M, \mathbb{R}^\infty)/\text{Diff}(M)}$$

We now turn to the general construction of classifying spaces. There are several possibilities; we use the one introduced by Milnor.

Definition 2.7.6 Join *Let J be a set and X_j be a topological space with $j \in J$. The join of $\{X_j\}_{j \in J}$ is defined as*

$$*_j \in J X_j := \{(t_j, x_j)_{j \in J} \mid t_j \in [0, 1], x_j \in X_j, \sum_j t_j = 1 \text{ and only finitely many } t_j \neq 0\} / \sim,$$

with

$$(t_j, x_j)_{j \in J} \sim (t_{j'}, x_{j'})_{j' \in J}, \quad \Leftrightarrow \quad \forall j \in J : t_j = t_{j'} \text{ and if } t_j \neq 0, \text{ then } x_j = x_{j'}.$$

The join $X := *_j \in J X_j$ is given the coarsest topology such that the following projection maps are continuous

$$t_j : X \longrightarrow [0, 1], \quad x_j : t_j^{-1}(0, 1] \longrightarrow X_j.$$

Theorem 2.7.7 Milnor Construction of the Classifying Space *The following is a functor*

$$G \longmapsto (EG \rightarrow BG),$$

i.e. every topological group G has a universal G -bundle.

Proof: We shall see, that the following choices give the desired properties of the classifying space

$$EG := *_i \in \mathbb{N} G, \quad BG := EG/G,$$

where the latter arises from the free right G -action $(t_j, g_j) \cdot g := (t_j, g_j \cdot g)$ on EG .

- We need to check local triviality of $\pi : EG \rightarrow BG$. The sets $U_i := t_i^{-1}(0, 1] \subseteq EG$ are G -stable and open, which also lets $\pi(U_i) \subseteq BG$ be open. The local trivializations are

$$\begin{array}{ccc} U_i & \xrightarrow{h} & \pi(U_i) \times G \\ \downarrow \pi & \swarrow & \\ \pi(U_i) & & \end{array} \quad h((t_j, g_j)_{j \in J}) := (\pi((t_j, g_j)_{j \in J}), g_i)$$

which are G -equivariant. The inverse map is induced by a section

$$s : \pi(U_i) \longrightarrow U_i, \quad [((t_j, g_j)_{j \in J})] \mapsto (t_j, g_j g_i^{-1})_{j \in J}.$$

- $EG \cong *$ is contractible. Set $0 := (0, g)$. The two maps $EG \rightarrow EG$ that associate to $(t_n, g_n)_{n \in \mathbb{N}}$ the images

$$(0, (t_1, g_1), (t_2, g_2), \dots), \quad ((1, e), 0, 0, \dots),$$

respectively, are homotopic, via the homotopy

$$((t, e), ((1-t)t_1, g_1), ((1-t)t_2, g_2), \dots).$$

Moreover, the formula $((tt_1, g_1), ((1-t)t_1, g_1), (t_2, g_2), \dots)$ defines a homotopy H_1 from the map $(0, (t_1, g_1), (t_2, g_2), \dots)$ to the identity. In a similar way, we define homotopies H_2, H_3, \dots

$$(0, (t_1, g_1), (t_2, g_2), \dots) \xrightarrow{H_1} ((t_1, g_1), 0, (t_2, g_2), \dots) \xrightarrow{H_2} ((t_1, g_1), (t_2, g_2), 0, (t_3, g_3), \dots) \xrightarrow{H_3} \dots$$

If we reparametrize the homotopies, such that the first takes time $[0, \frac{1}{2}]$, the second $[0, \frac{1}{4}]$ and so on, then the infinite concatenation is well defined and gives a homotopy from identity to the map $(0, (t_1, g_1), (t_2, g_2), \dots)$.

□

Theorem 2.7.8

- Let $\pi : P \rightarrow X$ be a G -principal bundle and X be paracompact, then there is a countable cover $\{U_i\}_{i \in \mathbb{N}}$, with a subordinate partition of unity $\{\lambda_i\}_{i \in \mathbb{N}}$ and a family of G -equivariant maps $\{\varphi_i\}_{i \in \mathbb{N}}$, where

$$\varphi_i : P|_{U_i} \longrightarrow G, \quad \text{and} \quad (\pi, \varphi_i)\varphi_i : P|_{U_i} \xrightarrow{\cong} U_i \times G,$$

that induce the map

$$\begin{aligned} P &\longrightarrow EG. \\ p &\longmapsto (\lambda_i(\pi(p)), \varphi_i(p))_{i \in \mathbb{N}} \end{aligned}$$

- Vice versa, if $f : P \rightarrow EG$ is G -equivariant and $f^{-1}(t_j^{-1}(0, 1])$ G -stable and open in P , then $V_j := \pi(f^{-1}(t_j^{-1}(0, 1]))$ is an open cover of X and the following commutes

$$\begin{array}{ccc} P & \xrightarrow{f} & EG & \xrightarrow{t_j} & \mathbb{R} \\ \pi \downarrow & & & \nearrow \lambda_j & \\ X & & & & \end{array}$$

3 Characteristic Classes

3.1 Definition and Basic Properties

Characteristic classes correspond to the cohomology of BG .

Definition 3.1.1 Characteristic Class *Let G be a topological group and R a commutative ring. A characteristic class for G -bundles with coefficients in R of degree n is a natural transformation of functors*

$$c : \text{Prin}_G(-) \longrightarrow H_{\text{sing}}^n(-; R),$$

where $\text{Prin}_G(-), H_{\text{sing}}^n(-; R) : \text{Top} \rightarrow \text{Set}$.

Remark 3.1.2 Yoneda Lemma and Cohomology of BG *The Yoneda lemma gives*

$$\text{Prin}_G(-) \cong [-; BG],$$

which lets a characteristic class take values in the cohomology of BG :

$$c(EG \rightarrow BG) \in H^n(BG; R).$$

I.e. the characteristic classes form a ring, which is $H^\bullet(BG; R)$.

Remark 3.1.3 Conventions *In the following, we shall consider \mathbb{K} -vector bundles $V \rightarrow X$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. When possible, we shall treat them on a common ground and thus adopt the conventions listed in the following table.*

\mathbb{K}	Structure Group	Coefficient Ring \mathbb{F}	d	Characteristic Class
\mathbb{R}	$O(n)$	\mathbb{Z}_2	1	Stiefel – Whitney $w_i(V) \in H^i(X, \mathbb{Z}_2)$
\mathbb{C}	$U(n)$	\mathbb{Z}	2	Chern $c_i(V) \in H^{2i}(X, \mathbb{Z})$
\mathbb{H}	$Sp(n)$	\mathbb{Z}	4	Pontrjagin

Be aware however, that the Pontrjagin classes arising in the quaternionic case $\mathbb{K} = \mathbb{H}$, are not the ones one usually finds in the literature.

In the complex case, we shall make use of the complex base (v_1, \dots, v_n) , which induces the positively oriented real base $(v_1, iv_1, \dots, v_n, iv_n)$

Remark 3.1.4 Goals *The goal of this section is to calculate $H^*(X; R)$ for the following pairs*

$$(BO(n); \mathbb{Z}_2), (BU(n); \mathbb{Z}), (BSO(n); \mathbb{Z}_2), (BSO(n); \mathbb{Q}), (BO(n); \mathbb{Q}).$$

Also we shall try to better understand the way in which elements of $H^(BG; R)$ correspond to characteristic classes.*

Definition and Lemma 3.1.5 Tautological Bundle *For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we set*

$H := \{(l, v) \mid l \in \mathbb{K}P^\infty, v \in l\}$ *and the canonical projection gives us the so called tautological bundle*

$$H \longrightarrow \mathbb{K}P^\infty.$$

Its dual $L := H^$ gives the generator $x := e(L) \in H^d(\mathbb{K}P^\infty; \mathbb{F})$ of the cohomology ring*

$$H^\bullet(\mathbb{K}P^\infty; \mathbb{F}) \cong \mathbb{F}[x].$$

Definition 3.1.6 Outer Tensor Product *Given two vector bundles $V \rightarrow X, W \rightarrow Y$, we define their outer tensor product as*

$$V \boxtimes W := [\text{pr}_X^* V \otimes \text{pr}_Y^* W] \longrightarrow X \times Y.$$

The following definition of the characteristic classes is due to Dold and can be found in the book [5].

Definition 3.1.7 Chern – and Stiefel – Whitney Classes Consider the $\mathbb{K} = \mathbb{R}, \mathbb{C}$ vector bundle $V \rightarrow X$, then we have the Künneth isomorphism

$$\begin{aligned} H^{dn}(X \times \mathbb{K}P^\infty; \mathbb{F}) &\xrightarrow{\cong} \bigoplus_{k=0}^n \left[H^{dk}(X; \mathbb{F}) \otimes_{\mathbb{F}} H^{d(n-k)}(\mathbb{K}P^\infty; \mathbb{F}) \right] \\ e(V \boxtimes L) &\mapsto \sum_{k=0}^n \left[a_k(V) \otimes x^{n-k} \right] \end{aligned}$$

which defines the characteristic classes $a_k(V) \in H^{dk}(X; \mathbb{F})$, which we call

$$\begin{aligned} \mathbb{K} = \mathbb{R} : \quad w_k(V) &:= a_k(V) \quad \text{Stiefel – Whitney Classes,} \\ \mathbb{K} = \mathbb{C} : \quad c_k(V) &:= a_k(V) \quad \text{Chern Classes.} \end{aligned}$$

Setting the higher classes to zero:

$$\text{if } k > n, \text{ then } w_k(V) := 0, \quad c_k(V) := 0,$$

we define the total Stiefel-Whitney and Chern classes as

$$w(V) := \sum_{k=0}^{\infty} w_k(V), \quad c(V) := \sum_{k=0}^{\infty} c_k(V).$$

Theorem 3.1.8 Properties of the Chern – Classes Let $V \rightarrow X$ and $W \rightarrow X$ be \mathbb{R} vector bundles, then the Chern classes of V, W have the following properties:

- 1.) $f^*c(V) = c(f^*V)$ for all continuous maps $f : Y \rightarrow X$,
- 2.) $c_0(V) = 1$,
- 3.) $c(V \oplus W) = c(V)c(W)$,
- 4.) $c(L) = 1 + x$.

Remark 3.1.9 Analogous statements with analogous proofs hold for $w_i(V)$ in the real case.

Proof:

- 1.) Follows from the naturality of the Künneth isomorphism.
- 2.) Let $p \in X$, then we can consider it as a map $p : \{*\} \rightarrow X$ and it suffices to show $p^*c_0(V) = 1 \in H^0(*)$. But we know $p^*c_0(V) = c_0(p^*X) = c_0(\mathbb{C}^n \rightarrow \{*\})$, and $e(\mathbb{C}^n \boxtimes L) = e(\bigoplus_{i=1}^n L) = e(L)^n = x^n$. Now equating the coefficients gives the desired result.
- 3.) Let V have rank n and W rank m , then on one hand we have

$$e((V \oplus W) \boxtimes L) = e((V \boxtimes L) \oplus (W \boxtimes L)) = \sum_{k,l=0}^{n,m} (c_k(V) \times x^{n-k}) \cup (c_l(W) \times x^{m-l}),$$

on the other, we also have $e((V \oplus W) \boxtimes L) = \sum_{k=0}^{n+m} c_k(V \oplus W) \times x^{n+m-k}$.

- 4.) Consider $e(L \boxtimes L) \in H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ and set $x_1 := e(L \boxtimes \mathbb{C})$, $x_2 := e(\mathbb{C} \boxtimes L)$, which gives

$$e(L \boxtimes L) = c_0(L) \times x_2 + c_1(L) \times 1 = 1 \times x_2 + z x_1 \times 1, \quad z \in \mathbb{Z}.$$

Take the interchanging map $t : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, $t(x, y) := (y, x)$, then

$$t^*x_1 = x_2, \quad t^*x_2 = x_1 \quad t^*(L \boxtimes L) \cong L \boxtimes L,$$

and thus $z = 1$.

□

Remark 3.1.10

- In the literature (e.g. [4]), the above properties (1-4) are usually stated as axioms.
- We have $\mathbb{C}P^1 \cong S^2$ and $\langle c_1(H); [\mathbb{C}P^1] \rangle = -1$. This convention was introduced by Hirzebruch, in order to recover the usual Riemann-Roch formula for Riemann surfaces.

Theorem 3.1.11 Further Properties

- 5.) If V, W are line bundles, then $c_1(V) + c_1(W) = c_1(V \times W)$.
- 6.) If $\text{rank}(V) = n$, then $c_n(V) = e(V)$.

Proof:

6.) Let $z : \{*\} \rightarrow \mathbb{C}P^\infty$ be the basepoint, then

$$(1 \times z)^* e(V \boxtimes L) = (1 \times z)^* \left(\sum_{k=0}^n c_k(V) \times x^{n-k} \right) = \sum_{k=0}^n c_k(V) \times z^* x^{n-k} = c_n(V),$$

on the other hand we have $(1 \times z)^* e(V \boxtimes L) = e(V \boxtimes z^* L) = e(V)$.

5.) Let $f, g : X \rightarrow \mathbb{C}P^\infty$ be continuous with $f^* L = V, g^* L = W$, then due to naturality, it suffices to prove $c_1(L \boxtimes L) = c_1(L) \times 1 + 1 \times c_1(V)$, and we indeed have

$$c_1(L \boxtimes L) \stackrel{(6)}{=} e(L \boxtimes L) = c_1(L) \times 1 + 1 \times c_1(V).$$

□

Corollary 3.1.12 We have $\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$ giving rise to the group isomorphism

$$\begin{array}{ccc} H^2(X; \mathbb{Z}) & \cong [X, K(\mathbb{Z}, 2)] \cong [X, \mathbb{C}P^\infty] & \cong (\text{Prin}_{U(1)}(X), \otimes) \\ c_1(L) & \longmapsto & L. \end{array}$$

Similarly, in the real case

$$\begin{array}{ccc} H^1(X; \mathbb{Z}/2) & \cong [X, K(\mathbb{Z}/2, 2)] \cong [X, \mathbb{R}P^\infty] & \cong (\text{Prin}_{O(1)}(X), \otimes) \\ c_1(L) & \longmapsto & L. \end{array}$$

Corollary 3.1.13 A real line bundle $L \rightarrow X$ is trivial iff $w_1(L) = 0$. Similarly, a complex line bundle $L \rightarrow X$ is trivial iff $c_1(L) = 0$.

The following is a topological version of the technique, omnipresent in the theory of compact Lie groups, to reduce statements on Lie groups to the maximal torus.

Theorem 3.1.14 Splitting principle Let $V \rightarrow X$ be a \mathbb{K} -vector bundle of rank n , then there exists a space Q and a map $f : Q \rightarrow X$, such that

- f^* is injective in $H^*(-; \mathbb{F})$,
- $f^* V = V_1 \oplus \dots \oplus V_n$ with $\text{rank}(V_i) = 1$.

Proof: Consider the projective bundle $q : PV \rightarrow X$ and the associated tautological line bundle

$$q^*V \supseteq H_V := \{(l, v) \mid l \in PV, v \in l\} \longrightarrow PV.$$

Take the set $\mathcal{B} := \{1, c_1(H_V), c_1(H_V)^2, \dots, c_1(H_V)^{n-1}\} \subseteq H^*(PV)$, then for all $x \in X$ the inclusion $j_x : PV_x \hookrightarrow PV$ lets $j_x^*\mathcal{B}$ be a base of $H^*(PV_x = \mathbb{C}P^{n-1})$ (Leray-Hirsch). Thus we have that q^* is injective in cohomology and we conclude

$$q^* \cong H_V \oplus q^*V/H_V.$$

Induction over the rank of V now gives the claim. □

Remark 3.1.15

- We can iterate the above construction

$$\begin{array}{ccccc} \dots & \xrightarrow{q_4} & P(q_2^*(q^*V/H_V)/H_{q^*V/H_V}) & \xrightarrow{q_3} & P(q^*V/H_V) & \xrightarrow{q_2} & PV & \xrightarrow{q} & X \\ & & l_1 \subseteq l_2 \subseteq l_3 & \longmapsto & l_1 \subseteq l_2 & \longmapsto & l & & \end{array}$$

with $\dim(l_i) = i$. After n -steps, we arrive at $Q = \{(x, l_o \subseteq \dots \subseteq l_n) \mid x \in X, l_i \subseteq V_x\}$, thus each fiber is a flag manifold

$$F_k(\mathbb{C}^n) := \{0 \subseteq X_1 \subseteq \dots \subseteq K_k \subseteq \mathbb{C}^k \mid \dim(X_i) = i\},$$

on which $U(n)$ acts transitively and has the stabilizer subgroup

$$\begin{pmatrix} z_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & z_k & 0 \\ 0 & \dots & 0 & (*) \end{pmatrix}.$$

If $k = n$, then $F_n(\mathbb{C}^n) = U(n)/T(n)$, where $T(n)$ is the maximal torus.

For a $GL_n(\mathbb{C})$ -principal bundle

$$P \times_{GL_n(\mathbb{C})} F_n(\mathbb{C}^n) \longrightarrow X$$

is injective in cohomology and gives

$$\begin{array}{ccc} EU(n) \times_{U(n)} U(n)/T(n) & \longrightarrow & BU(n) \\ \downarrow = & \nearrow & \\ BT(n) & & \end{array}$$

- Let $c_1(H_V)$, then $p^*V = H_V \oplus p^*V/H_V$ and $c(p^*V) = (1+x)c(p^*V/H_V)$, which gives

$$c(p^*V/H_V) = \sum_{k=0}^{\infty} x^k c(p^*V).$$

- We always have $\#\text{Weyl}(G) = \chi(G/T)$, which is proved with Poincaré-Hopf, and thus

$$H^*(BG; \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{\#\text{Weyl}(G)}.$$

Theorem 3.1.16 Uniqueness The splitting principle implies that if \tilde{c} is a characteristic class for \mathbb{C} -vector bundles, which satisfies (1–4), then $\tilde{c} = c$.

Theorem 3.1.17 Let V be an \mathbb{R} -vector bundle of rank n , then $w_1(\Lambda^n V) = w_1(V)$.

Proof: The splitting principle shows, that it is sufficient to prove the claim for a sum of line bundles $V = V_1 \oplus \dots \oplus V_n$, and then $w(V) = \prod_{i=1}^n (1 + w_1(V_i))$, and

$$w_1(V) = w_1(V_1) + \dots + w_1(V_n) = w_1(\Lambda^1 V_1 \otimes \dots \otimes \Lambda^1 V_n) = w_1(\Lambda^n(V_1 \oplus \dots \oplus V_n)) = w_1(\Lambda^n V).$$

□

Corollary 3.1.18 Orientability A rank n vector bundle $V \rightarrow X$ is orientable, iff $w_1(V) = 0$.

3.2 Universal Classes

Lemma 3.2.1 *Let $W \rightarrow X$ be a \mathbb{C} -vector bundle, then*

$$c_k(\overline{W}) = (-1)^k c_k(W).$$

Proof: Let W be one dimensional, then $\overline{W} \cong W^*$ and

$$c_1(W^* \otimes W) = c_1(W^*) + c_1(W) = c_1(\mathbb{C}) = 0.$$

If $V = \oplus_{i=1}^n W_i$ with W_i being line bundles, then

$$c(V) = \prod_{i=1}^n (1 + c_1(W_i)), \quad c(\overline{V}) = \prod_{i=1}^n (1 - c_1(W_i))$$

and the general case follows from the splitting principle. □

Remark 3.2.2 *For the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ of an \mathbb{R} -vector bundle $V \rightarrow X$ it also follows that $c_{2l+1}(\overline{V}_{\mathbb{C}}) = -c_{2l+1}(V_{\mathbb{C}})$, and thus*

$$2c_{2l+1}(\overline{V}_{\mathbb{C}}) = 0.$$

Definition 3.2.3 Pontrjagin – Classes *The k -th Pontrjagin-class of an \mathbb{R} -vector bundle $V \rightarrow X$ is defined as*

$$p_k(V) := (-1)^k c_{2k}(V_{\mathbb{C}}) \in H^{4k}(X; \mathbb{Z}).$$

If V has rank n , we define $p_0(V) = 1$ and

$$p(V) := \sum_{k=0}^{\infty} p_k(V).$$

Theorem 3.2.4 Properties of Pontrjagin – Classes

- 1.) $p(f^*V) = f^*p(V)$
- 2.) $2p(V \oplus W) = 2p(V)p(W)$
- 3.) For $\text{rank} V = 2m$ and V oriented: $e(V)^2 = p_m(V)$.
- 4.) If we start with a \mathbb{C} -vector bundle V , then

$$p_k(V_{\mathbb{R}}) = (-1)^k \sum_{p+q=k} c_p(V)c_q(\overline{V}).$$

Remark 3.2.5 *From 4.), we see that for complex vector bundles, Pontrjagin classes do not give any new information.*

Proof:

- 1.) Follows from the naturality of the Chern classes.
- 2.)

$$\begin{aligned} 2p(V \oplus W) &= 2 \sum_{k=0}^{\infty} p_k(V \oplus W) = \sum_{k=0}^{\infty} (-1)^k 2c_{2k}(V_{\mathbb{C}} \oplus W_{\mathbb{C}}) \\ &= \sum_{k=0}^{\infty} (-1)^k 2 \sum_{p+q=2k} c_p(V_{\mathbb{C}})c_q(W_{\mathbb{C}}) \\ &= \sum_{k=0}^{\infty} (-1)^k 2 \sum_{i+j=k} c_{2i}(V_{\mathbb{C}})c_{2j}(W_{\mathbb{C}}) = 2p(V)p(W) \end{aligned}$$

3.) The isomorphism $V \oplus V \cong V_{\mathbb{C}}$ given by $(v, w) \mapsto v + iw$ is not orientation preserving. We have the basis

$$\begin{aligned} (v_1, \dots, v_{2m}, iv_1, \dots, iv_{2m}) & \text{ of } V \oplus V, \\ (v_1, iv_1, \dots, v_{2m}, iv_{2m}) & \text{ of } V_{\mathbb{C}}. \end{aligned}$$

That is, we need $2m(2m-1)/2 = m(2m-1)$ transpositions, in order to identify the two basis and we thus get the following sign:

$$e(V_{\mathbb{C}}) = (-1)^{m(2m-1)} e(V \oplus V).$$

Now we can get to the actual calculation:

$$\begin{aligned} p_m(V) &= (-1)^m c_{2m}(V_{\mathbb{C}}) = (-1)^m e(V_{\mathbb{C}}) = (-1)^{m+m(2m-1)} e(V \oplus V) \\ &= e(V \oplus V) = e(V)^2 \end{aligned}$$

where we have used $m + m(2m-1) \equiv 0 \pmod{2}$.

4.) $p(V_{\mathbb{R}}) = \sum_{k=0}^{\infty} (-1)^k c_{2k}(V_{\mathbb{C}}) = \sum_{k=0}^{\infty} (-1)^k c_{2k}(V) c_{2k}(\bar{V})$.

□

Proof:

" \Rightarrow " This is easy, due to the fact, that a section of $\Lambda^n V$ is a volume form.

" \Leftarrow " This follows immediately from Corollary 3.1.13.

□

It will be of crucial importance to understand the characteristic classes of the tangent bundle of the complex projective space. For that, we need to determine the tangent bundle of the projective space first.

Theorem 3.2.6 *Let V be a finite-dimensional \mathbb{C} -vector space, then there is a bundle isomorphism*

$$T(PV) \cong \text{Hom}(H, \underline{V})/\mathbb{C}, \quad \underline{V} := X \times V.$$

Proof: Let $n+1$ be the dimension of V . Consider the bundle maps

$$\begin{aligned} \phi : PV \times \text{End}(V) &\longrightarrow T(PV) \\ (l, f) &\longmapsto \left. \frac{d}{dt} \right|_{t=0} (e^{tf} \cdot l) \in T_l(PV) \end{aligned}$$

and

$$\text{End}(V) \xrightarrow{|_H} \text{Hom}(H, V) \longrightarrow \text{Hom}(H, V)/\text{Hom}(H, H). \tag{3}$$

given by restriction and quotienting. Consider the kernel of ϕ . If $\phi(l, f) = 0$, then $l \subset V$ is an eigenspace of f and thus f -invariant and hence contained in the kernel of ϕ . By a dimension count, ϕ is surjective and the kernels of ϕ and ψ agree. As ψ is an epimorphism, the result follows.

□

Lemma 3.2.7 *It holds that $p(T\mathbb{C}P^n) = (1 + x^2)^{n+1}$.*

Proof: We have $\mathbb{C} \oplus T\mathbb{C}P^n = \underbrace{L \oplus \cdots \oplus L}_{n+1}$ and thus $p(T\mathbb{C}P^n) = p(L)^{n+1}$, where $p(L) = (1 + x^2)$, since $c(L_{\mathbb{C}}) = (1 - x)(1 + x) = (1 - x^2)$.

□

Definition 3.2.8 Characteristic Numbers *Let M^n be R -oriented and $c \in H^n(BSO(n); R)$ a characteristic class, then its associated characteristic number is defined as*

$$\langle c(TM), [M] \rangle =: c([M]).$$

Theorem 3.2.9 *Consider the inclusion $j^* : H^n(BSO(n+1); R) \longrightarrow H^n(BSO(n); R)$ and let $c \in H^n(BSO(n+1); R)$ be a characteristic class. If M_0, M_1 are oriented bordant, then*

$$(j^* c)([M_0]) = (j^* c)([M_1]).$$

Remark 3.2.10 *Let M^n be a manifolds, whose tangent bundle arises as the pullback by $f_{TM} : M \rightarrow BSO(n)$, then $(f_{TM})_*[M] \in H_n(BSO(n))$, and*

$$\langle c, (f_{TM})_*[M] \rangle = \langle (f_{TM})^* c, [M] \rangle = c([M]).$$

Remark 3.2.11 *The theorem can be applied to polynomials of Pontrjagin classes, but not to the Euler class.*

3.3 Cohomology of classifying spaces

We need one final ingredient for the computation of the cohomology of classifying spaces, namely the transfer for finite coverings.

Remark 3.3.1 *Let G be a finite group. For a G -principal bundle $p : X \longrightarrow Y$, we consider a singular simplex $c : \Delta^n \rightarrow Y$. Because the simplex is simply-connected, there exist $|G|$ different lifts \tilde{c} of c to X and two lifts differ by some $g \in G$.*

Definition 3.3.2 Transfer Map *Let G be a finite group and let $p : X \longrightarrow Y$ be a G -principal bundle. We define the transfer map*

$$\text{trf}_p : C_*(Y) \longrightarrow C_*(X)$$

by the following formula

$$\text{trf}_p(c) := \sum_{\tilde{c} \text{ lift of } c} \tilde{c}.$$

This equals $\sum_{g \in G} g\tilde{c}$ if a specific lift is chosen.

Remark 3.3.3 Transfer Map *Since trf_p is a chain map, we get the induced map*

$$\text{trf}_p : H_n(Y; R) \longrightarrow H_n(X; R),$$

Considering the induced map of chain complexes

$$p_* : C_*(X) \longrightarrow C_*(Y),$$

we have

$$\text{trf}_p \circ p_*(c) = \sum_{g \in G} gc, \quad p_* \circ \text{trf}_p(c) = |G|c. \tag{4}$$

By dualizing, we get an induced map on cochain complexes (and hence in cohomology)

$$\text{trf}_p^* : C^*(X; R) \longrightarrow C^*(Y; R).$$

Equation 4 implies

$$\text{trf}_p^* \circ p^*(y) = |G|y, \quad p^* \circ \text{trf}_p^*(y) = \sum_{g \in G} gy$$

on the cochain level. Because $(pg)^*y = p^*y$, p^* maps into the G -invariant part

$$C^*(Y; R) \xrightarrow{p^*} C^*(X; R)^G.$$

This map is injective (true for any covering), and it surjective if $R = \mathbb{Q}$ because

$$x = \frac{1}{|G|} \sum_g gx = \frac{1}{|G|} p^* \circ \text{trf}_p^*(x) = p^* \left(\frac{1}{|G|} \text{trf}_p^*(x) \right),$$

Because G is finite and we are considering \mathbb{Q} -vector spaces, $H^*(C^*(X; \mathbb{Q})^G) \cong H^*(X; \mathbb{Q})^G$. Altogether, we have proven that

$$p^* : H^*(Y; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})^G$$

is an isomorphism.

Theorem 3.3.4 Ring – Isomorphisms *The following ring homomorphisms are isomorphisms*

- 1.) $\mathbb{Z}[c_1, \dots, c_n] \longrightarrow H^*(BU(n); \mathbb{Z}),$
- 2.) $\mathbb{Z}[c_2, \dots, c_m] \longrightarrow H^*(BSU(n); \mathbb{Z}),$
- 3.) $\mathbb{Z}_2[w_1, \dots, w_n] \longrightarrow H^*(BO(n); \mathbb{Z}_2),$
- 4.) $\mathbb{Z}_2[w_2, \dots, w_n] \longrightarrow H^*(BSO(n); \mathbb{Z}_2),$
- 5a.) $\mathbb{Q}[p_1, \dots, p_m] \longrightarrow H^*(BO(2m+1); \mathbb{Q}),$
- 5b.) $\mathbb{Q}[p_1, \dots, p_m] \longrightarrow H^*(BO(2m); \mathbb{Q}),$
- 6.) $\mathbb{Q}[p_1, \dots, p_m] \longrightarrow H^*(BSO(2m+1); \mathbb{Q}),$
- 7.) $\frac{\mathbb{Q}[p_1, \dots, p_m, e]}{(p_m - e^2)} \longrightarrow H^*(BSO(2m); \mathbb{Q}).$

Proof: All the proofs are by induction over n or m respectively.

- 1.) The key fact needed for the inductive proof is the existence of a homotopy commutative diagram

$$\begin{array}{ccc} BU(n-1) & \xrightarrow{\cong} & SV_n \\ \downarrow & & \downarrow p \\ BU(n) & \xrightarrow{id} & BU(n), \end{array}$$

where SV_n is the unit sphere bundle of the universal vector bundle and the left vertical map is induced by the inclusion $U(n-1) \rightarrow U(n)$. A similar statement is true for the orthogonal groups.

Now we evoke the Gysin-Sequence

$$\begin{array}{ccc} \mathbb{Z}[c_1, \dots, c_{n-1}] & \xrightarrow{(*)} H^*(BU(n-1)) & \xrightarrow{p!} H^*(BU(n)) \\ & \searrow \exists j & \swarrow p^* \\ & & H^*(BU(n)) \end{array}$$

$\swarrow \cdot c_n$

where $(*)$ is the induction hypothesis and j exists, since p^* is surjective (because the Chern class c_i on $BU(n-1)$ extends to $BU(n)$) and every surjection on a polynomial ring has a split.

Due to the surjectivity of p^* , we have that $p_! = 0$ and thus $\cdot c_n$ is injective, which means for a $x \in H^*(BU(n))$, that we have

$$x = c_n \cdot y + jp^*(x)$$

for some y if smaller degree. We can now repeat this argument with y .

2,3,4) Are proved similarly.

6,7) (i) Here we obtain from the Gysin sequence and the inductive assumption a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}[p_1, \dots, p_{m-1}] \cong H^*(BSO(2m-1)) & \xrightarrow{p_!} & H^*(BSO(2m)) \\ & \swarrow p^* & \searrow \cdot e \\ & H^*(BSO(2m)) & \end{array}$$

Again, p^* is surjective and a similar argument as before applies.

(ii) For $p: BSO(2m) \rightarrow BSO(2m+1)$ the induced map p^* is not surjective, since e is not in the image. Instead, p^* is injective.

$$\begin{array}{ccc} \frac{\mathbb{Q}[p_1, \dots, p_m, e]}{(p_m^2 - e)} \cong H^*(BSO(2m)) & \xrightarrow{p_!} & H^*(BSO(2m+1)) \\ & \swarrow p^* & \searrow \cdot e=0 \\ & H^*(BSO(2m+1)) & \end{array}$$

We need to determine the kernel of the Gysin map $p_!$. For that, one uses the identity $p_!(p^*x \cdot y) = x \cdot p_!(y)$. This implies that if x is a polynomial in the Pontrjagin classes, then $p_!(x) = 0$ and $p_!(ex) = p_!(e)x$. Therefore the kernel is the ideal $eH^*(BSO(2m))$, as was to be shown.

5.) $BO(n) \cong EO(n) \times_{O(n)} O(n)/SO(n)$, so $BSO(n) \rightarrow BO(n)$ is a two sheeted covering.

Identifying

$$BSO(n) = \{(V, o) \mid V \subseteq \mathbb{R}^\infty, \dim(V) = n, o \text{ orientation of } V\},$$

we have a Deck-transformation

$$T: BSO(n) \rightarrow BSO(n), \quad T(V, o) = (V, -o).$$

We have a \mathbb{Z}_2 action $\mathbb{Z}_2 \times H^*(BSO(n); \mathbb{Q}) \rightarrow H^*(BSO(n); \mathbb{Q})$ and get the isomorphism

$$H^*(BO(n); \mathbb{Q}) \xrightarrow{\cong} H^*(BSO(n); \mathbb{Q})^{\mathbb{Z}_2}.$$

The fact that $T(e) = -e$ and $T(p_i) = p_i$, finishes the proof.

□

3.4 Multiplicative Sequences

Theorem 3.4.1 *Let $V \rightarrow X$ be an oriented vector bundle and*

$$F(V) \in H^{**}(X; \mathbb{Q}) := \prod_{k \geq 0} H^k(X; \mathbb{Q})$$

such that

$$F(X \times \mathbb{R}) = 1, \quad F(V \oplus W) = F(V)F(W),$$

then the following is a ring homomorphism

$$\begin{aligned} \Omega_* &\longrightarrow \mathbb{Q}. \\ [M] &\longmapsto \langle F(TM), [M] \rangle \end{aligned}$$

Definition 3.4.2 Multiplicative Characteristic Class *A multiplicative characteristic class with values in \mathbb{Q} is a natural transformation*

$$F : \text{Prin}_G(-) \longrightarrow H^{**}(-; \mathbb{Q}).$$

In the following we shall need the set of multiplicative characteristic classes of \mathbb{K} -vector bundles with values in \mathbb{Q} that fulfill certain additional requirements. We set

$$\text{MCC}_{\mathbb{K}} := \{F : \text{Prin}_{\text{Gl}_n(\mathbb{K})}(-) \longrightarrow H^{**}(-; \mathbb{Q}) \mid F \text{ natural, } F(X \times \mathbb{K}) = 1, F(V \otimes_{\mathbb{K}} W) = F(V)F(W)\}$$

Remark 3.4.3 *Let $F \in \text{MCC}_{\mathbb{C}}$ and consider the dual canonical bundle $L \rightarrow \mathbb{C}P^{\infty}$, then $f(x) := F(L) \in H^{**}(\mathbb{C}P^{\infty}; \mathbb{Q}) = \mathbb{Q}[[x]]$ and*

$$f(x) = 1 + f_1x + f_2x^2 + \dots \in 1 + x\mathbb{Q}[[x]].$$

Theorem 3.4.4 Hirzebruch *The following maps are bijections. In the complex case*

$$\begin{aligned} \text{MCC}_{\mathbb{C}} &\longrightarrow 1 + x\mathbb{Q}[[x]], \\ F &\longmapsto F(L) \end{aligned}$$

and in the real case

$$\begin{aligned} \text{MCC}_{\mathbb{R}} &\longrightarrow 1 + x^2\mathbb{Q}[[x^2]]. \\ F &\longmapsto F(L_{\mathbb{R}}) \end{aligned}$$

Proof: We shall only prove the complex case.

- First we prove injectivity. Let $F_0(L) = F_1(L)$, then due to the splitting principle, we have

$$F_0(p^*V) = F_0(L_1 \oplus \dots \oplus L_n) = F_0(L_1) \cdots F_0(L_n) = F_1(L_1) \cdots F_1(L_n) = F_1(p^*V).$$

- Now we come to surjectivity. Let $f \in 1 + x\mathbb{Q}[[x]]$ and $V = \bigoplus_{i=1}^n L_i$ for line bundles L_i . Then we define

$$F(V) := \prod_{i=1}^n F(L_i) := \prod_{i=1}^n f(c_1(L_i)) \equiv \prod_{i=1}^n f(x_i).$$

The main theorem on symmetric polynomials, says that for monomials σ_i of even order, there is a $K_r^n(\sigma_1, \dots, \sigma_n)$ such that

$$K_r^n(\sigma_1(x_1, \dots, x_n), \dots, \sigma_n(x_1, \dots, x_n)) = \left(\prod_{i=1}^n f(x_i) \right)_{2r} \in H^{2r}(X; \mathbb{Q}),$$

and

$$c(V) = \prod_{i=1}^n c(L_i) = \prod_{i=1}^n (1 + x_i) = \sum_{i=1}^n \sigma_i(x_1, \dots, x_n)$$

so we have $c_i = \sigma_i(x_1, \dots, x_n)$ and thus

$$K_r^n(c_1(V), \dots, c_n(V)) = \left(\prod_{i=1}^n f(x_i) \right)_{2r}.$$

□

Remark 3.4.5 *So for every $f \in 1 + x^2\mathbb{Q}[[x^2]]$, we get an element $F \in \text{MCC}_{\mathbb{R}}$ and thus the ring homomorphism*

$$\Omega_*^{SO} \otimes \mathbb{Q} \rightarrow \mathbb{Q}, \quad [M] \mapsto F([M]).$$

In the following, we shall consider the new series

$$\begin{aligned} g(t) &:= \sum_{n=0}^{\infty} F([\mathbb{C}P^n])t^n = \sum_{n=0}^{\infty} \langle F(T\mathbb{C}P^n), [\mathbb{C}P^n] \rangle t^n = \sum_{n=0}^{\infty} \langle F(L^{\oplus(n+1)}), [\mathbb{C}P^n] \rangle \\ &= \sum_{n=0}^{\infty} \langle f(x)^{n+1}, [\mathbb{C}P^n] \rangle t^n = \sum_{n=0}^{\infty} \varphi_n^{(f)} t^n \end{aligned}$$

where $\varphi_n^{(f)}$ is the n -th coefficient of $f(x)^{n+1}$.

Theorem 3.4.6 Hirzebruch *Let f and g be as above, then setting*

$$q(x) := \frac{x}{f(x)}, \quad h(x) := q^{-1}(x) \quad \text{i.e. } h(q(x)) = x,$$

it holds that

$$g(x) = h'(x).$$

Proof: We take f to have a positive convergence radius. Then, since $\varphi_n^{(f)}$ was the n -th coefficient of $f(x)^{n+1}$, we have

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} \varphi_n^{(f)} t^n = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_c \frac{f(x)^{n+1}}{x^{n+1}} dx \right) t^n = \frac{1}{2\pi i} \int_c \frac{1}{1 - \frac{f(x)t}{x}} \frac{f(x)}{x} dx \\ &= \frac{1}{2\pi i} \int_c \frac{1}{1 - \frac{t}{g(x)}} \frac{1}{q(x)} dx = \frac{1}{2\pi i} \int_c \frac{dx}{q(x) - t} = \frac{1}{2\pi i} \int_c \frac{h'(z)}{z - t} dz = h'(z) \end{aligned}$$

□

Corollary 3.4.7

1.) *There is exactly one $f \in 1 + x^2\mathbb{Q}[[x^2]]$, such that for given $a_n \in \mathbb{Q}$ and $a_0 = 1$, we have*

$$\varphi_n^{(f)} = a_n \quad \forall n \in \mathbb{N}.$$

2.) *The following homomorphism is injective*

$$\begin{aligned} \mathbb{Q}[y_1, y_2, \dots] &\longrightarrow \Omega_* \otimes \mathbb{Q} \\ y_i &\longmapsto [\mathbb{C}P^{2i}]. \end{aligned}$$

3.) *If $\varphi_n^{(f)} = 1$ for all $n \in \mathbb{N}$, then $g(t) = \sum_{k=0}^{\infty} t^{2k} = \frac{1}{1-t^2}$, which gives $h(t) = \text{arctanh}(t)$, $q(x) = \tanh(x)$ and*

$$f(x) = \frac{x}{q(x)} = \frac{x}{\tanh(x)}.$$

3.5 Bordism vs. Homotopy: The Pontrjagin-Thom Construction

Definition 3.5.1 *Let M^m be a smooth manifold, $V \rightarrow X$ a smooth vector bundle of rank k and $A \subseteq M$ a closed subset. We define*

$$L(M, A; V) := \{(N, g, \varphi) \mid N \subseteq M \text{ closed codim } k, N \cap A = \emptyset, g : N \rightarrow X, \varphi : \nu_N^M \xrightarrow{\cong} g^*V\}.$$

Let everything be as in the definition. Consider the Thom space $\text{Th}(V) := D_1V/SV$ with $\infty = \{SV\}$. To any given continuous map

$$f : (M, A) \longrightarrow (\text{Th}(V), \infty),$$

we can associate an element $(f^{-1}(0), f|_{f^{-1}(0)}, \varphi) \in L(M, A; V)$. However we need to say what φ is supposed to be. For that, we consider a tubular neighborhood $U = D_{1/2}V$ of M in D_1V where $f^{-1}(U) \subseteq M \setminus A$. Making use of a smooth approximation, we can assume, that $f|_{f^{-1}(0)}$ is smooth. If f is transverse to the zero section: $f \pitchfork 0$, we have

$$f^{-1}(0) \subset M \setminus A, \quad \dim(f^{-1}(0)) = m - k, \quad \nu_N^M = (f|_N)^* \nu_0^U \cong (f|_N)^*V.$$

So there is a canonical map $\varphi : \nu_N^M \xrightarrow{\cong} (f|_N)^*V$.

Lemma 3.5.2 *There exists a map*

$$\begin{aligned} \{f : (M, A) \rightarrow (\text{Th}(V), \infty) \mid f \pitchfork 0\} &\longrightarrow L(M, A; V), \\ f &\longmapsto (f^{-1}(0), f|_{f^{-1}(0)}, \varphi). \end{aligned}$$

Let two smooth functions $f_0, f_1 \in \{f : (M, A) \rightarrow (\text{Th}(V), \infty) \mid f \pitchfork 0\}$ be homotopic relative to A with the homotopy

$$F : [0, 1] \times (M, A) \rightarrow (\text{Th}(V), \infty)$$

being constant on $[0, \varepsilon], [1 - \varepsilon, 1]$ and w.l.o.g. $F|_{F^{-1}(U)}$ smooth and $F \pitchfork 0$. We further consider the codimension k submanifold $W := F^{-1}(0) \subseteq [0, 1] \times M$ for which $W \cap ([0, 1] \times A) = \emptyset$ together with maps $G : W \rightarrow X$ and $\psi : \nu_W^{[0,1] \times M} \rightarrow G^*M$, such that

$$G|_{N_i} = f_i|_{N_i}, \quad \nu_W^{[0,1] \times M}|_{N_i} = \nu_{N_i}^M, \quad \psi|_{N_i} = \varphi_i,$$

where (N_i, g_i, φ_i) are the data obtained from f_i under the above map.

Lemma 3.5.3 *There is an induced map*

$$T : [(M, A), (\text{Th}(V), \infty)] \longrightarrow L(M, A; V) / \sim,$$

where $(N_0, g_0, \varphi_0) \sim (N_1, g_1, \varphi_1)$, iff there exists a closed codimension k submanifold $W \subseteq [0, 1] \times M$ with $W \cap ([0, 1] \times A) = \emptyset$ and maps $G : W \rightarrow X$, $\psi : \nu_W^{[0,1] \times M} \rightarrow G^*M$, such that

$$G|_{N_i} = g_i|_{N_i}, \quad \psi|_{N_i} = \varphi_i.$$

Theorem 3.5.4 Pontrjagin – Thom *T is bijective.*

Proof: We will construct an inverse of T :

$$P : L(M, A; V) \longrightarrow [(M, A), (\text{Th}(V), \infty)].$$

Now (N, f, φ) is given. We chose a tubular neighborhood $N \subset U \subseteq M \setminus A$ and consider the map

$$M \longrightarrow \overline{U}/\partial U = M/(M \setminus U), \quad x \mapsto \begin{cases} x, & \text{if } x \in U \\ \infty, & \text{if } x \notin U. \end{cases}$$

For a tubular map $\nu_N^M \xrightarrow{\cong} U$ we get the induced map $\bar{U}/\partial U \xrightarrow{\cong} \text{Th}(\nu_N^M)$, which induces the first of the following two maps

$$(M, A) \longrightarrow (\text{Th}(\nu_N^M), \infty) \longrightarrow (\text{Th}(V), \infty),$$

the second one being induced by $g : N \rightarrow X$ and $\varphi : \nu_N^M \xrightarrow{\cong} g^*V$. Defining $\tilde{L}(M, A; V) := \{(N, g, \varphi, t) \mid (N, g, \varphi) \in L(M, A; V), t : \nu_N^M \rightarrow M \setminus A \text{ tubular map}\}$, the above concatenation induces a map

$$\tilde{L}(M, A; V)/\sim \longrightarrow [(M, A), (\text{Th}(V), \infty)],$$

where $(N_0, g_0, \varphi_0, t_0) \sim (N_1, g_1, \varphi_1, t_1)$, iff there is a (W, G, ψ, T) with $(W, G, \psi,)$ as above and $T : \nu_W^{[0,1] \times M} \rightarrow [0, 1] \times (M \setminus A)$ with a tubular map such that $T|_{N_i} = t_i$.

Adding the fact that $\tilde{L}(M, A; V) \cong L(M, A; V)$, proves the claim. □

Example 3.5.5 Let $M = B \times S^n$ and $A = B \times \{*\}$, then we have

$$L(M, A; V)/\sim \cong [B \times (S^n, *); (\text{Th}(V), \infty)] = [(S^n, *); \text{map}(B, \text{Th}(V))] = \pi_n(\text{map}(B, \text{Th}(V)))$$

3.6 Pontrjagin-Thom Construction and Homology

Take $V \rightarrow X$ to be an oriented vector bundle of rank k and $f : S^{n+k} \rightarrow \text{Th}(V)$ with $f \pitchfork 0$. Setting $M := f^{-1}(0)$, we get an induced map $c : S^{n+k} \rightarrow \text{Th}(\nu_M^{S^{n+k}})$. We orient M , such that $TM \oplus \nu_M^{S^{n+k}} = M \times \mathbb{R}^{n+k}$. With $\tau \in H^k(\text{Th}(\nu_M^{S^{n+k}}))$, it holds that

$$\tau \cap c_*[S^n] \in H_n(\nu) \cong H_n(M),$$

which gives $\tau \cap c_*[S^n] = (-1)^{nk}[M]$.

Corollary 3.6.1 Let W be an oriented bordism from M_0 to M_1 and $j_i : M_i \hookrightarrow W$ the corresponding inclusions, then

$$(j_0)_*[M_0] = (j_1)_*[M_1].$$

Proof: Embedding W in $[0, 1] \times \mathbb{R}^n$ for some $n \in \mathbb{N}$, gives $c : ([0, 1] \times \mathbb{R}^n)^+ \rightarrow \text{Th}(\nu_W^{\mathbb{R}^{n+1}})$ and we have

$$([0, 1] \times \mathbb{R}^n)^+ \cong [0, 1]_+ \wedge S^n \cong ([0, 1] \times S^n)/([0, 1] \times \{*\}).$$

□

Definition 3.6.2 Hurewicz – Homomorphism Let $V \rightarrow X$ be a vector bundle of rank k , then the Hurewicz-homomorphism hur is defined as the following map, factoring over the Thom-isomorphism ($x \mapsto \tau \cap x$):

$$\begin{array}{ccc} \pi_{n+k} \text{Th}(V) & \xrightarrow{\text{hur}} & H_n(X) \\ & \searrow & \nearrow \cong \\ & H_{n+k} \text{Th}(V) & \end{array}$$

That is for $f : S^{n+k} \rightarrow \text{Th}(V)$ with $f \pitchfork 0$ and $M := f^{-1}(0)$, it holds that

$$\text{hur}([f]) = (\tau \cap f_*[S^n]) = (-1)^{nk} f_*[M].$$

4 Spectra and the Bordism Ring

4.1 Spectra

Definition 4.1.1 Stable Vector Bundle A vector bundle $V \rightarrow X$ is called stable, iff it has the following additional data:

- 1.) A filtration $X_0 \subset X_1 \subset \dots \subset X$ of X ,
- 2.) rank n vector bundles $V_n \rightarrow X_n$,
- 3.) isomorphisms $\varepsilon_n : V_n \oplus \mathbb{R} \rightarrow V_{n+1}|_{X_n}$.

Two stable vector bundles V_0, V_1 over X are called equivalent (concordant), iff there is a stable vector bundle V over $X \times [0, 1]$, such that $V_{X \times 0} = V_0$ and $V_{X \times 1} = V_1$.

Example 4.1.2 Stable normal bundle.

Definition 4.1.3 (Ω) – Spectrum A spectrum is a sequence of pointed spaces $\{X_n\}_{n \in \mathbb{N}}$ and maps

$$\varepsilon_n : \Sigma X_n \rightarrow X_{n+1}.$$

A spectrum $\{X_n, \varepsilon_n\}_{n \in \mathbb{N}}$ is called Ω – spectrum, iff ε_n is a homotopy equivalence for all $n \in \mathbb{N}$.

Remark 4.1.4 An equivalent definition for a spectrum is, to prescribe maps $X_n \rightarrow \Omega X_{n+1}$.

Definition 4.1.5 Thom Spectrum Let $V \rightarrow X$ be a stable vector bundle, The Thom spectrum $\text{Th}(V)$ is defined by setting $\text{Th}(V)_n := \text{Th}(V_n)$ and taking the maps

$$\Sigma(\text{Th}(V)_n) = \text{Th}(V_n \oplus \mathbb{R}) \xrightarrow{\varepsilon_n^*} \text{Th}(V_{n+1}) = \text{Th}(V)_{n+1}.$$

where $\varepsilon_n : V_n \oplus \mathbb{R} \rightarrow V_{n+1}|_{X_n}$ is the isomorphism, which is part of the data of the given stable vector bundle.

Definition 4.1.6 Eilenberg – MacLane Spectrum $E_n = K(\mathbb{Z}; n)$

Theorem 4.1.7 Generalized (Co)homology Theories Let $\{E_n, \varepsilon_n\}_{n \in \mathbb{N}}$ be a spectrum and X be a space, then the following maps

$$\pi_{n+k}(E_k \wedge X_+) \xrightarrow{(\Sigma \wedge \text{id})^*} \pi_{n+k+1}(\Sigma E_k \wedge X_+) \xrightarrow{(\varepsilon_k \wedge \text{id})^*} \pi_{n+k+1}(E_{k+1} \wedge X_+)$$

make it possible to define the generalized homology theory

$$E_n(X) := \varinjlim_k \pi_{n+k}(E_k \wedge X_+).$$

For a pair (X, A) , we can define the generalized (co)homology theories

$$\begin{aligned} E_n(X, A) &:= \varinjlim_k \pi_{n+k}(E_k \wedge X/A), \\ E^n(X, A) &:= \varinjlim_k [(X, A), (\Omega^{k-n} E_k, *)]. \end{aligned}$$

Example 4.1.8 By the above construction, the Eilenberg-MacLane spectrum $E_n = K(\mathbb{Z}; n)$, gives

$$E^n(X, A) = \varinjlim_k [(X, A), (\Omega^{k-n} K(\mathbb{Z}; k), *)] = \varinjlim_k [(X, A), K(\mathbb{Z}; n)] = H^n(X, A; \mathbb{Z}).$$

Definition 4.1.9 Homotopy Groups of a Spectrum Let $E = \{E_n, \varepsilon_n\}_{n \in \mathbb{N}}$ be a spectrum, then we define its homotopy groups by

$$\pi_n(E) := \varinjlim_k \pi_{n+k}(E_k) = E_n(*).$$

Theorem 4.1.10 Brown Representation Theorem On the category on CW-complexes, all generalized (co)homology theories have an associated spectrum, which gives rise to them by the above construction.

Definition 4.1.11 Thom Spectrum We consider the universal bundle $ESO(n) \rightarrow BSO(n)$ and set

$$V_n := ESO(n) \times_{SO(n)} \mathbb{R}^n,$$

which lets $V := \varinjlim_n V_n$ be a stable vector bundle. The Thom spectrum is now defined as

$$MSO := \text{Th}(V) = \varinjlim_n MSO(n),$$

with $MSO(n) = \text{Th}(V_n)$.

Definition 4.1.12 The to the Thom spectrum associated generalized (co)homology theories are denoted

$$MSO_n(X) := \varinjlim_k \pi_{n+k}(MSO(k) \wedge X_+), \quad MSO^n(X) := \varinjlim_k [X_+, \Omega^{k-n} MSO(k)].$$

Remark 4.1.13 Our goal is to calculate the bordism ring. In order to do so, the following isomorphism will come in very handy

$$\pi_*(MSO) \cong \Omega_*^{SO},$$

and explains our interest in the Thom spectrum.

Theorem 4.1.14 Pontrjagin – Thom Let M^n be a closed manifold, then

$$\pi_{n+k}(MSO(k) \wedge X_+) = \{(M^n, (f, g), \varphi) \mid M \subseteq \mathbb{R}^{n+k}, f : M \rightarrow BSO(k), g : M \rightarrow X, \varphi : \nu_M \cong f^* V_{\mathbb{R}}\} / \sim,$$

and

$$\varinjlim_k \pi_{n+k}(MSO(k) \wedge X_+) = \{(M^n, g, o) \mid g : M \rightarrow X, o \text{ orientation on } M^n\} / \sim.$$

Where in both cases \sim means bordant.

Remark 4.1.15

1.) The following holds:

$$\begin{aligned} MSO^n(X) &= \varinjlim_k [X_+, \Omega^{k-n} MSO(k)] = \varinjlim_k [\Sigma^{k-n} X_+, MSO(k)] \\ &= \varinjlim_k [(S^{k-n} \times X; \{*\} \times X), MSO(k)] \\ &\stackrel{\text{PT}}{=} \varinjlim_k L((S^{k-n} \times X; \{*\} \times X), V_k) \\ &= \varinjlim_k \{M \subseteq \mathbb{R}^{k-n} \mid \dim(M) = \dim(X) - n, \nu_M \text{ oriented pr } : M \rightarrow X \text{ proper}\} / \sim \\ &= \{p : M \rightarrow X \mid p \text{ proper, oriented}\} / \sim. \end{aligned}$$

2.) If M^n is closed and oriented, then

$$MSO_k(M) \xrightarrow{\cong} MSO^{n-k}(X).$$

4.2 Calculation of the Oriented Bordism Ring

Theorem 4.2.1 *The following map is an isomorphism*

$$\begin{aligned} \mathbb{Q}[y_1, y_2, \dots] &\longrightarrow \Omega_* \otimes \mathbb{Q}. \\ y_n &\longmapsto [\mathbb{C}P^n] \end{aligned}$$

Proof: We have already seen injectivity. Let $p(n)$ be the number of partitions of $n \in \mathbb{N}$, then

$$\dim \mathbb{Q}[y_1, y_2, \dots]_i = \begin{cases} 0, & \text{if } i \notin 4\mathbb{N} \\ p(i/4), & \text{if } i \in 4\mathbb{N}. \end{cases}$$

The surjectivity can be seen as follows:

$$\begin{aligned} \Omega_n \otimes \mathbb{Q} &= \pi_n(MSO) \otimes \mathbb{Q} \\ &= \varinjlim_k \pi_{n+k}(MSO_k) \otimes \mathbb{Q} \xrightarrow{\text{hur}} \varinjlim_k H_{n+k}(MSO_k; \mathbb{Q}) \stackrel{\text{Thom}}{\cong} \varinjlim_k H_n(BSO(k); \mathbb{Q}). \end{aligned}$$

Now $H_n(BSO(k); \mathbb{Q})$ is dual to $H^n(BSO(k); \mathbb{Q})$, which has the same rank as $\mathbb{Q}[y_1, y_2, \dots]$. □

4.3 The Signature

In the following, let M^{4k} be a closed and oriented manifold. We consider the following symmetric, bilinear, non-degenerate form

$$\begin{aligned} \beta : H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) &\longrightarrow \mathbb{R}. \\ (x, y) &\longmapsto \langle x \cup y, [M] \rangle \end{aligned}$$

With Sylvester's theorem, we know, that there is a basis $\{v_i\}_{i \in I}$, such that

$$\beta = \text{diag}(\underbrace{1, \dots, 1}_r, \underbrace{-1, \dots, -1}_s),$$

with r, s well defined.

Definition 4.3.1 Signature *Let everything be as above, however M^n be of arbitrary dimension n . Then we define the signature of M to be*

$$\text{sign}(M) := \begin{cases} \text{sign}(\beta), & \text{if } n \in 4\mathbb{N} \\ 0, & \text{if } n \notin 4\mathbb{N}, \end{cases} \quad \text{with } \text{sign}(\beta) := r - s.$$

In order to prove some of the properties of the signature, we shall need a little linear algebra.

Lemma 4.3.2 *Let (V, β) be a \mathbb{R} -vector space with symmetric bilinear form and $U \subseteq V$ be an isotropic subspace (i.e. $\beta|_{U \times U} = 0$). If $\dim U = \frac{1}{2} \dim V$, then $\text{sign}(\beta) = 0$.*

Proof: We chose a complement W of U and a basis w_1, \dots, w_n of W , such that $(w_i, w_j) = \delta_{ij} a_i$ with $a_i \in \{\pm 1, 0\}$. Now (u_i, w_j) exists due to non-degeneracy and we get the matrix

$$\left(\begin{array}{cc|cc} \mathbf{1} & & & \\ & -\mathbf{1} & & \mathbf{1} \\ & & 0 & \\ \hline & & & \\ & \mathbf{1} & & 0 \end{array} \right)$$

thus $\text{sign}(V) = 0$.

□

Lemma 4.3.3 *Let (V_0, ω_0) , (V_1, ω_1) be symplectic vector spaces, $(V_0 \otimes V_1, \omega_0 \otimes \omega_1)$ symmetric, then*

$$\text{sign}(V_0 \otimes V_1, \omega_0 \otimes \omega_1) = 0.$$

Proof: Take an isotropic subspace $U \subseteq V_0$ with $\dim U = \frac{1}{2} \dim V_0$, then $(\omega_0 \otimes \omega_1)|_{U \otimes V_1}$ is isotropic.

□

Lemma 4.3.4 Properties of the Signature

- 1.) $\text{sign}(-M) = -\text{sign}(M)$,
- 2.) $\text{sign}(M \sqcup N) = \text{sign}(M) + \text{sign}(N)$,
- 3.) $\text{sign}(\mathbb{C}P^{2n}) = 1$,
- 4.) $\text{sign}(\partial M) = 0$,
- 5.) $\text{sign}(M \times N) = \text{sign}(M) \cdot \text{sign}(N)$.

Proof:

1.)-3.) These are easy.

4.) Let $\dim(\partial M) = 4n$ and $j : \partial M \hookrightarrow M$ be the inclusion. Then

$$\begin{array}{ccccc} H^{2n}(M) & \xrightarrow{j^*} & H^{2n}(\partial M) & \xrightarrow{\delta} & H^{2n+1}(M, \partial M) \\ & & \downarrow \cong & & \downarrow \cong \\ & & H_{2n}(\partial M) & \xrightarrow{j_*} & H_{2n}(M) \end{array}$$

and we have $\langle j^*x \cup j^*y, [M] \rangle = \langle x \cup y, j_*[\partial M] \rangle = 0$, since $j_*[\partial M] = 0$. Thus $\text{Im}(j^*)$ is isotropic. Also we have

$$\dim \text{Im} j^* = \dim \ker \delta = \dim \ker j_* = \dim H_{2n}(\partial M) - \dim \text{Im} j_* = \dim H_{2n}(\partial M) - \dim \text{Im} j^*,$$

which gives $\dim \text{Im} j^* = \frac{1}{2} \dim H_{2n}(\partial M)$.

5.) Let $\dim M = n$ and $\dim N = 4k - n$. We define $A_s := H^s(M) \otimes H^{2k-s}(N)$, which gives

$$H^{2k}(M \times N) = \bigoplus_{0 \leq s < n/2} (A_s \oplus A_{n-2}) \oplus A_{n/2}.$$

The form is zero on $A_s \times A_t$, if $s + t \neq n$, thus the above decomposition is orthogonal. Now $A_s \subseteq A_s \oplus A_{n-s}$ is isotopic and thus

$$\text{sign}(H^{2k}(M \times N)) = \text{sign}(A_{n/2}).$$

- If n is odd we have $A_{n/2} = 0$ and thus $\text{sign}(M \times N) = 0 = \text{sign}(M) \cdot \text{sign}(N)$.
- If $n = \dim M = 4p + 2$, $\dim(N) = 4q + 2$, we have $p + q - 1 = k$ and

$$\text{sign}(H^{2k}(M \times N)) = \text{sign}(A_{2p+1}) = \text{sign}(H^{2p+1}(M) \otimes H^{2q+1}(N)) = 0$$

since both are symplectic forms.

- If $\dim M = 4p$, $\dim N = 4q$, we have

$$\text{sign}(M \times N) = \text{sign}(H^{2p}(M) \otimes H^{2q}(N)) = \text{sign}(M) \cdot \text{sign}(N).$$

□

Theorem 4.3.5 Hirzebruch *Let $L \in H^*(BSO; \mathbb{Q})$ associated to $\frac{x}{\tanh x}$, then*

$$\text{sign}(M) = \langle L(TM), [M] \rangle.$$

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