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ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 13
Abgabe: Mittwoch, 22.7., in der Vorlesung

Exercise 13.1. Compute the cohomology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}/2$ and in $\mathbb{Z}$, using the universal coefficient theorem.

Exercise 13.2. Let G be a group and R be a commuative ring with unit. The *group ring* RG of G is defined as follows: as an abelian group, RG is the free R-module generated by the elements of G. Given two elements $\sum_{g\in G} a_g g$, $\sum_{g \in G} b_g g$ with $a_g, b_g \in R$ (only finitely many a_g, b_g are nonzero), their product is defined to be $\sum_{g\in G}(\sum_{h,k\in G; hk=g} a_h b_k) g$. Show that this is a ring with unit (not commuative unless G is). Show that a left RG-module is the same as a representation of G in R-modules. Let V be an RG-module. Denote by $V^G \subset V$ be the R-module consisting of all $v \in V$ with $gv = v$ for all $v \in V$ (the *invariant subspace of* V. Show that $V \mapsto V^G$ defines an additive left-exact functor $RG - \mathbf{Mod} \to R - \mathbf{Mod}$.

Exercise 13.3. Let X be a topological space with a free right-G-action, $Y := X/G$ such that the quotient map $q : X \to Y$ is a covering. The G-action on X turns the singular cochain complex $C^*(X; A)$ with coefficients in the abelian group into a chain complex of left $\mathbb{Z}G - \mathbf{Mod}$ and so we an talk about the invariant subcomplex $C^*(X; A)^G$. Show that q induces an isomorphism $C^*(Y; A) \to C^*(X; A)^G$.

Exercise 13.4. Let G be a finite group. Let R be a commutative ring in which $|G|$ in invertible. Show that the functor $RG - \mathbf{Mod} \to R - \mathbf{Mod}$; $V \to V^G$ is exact. Hint: you may need the operator $v \mapsto \frac{1}{|G|}\sum_{g\in G} gv$ which sends $V \to V^G$. Show: in the situation of exercise 13.3, the quotient map q induces an isomorphism $H^*(Y; R) \cong H^*(X; R)^G$. Hint: exercise 12.2. Show, by an example, that both the finiteness of G and the invertibility of $|G|$ are essential for this isomorphism to hold.

Exercise 13.5. Let $f : X \to Y$ be a finite covering, say of degree k. Recall the transfer $f^! : C_*(Y) \to C_*(X)$, which is a chain map. There is an induced map on singular cochain complexes $f_! : C^*(X) \to C^*(Y)$ (arbitrary coefficents) which in turn induces a map $f_! : H^*(X) \to H^*(Y)$. Show that the composition $f_! \circ f^* : H^*(Y) \to H^*(Y)$ is equal to multiplication by k.